Serre subcategories of artinian modules

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Throughout this talk

- R : a commutative Noetherian ring.
- $\mathrm{mod}(R)$: the category of finitely generated R-modules.
- $\mathrm{Art}(R)$: the category of artinian R-modules.
- a subcategory: a nonempty full subcategory which is closed under isomorphism.

Definition.

A subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is said to be

- **wide** if it is closed under kernels, cokernels and extensions.
- **Serre** if it is wide and closed under subobjects. (if and only if $\forall\ 0 \to L \to M \to N \to 0$ in $\mathcal{A}$, $M \in \mathcal{X} \Leftrightarrow L, N \in \mathcal{X}$.)

§0. Classification theory of subcategories.

[Hopkins, 1987], [Neeman,1992], [Thomason, 1997]

Classifying **thick** subcategories of the derived category.
(A thick subcategory : closed under direct summands and exact triangle.)

[Gabriel, 1962]

He gives a bijection:

$$\left\{ \begin{array}{c} \text{Serre subcategories} \\ \text{of } \mathrm{mod}(R) \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \begin{array}{c} \text{specialization closed subsets} \\ \text{of } \mathrm{Spec}\ R \end{array} \right\}$$

[Takahashi, 2008], [Krause, 2008]

$\exists$ 1-1 correspondences:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{subcategories of } \mathrm{mod}(R) \\ \text{closed under} \\ \text{submodules and extensions} \end{array} \right\} & \begin{array}{c} \xrightarrow{\Psi} \\ \xleftarrow[\Phi]{} \end{array} & \left\{ \text{subsets of } \mathrm{Spec}\ R \right\} \\ \uparrow\subseteq & & \uparrow\subseteq \\ \left\{ \begin{array}{c} \text{Serre subcategories} \\ \text{of } \mathrm{mod}(R) \end{array} \right\} & \begin{array}{c} \xrightarrow{\Psi} \\ \xleftarrow[\Phi]{} \end{array} & \left\{ \begin{array}{c} \text{specialization closed} \\ \text{subsets of } \mathrm{Spec}\ R \end{array} \right\} \end{array}$$

where $\Psi(\mathcal{M}) = \cup_{M\in\mathcal{M}} \mathrm{Ass}_R M$ and $\Phi(\mathcal{S}) = \{M \in \mathrm{mod}(R) | \mathrm{Ass}_R M \subseteq \mathcal{S}\}$.

In addition, Takahashi pointed out that

[Takahashi, 2008]

Every wide subcategory of $\mathrm{mod}(R)$ is a Serre subcategory of $\mathrm{mod}(R)$.

In this talk we want to consider the artinian analogue of these result.

Every wide subcategory of $\mathrm{Art}(R)$ is a Serre subcategory of $\mathrm{Art}(R)$.

$\exists$ 1-1 correspondences:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{subcategories of } \mathrm{Art}(R) \\ \text{closed under} \\ \textbf{quotient} \text{ and extensions} \end{array} \right\} & \begin{array}{c} \xrightarrow{\Psi} \\ \xleftarrow[\Phi]{} \end{array} & \left\{ \begin{array}{c} \text{subsets of} \\ \textbf{closed prime ideals} \text{ of } \hat{R} \end{array} \right\} \\ \uparrow\subseteq & & \uparrow\subseteq \\ \left\{ \begin{array}{c} \text{Serre subcategories} \\ \text{of } \mathrm{Art}(R) \end{array} \right\} & \begin{array}{c} \xrightarrow{\Psi} \\ \xleftarrow[\Phi]{} \end{array} & \left\{ \begin{array}{c} \text{specialization closed} \\ \text{subsets of} \\ \textbf{closed prime ideals} \text{ of } \hat{R} \end{array} \right\} \end{array}$$

§1. Wide subcategories of $\mathrm{Art}(R)$.

Let $M \in \mathrm{Art}(R)$. We denote by $\mathrm{Soc}(M)$ the sum of simple submodules of M. Since $\mathrm{Soc}(M)$ is artinian,

$$\mathrm{Soc}(M) = \oplus_{i=1}^{s} (R/\mathfrak{m}_i)^{n_i}.$$

Set $J_M = \bigcap_{i=1}^{s} \mathfrak{m}$ and $\hat{R}^{(M)} = \varprojlim R/J_M^n$.

Lemma 1 ([Sharp, 1992])

$$\forall x \in M, \exists k \in \mathbb{N} \text{ s.t. } (J_M)^k x = 0.$$

Hence M has the natural structure of $\hat{R}^{(M)}$-modules in such a way that $N \subset M$ is an R-submodule if and only if it is an $\hat{R}^{(M)}$-submodule.

