SOURCE ALGEBRAS AND COHOMOLOGY OF BLOCK IDEALS OF FINTIE GROUP ALGEBRAS

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1. Block ideals

We shall argue under the following situation:

- G is a finite group
- k is an algebraically closed field of characteristic $p \mid |G|$
- B is a block ideal of kG with a defect group D:
  (1) B is an indecomposable ideal $\mid kG$,
  (2) the indecomposable $k[G\times G^{\mathrm{op}}]$-module B has vertex $\Delta(D)$.

Definition. (1) X is a source module of B:
an indec. $k[G\times D^{\mathrm{op}}]$-direct summand of B with vtx $\Delta(D)$.
(2) $A = X^*\otimes_B X$ is *a source algebra* of B.

Theorem (Puig)**.** *A and B are Morita equivalent.*

Definition. (1) $\exists!$ block ideal b_D of $C_G(D)$ associated with X.
(D, b_D) is a Sylow B-subpair.
(2) Brauer category $\mathcal{F}_{(D,b_D)}(B,X)$:

- object (Q, b_Q) of $Q\leqslant D$ and a block b_Q of $kC_G(Q)$
- morphism $(Q, b_Q)\longrightarrow(R, b_R)$; conjugation by an element in G.

Definition (Linckelmann [2])**.** The cohomology ring of B w.r.t D and X:

$$H^*(G,B;X) = \{\,\zeta\in H^*(D,k)\mid \zeta \text{ is } \mathcal{F}_{(D,b_D)}(B,X)\text{-stable}\,\}.$$

Theorem 1 (Linckelmann [2], Sasaki [3])**.** *For* $\zeta\in H^*(D,k)$

$$\zeta\in H^*(G,B;X)\iff \delta_D\zeta\in HH^*(kD) \text{ is } A\text{-stable}.$$

Problem. To know module structure of $A = X^*\otimes_B X$.

Note that A is isomorphic to a direct sum of some $k[DgD]$s because

$$A\mid kG = \bigoplus_{DgD\in D\backslash G/D} k[DgD].$$

Theorem. (1)

$$A\simeq\left(\bigoplus_{g\in N_G(D,b_D)/DC_G(D)} k[Dg]\right)\oplus N,$$

where N is a direct sum of $k[DxD]$s with $x\in G\smallsetminus N_G(D)$.
(2) *No two of $k[Dg]$s, $g\in N_G(D,b_D)/DC_G(D)$, are isomorphic.*

We have had almost no information on N above!

Theorem 2 (Sasaki, 2013)**.** *Let $(P,b_P),(Q,b_Q)\subseteq(D,b_D)$; assume that $C_D(P)$ is a defect group of b_P or $C_D(Q)$ is a defect group of b_Q. For $g\in G$ with ${}^g(P,b_P)=(Q,b_Q)$ if the map*

$$t_g : H^*(D,k)\longrightarrow H^*(D,k);\zeta\mapsto \mathrm{tr}^D\,\mathrm{res}_Q\,{}^g\zeta$$

does not vanish, then the following hold:
(1) *$Q = D\cap {}^gD$; hence $t_g = t_{DgD}$,*
(2) *the (kD,kD)-bimodule $k[DgD]$ is isomorphic to a direct summand of the source algebra A,*
(3) *a (kD,kD)-bimodule $k[Dg'D]$ is isomorphic to $k[DgD]$ if and only if $Dg'D = DcgD$ for some $c\in C_G(Q)$.*

2. Trace maps for cohomology rings of blocks

The (kD,kD)-bimodule A induces a transfer map on $H^*(D,k)$:

$$\begin{array}{ccc} H^*(D,k) & \xrightarrow{\delta_D} & HH^*(kD) \\ t\downarrow & \circlearrowleft & \downarrow t_A \\ H^*(D,k) & \xrightarrow[\delta_D]{} & HH^*(kD)\end{array}.$$

Conjecture.

$$H^*(G,B;X) = t(H^*(D,k)).$$

Example. If $N_G(D,b_D)$ controls the fusion of subpairs in (D,b_D), then the above do hold. For example

- D is abelian,
- D is normal in G, and so on.

The transfer map t is described as follows:

$$t : H^*(D,k)\longrightarrow H^*(D,k);\zeta\mapsto \sum_{A\simeq\bigoplus_{DgD} k[DgD]} \mathrm{tr}^D\,\mathrm{res}_{D\cap {}^gD}\,{}^g\zeta.$$

However we do not know

- which $k[DgD]$ is isomorphic to a direct summand of A,
- how to determine the multiplicity of a direct summand of A isomorphic to $k[DgD]$

so that it would be so difficult to write down the map t explicitly.
The following are observations.