Proof.

- Since $\mathrm{Soc}(M) = \oplus_{i=1}^{s}(R/\mathfrak{m}_i)^{n_i}$,

$$M \hookrightarrow \oplus_{i=1}^{s}(E_R(R/\mathfrak{m}_i))^{n_i}$$

where $E_R(R/\mathfrak{m}_i)$ is an injective hull of $R/\mathfrak{m}_i$.
- Hence, $\forall x \in M$, $\exists k$, $J_M^k = (\mathfrak{m}_1 \cdots \mathfrak{m}_s)^k x = 0$.
- Let $x \in M$ and $\hat{r} = (r_n + J_M^n)_{n\in\mathbb{N}} \in \hat{R}^{(M)}$. Suppose that $J_M^k x = 0$.
- Check that M has the structure of an $\hat{R}^{(M)}$-module such that

$$\hat{r}x := r_k x.$$

□

Strategy.

- $\forall M \in \mathrm{Art}(R)$, M can be regard as a module over "a certain complete semi-local ring" $\hat{R}$.
- **The Matlis duality theorem** holds over a noetherian complete semi-local ring.
- By using Matlis duality we replace the categorical property on a subcategory of $\mathrm{mod}(\hat{R})$ with that of $\mathrm{Art}(R)$.

$$\begin{array}{ccc} \mathrm{Art}(R) & \longrightarrow & \mathrm{mod}(\hat{R}) \\ \mathcal{X} & & \mathcal{X}^{\vee} \end{array}$$

Lemma 2

Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_s)$ be a noeth. complete semi-local ring and $E = \oplus_{i=1}^{s} E_R(R/\mathfrak{m}_i)$. For $\forall \mathcal{X} \subseteq \mathrm{Mod}(R)$, we denote by $\mathcal{X}^{\vee}$ $\{M^{\vee} \mid M \in \mathcal{X}\}$ where $(-)^{\vee} = \mathrm{Hom}_R(-, E)$. Then

$$\begin{array}{ccc} \mathrm{Art}(R) & \underset{(-)^{\vee}}{\overset{(-)^{\vee}}{\rightleftarrows}} & \mathrm{mod}(R) \\ \left\{ \begin{array}{c} \textit{closed under} \\ \textit{quotient modules and extensions} \end{array} \right\} & \rightleftarrows & \left\{ \begin{array}{c} \textit{closed under} \\ \textit{submodules and extensions} \end{array} \right\} \\ \{ \textit{ Wide subcategories } \} & \rightleftarrows & \{ \textit{ Wide subcategories } \} \\ \{ \textit{ Serre subcategories } \} & \rightleftarrows & \{ \textit{ Serre subcategories } \} \end{array}$$

Theorem 3

Let R be a noetherian ring. Every wide subcategory of $\mathrm{Art}(R)$ is a Serre subcategory of $\mathrm{Art}(R)$.

Remarks.

- For $M \in \mathrm{Mod}(R)$, we denote by $\mathrm{Wid}_R(M)$ **the smallest wide subcategory** of $\mathrm{Mod}(R)$ which contains M.
- If $M \in \mathrm{Art}(R)$, $\mathrm{Wid}_R(M) \subseteq \mathrm{Art}(R)$. Moreover,

$$\mathrm{Wid}_R(M) \cong \mathrm{Wid}_{\hat{R}^{(M)}}(M)$$

as subcategories of $\mathrm{Art}(\hat{R}^{(M)})$.

Proof.

- Let $\mathcal{X} \subseteq \mathrm{Art}(R)$ be wide. It is enough to show that $\mathcal{X}$ is **closed under submodules**.
- If not, $\exists M \in \mathcal{X}$, $\exists N \subseteq M$ s.t. $N \notin \mathcal{X}$. Consider $\mathrm{Wid}_R(M)$. Then

$$\mathrm{Wid}_R(M) \cong \mathrm{Wid}_{\hat{R}}(M)$$

in $\mathrm{Art}(\hat{R})$ where $\hat{R} = \hat{R}^{(M)}$.
- Applying the Matlis duality, we have that

$$\mathrm{Wid}_{\hat{R}}(M)^{op} \cong \mathrm{Wid}_{\hat{R}}(M)^{\vee} = \mathrm{Wid}_{\hat{R}}(M^{\vee})$$

is a **Serre** subcategory of $\mathrm{mod}(\hat{R})$, where $(-)^{\vee} = \mathrm{Hom}_{\hat{R}}(-, E)$.
- Since Serre subcategories are closed under quotient modules, $N^{\vee} \in \mathrm{Wid}_{\hat{R}}(M^{\vee})$. Hence

$$(N^{\vee})^{\vee} = N \in \{\mathrm{Wid}_{\hat{R}}(M^{\vee})\}^{\vee} = \mathrm{Wid}_{\hat{R}}(M) \cong \mathrm{Wid}_R(M) \subseteq \mathcal{X}.$$

This is a contradiction.

□

§2. Clssifying subcategories of $\mathrm{Art}(R)$.

Definition 4 (Attached prime ideal)

Let M be an R-modules.

- M is **secondary** $\Leftrightarrow$ $\forall a \in R$, $a_M : M \to M$ is either surjective or nilpotent.
- If M is secondary then $\mathfrak{p} = \sqrt{\mathrm{ann}_R(M)}$ is a prime ideal and M is said to be $\mathfrak{p}$-seconadry.
- A prime ideal $\mathfrak{p}$ is an **attached prime ideal** of M if M has a $\mathfrak{p}$-secondary quotient.

$$\mathrm{Att}_R M = \{ \text{ the attached prime ideals of } M \}.$$

Remark.

- Given a submodule $N \subseteq M$, we have

$$\mathrm{Att}_R M/N \subseteq \mathrm{Att}_R M \subseteq \mathrm{Att}_R(N) \cup \mathrm{Att}_R M/N.$$

- If M is artinian, then

$$M = S_1 + \cdots + S_r, \quad \text{where } S_i \ : \ \mathfrak{p}_i\text{-secondary}$$

Namely it has a secondary representation. Moreover,

$$\mathrm{Att}_R M = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}.$$

Observation.

Let $(R, \mathfrak{m})$ be a noetherian local ring and $\mathcal{X} \subseteq \mathrm{Art}(R)$ a Serre subcategory.

- By virtue of Lemma 1, we can consider $\mathcal{X}$ as a subcategory of $\mathrm{Art}(\hat{R})$ where $\hat{R}$ is an $\mathfrak{m}$-adic completion of R .
- Since $\mathcal{X}^\vee \subseteq \mathrm{mod}(\hat{R})$ is Serre, $\mathcal{X}^\vee$, hence $\mathcal{X}$, corresponds to the specialization closed subset of $\mathrm{Spec}\,\hat{R}$.

$$\left\{ \begin{array}{c} \text{Serre subcategories} \\ \text{of } \mathrm{Art}(R) \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \begin{array}{c} \text{specialization closed subsets} \\ \text{of } \mathrm{Spec}\,\hat{R} \end{array} \right\}$$

We should consider

a **larger set** than $\mathrm{Spec}\,R$ to classify subcategories of $\mathrm{Art}(R)$.

We want to treat all the artinian R-modules as modules over **the same completed ring**.

For this, we consider

$$\mathcal{T} = \{ \ I \mid \text{the length of } R/I \text{ is finite } \}.$$

Then $\{R/I, f_{I,I'}\}$ forms the inverse system.

$$I, I' \in \mathcal{T} \text{ and } I' \subseteq I \Rightarrow f_{I,I'} : R/I' \to R/I.$$

We denote $\varprojlim_{I \in \mathcal{T}} R/I$ by $\hat{R}_{\mathcal{T}}$.

Lemma 5

Every artinian R-module has the structure of an $\hat{R}_{\mathcal{T}}$-module. Consequently, we have

$$\mathrm{Art}(R) \cong \mathrm{Art}(\hat{R}_{\mathcal{T}}).$$

Proposition 6

As topological rings

$$\hat{R}_{\mathcal{T}} \cong \prod_{\mathfrak{n} \in \max(R)} \hat{R}_{\mathfrak{n}}.$$

In the rest of the talk, we identify $\hat{R}_{\mathcal{T}}$ with $\prod_{\mathfrak{n} \in \max(R)} \hat{R}_{\mathfrak{n}}$ and denote them by $\hat{R}$.

$\mathrm{Att}_{\hat{R}} M = \ ?$

Why do we consider closed prime ideals?

Let M be an artinian R-module. As an $\hat{R}(:= \varprojlim_{I \in \mathcal{T}} R/I)$-module, we can show that

$$\sqrt{\mathrm{ann}_{\hat{R}}(M)} = \cap_{I \in \mathcal{T}} (\sqrt{\mathrm{ann}_{\hat{R}}(M) + I}).$$

Namely the radical of $\mathrm{ann}_{\hat{R}}(M)$ is a closed ideal of $\hat{R}$.
Hence, for $M \in \mathrm{Art}(R)$, $\mathrm{Att}_{\hat{R}}(M)$ is a subset of the set of closed prime ideals of $\hat{R}$.