3. Blocks of tame representation type

Hereafter we let $p = 2$ and let B be a block of tame representation type with defect group D.
The defect group D is one of the followings:

- dihedral 2-group
$$\langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{-1}\,\rangle,\ n\geqslant 3;$$
- generalized quaternion 2-group
$$\langle\, x, y \mid x^{2^{n-2}} = y^2 = z, z^2 = 1, yxy^{-1} = x^{-1}\,\rangle,\ n\geqslant 3;$$
- semidihedral 2-group
$$\langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{-1+2^{n-2}}\,\rangle,\ n\geqslant 4.$$

In Kawai–Sasaki [1] we calculated cohomology rings of 2-blocks of tame representation type and of blocks with defect groups isomorphic to wreathed 2-groups of rank 2. We constructed a transfer map

$$\mathrm{Tr}_D^B : H^*(D,k) \to H^*(D,k)$$

such that

$$\mathrm{Tr}_D^B H^*(D,k) = H^*(G,B,X).$$

Let us assume that D is a semidihedral 2-group:

$$D = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{-1+2^{n-2}} \rangle, \ n \geqslant 4$$

and take subgroups $E, V \leqslant D$ as

$$E \simeq \text{four-group}, \quad V \simeq \text{quaternion group}.$$

Let

$$(E, b_E), (V, b_V) \subseteq (D, b_D).$$

The inertia groups $N_G(E, b_E)/C_G(E)$ and $N_G(V, b_V)/VC_G(V)$ determine the fusion of subpairs. Let us assume here that

$$N_G(E, b_E)/C_G(E) \simeq \mathrm{GL}(2,2), \ N_G(V, b_V)/VC_G(V) \simeq \mathrm{GL}(2,2)$$

and choose elements $g_0 \in N_G(E, b_E)$ and $g_1 \in N_G(V, b_V)$ such that

(1) g_0 induces an automorphism of E of order 3 and

(2) g_1 induces an outer automorphism V of order 3.

Let us define a (kD, kD)-bimodule M by

$$M = kD \oplus k[Dg_0D] \oplus k[Dg_1D].$$

Theorem 2 tells us the following.

Theorem 3. (1) $M \mid A$

(2) *The transfer map in* [1]

$$\mathrm{Tr}_D^B : H^*(D,k) \to H^*(D,k); \zeta \mapsto \zeta + \mathrm{tr}^D \mathrm{res}_E {}^{g_0}\zeta + \mathrm{tr}^D \mathrm{res}_V {}^{g_1}\zeta$$

is induced by M.

Moreover, a detailed analysis for fusion of subpairs shows us that the source algbra A induces the transfer map t_A as follows:

$$t_A : H^*(D,k) \to H^*(D,k); \zeta \mapsto \zeta + I_0 \mathrm{tr}^D \mathrm{res}_E {}^{g_0}\zeta + I_1 \mathrm{tr}^D \mathrm{res}_V {}^{g_1}\zeta.$$

Remark. (1) The coefficients I_0 and I_1 are still unknown!

(2) Similar results hold for othe blocks of tame representation type.

4. Blocks with wreathed defect groups

Here we assume that D is a wreathed 2-group of rank 2:

$$D = \langle a, b, t \mid a^{2^n} = b^{2^n} = t^2 = 1, \ ab = ba, \ tat = b \rangle, \ n \geq 2.$$

We can specify the subgroups $U, V \leqslant D$ such that the Inertial quotients of $(U, b_U), (V, b_V) \subseteq (D, b_D)$, which are both isomorphic to $\mathrm{GL}(2,2)$, determine the fusion of subpais. We assume here that

$$N_G(U, b_U)/C_G(U) \simeq \mathrm{GL}(2,2), \ N_G(V, b_V)/VC_G(V) \simeq \mathrm{GL}(2,2)$$

and choose elements $g_0 \in N_G(U, b_U)$ and $g_1 \in N_G(V, b_V)$ such that

(1) g_0 induces an automorphism of U of order 3 and

(2) g_1 induces an outer automorphism V of order 3.

Let us define $\mathrm{Tr}_D^B : H^*(D,k) \to H^*(D,k)$ as

$$\begin{aligned}\mathrm{Tr}_D^B : \zeta \mapsto \zeta &+ \mathrm{tr}^D \mathrm{res}_U {}^{g_0}\zeta + \mathrm{tr}^D \mathrm{res}_V {}^{g_1}\zeta \\ &+ \mathrm{tr}^D \mathrm{res}_{V \cap {}^{g_1}U} {}^{g_1 g_0}\zeta + \mathrm{tr}^D \mathrm{res}_{U \cap {}^{g_0}V} {}^{g_0 g_1}\zeta \\ &+ \mathrm{tr}^D \mathrm{res}_{V \cap {}^{g_1}U \cap {}^{g_1 g_0}V} {}^{g_1 g_0 g_1}\zeta.\end{aligned}$$

Then it follows that

$$\mathrm{Im}\,\mathrm{Tr}_D^B = H^*(G,B;X).$$

- The first five terms above come from direct summands of A but
- Theorem 2 cannnot be applied to the last one; it is unknown whether the last one is induced by a direct summand of A or not.

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References

[1] H. Kawai and H. Sasaki, Cohomology algebras of 2-blocks of finite groups with defect groups of rank two, J. Algebra **306** (2006), no. 2, 301–321.

[2] M. Linckelmann, Transfer in Hochschild cohomology of blocks of finite groups, Algebr. Represent. Theory **2** (1999), 107–135.

[3] H. Sasaki, Cohomology of block ideals of finite group algebras and stable elements, Algebr. Represent. Theory **16** (2013), 1039–1049.