Proposition 7

$\forall \mathfrak{P} \in \mathrm{Spec} \prod_{\mathfrak{n} \in \max(R)} \hat{R}_{\mathfrak{n}}$: *a closed prime ideal,*

$$\mathfrak{P} = \mathfrak{p} \times \prod_{\mathfrak{n} \in \max(R), \mathfrak{m} \neq \mathfrak{n}} \hat{R}_{\mathfrak{n}}$$

for some prime ideal $\mathfrak{p} \in \mathrm{Spec}\,\hat{R}_{\mathfrak{m}}$. *Thus,*

$$\{ \text{ closes prime ideals of } \prod_{\mathfrak{n} \in \max(R)} \hat{R}_{\mathfrak{n}} \} = \coprod_{\mathfrak{n} \in \max(R)} \mathrm{Spec}\,\hat{R}_{\mathfrak{n}}$$

Lemma 8

Let $M \in \mathrm{Art}(R)$. Assume that $\mathrm{Ass}_R M = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$. Then

$$M = \oplus_{i=1}^{s} \Gamma_{\mathfrak{m}_i}(M).$$

Here $\Gamma_{\mathfrak{m}}(M) = \{x \in M | \exists k \text{ s.t. } \mathfrak{m}^k x = 0\}$.

Proposition 9

Let M be an $\mathfrak{m}$-torsion R-module. Then

$$\mathrm{Att}_{\hat{R}} M = \mathrm{Att}_{\hat{R_\mathfrak{m}}} M$$

as a subset of $\coprod_{\mathfrak{n} \in \max(R)} \mathrm{Spec}\ \hat{R}_\mathfrak{n}$.

Corollary 10

Let $M \in \mathrm{Art}(R)$. Then

$$\mathrm{Att}_{\hat{R}} M = \coprod_{\mathfrak{m} \in \mathrm{Ass}_R M} \mathrm{Att}_{\hat{R_\mathfrak{m}}} \Gamma_\mathfrak{m}(M)$$

as a subset of $\coprod_{\mathfrak{n} \in \max(R)} \mathrm{Spec}\ \hat{R}_\mathfrak{n}$.

A key to classify the subcategories.

Theorem 11 ([Takahashi, 2008], [Krause, 2008])

Let M and $N \in \mathrm{mod}(R)$. Then

$$M \in \text{sub-ext}_R(N) \Leftrightarrow \mathrm{Ass}_R M \subseteq \mathrm{Ass}_R N.$$

Theorem 12

Let M and $N \in \mathrm{Art}(R)$. Then

$$M \in \text{quot-ext}_R(N) \Leftrightarrow \mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N.$$

Lemma 13

Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_s)$ be a complete semi-local ring and set $E = \oplus_{i=1}^{s} E_R(R/\mathfrak{m}_i)$. For $M \in \mathrm{Art}(R)$, we have

$$\mathrm{Att}_R M = \mathrm{Ass}_R \mathrm{Hom}_R(M, E).$$

Proof of Thm 12.

- Suppose that $M \in \text{quot-ext}_R(N)$ (hence, $M \in \text{quot-ext}_{\hat{R}}(N)$). Then

$$\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N.$$

- Conversely suppose that $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N$. Since

$$M = \oplus_{\mathfrak{m} \in \mathrm{Ass}_R M} \Gamma_\mathfrak{m}(M), \quad N = \oplus_{\mathfrak{n} \in \mathrm{Ass}_R N} \Gamma_\mathfrak{n}(N)$$

and $\text{quot-ext}_R(N)$ is closed under **direct sums and direct summands**, we may assume that M and N are $\mathfrak{m}$-**torsion**.

- Consider M and N as artinian $\hat{R_\mathfrak{m}}$-moduls s.t. $\mathrm{Att}_{\hat{R_\mathfrak{m}}} M \subseteq \mathrm{Att}_{\hat{R_\mathfrak{m}}} N$. By the Matlis duality, we have

$$M^\vee,\ N^\vee \in \mathrm{mod}(\hat{R_\mathfrak{m}}) \text{ and } \mathrm{Ass}_{\hat{R_\mathfrak{m}}} M^\vee \subseteq \mathrm{Ass}_{\hat{R_\mathfrak{m}}} N^\vee.$$

Thus,

$$M^\vee \in \text{sub-ext}_{\hat{R_\mathfrak{m}}}(N^\vee).$$

- Hence

$$\begin{aligned} M^{\vee\vee} \cong M &\in \text{sub-ext}_{\hat{R_\mathfrak{m}}}(N^\vee)^\vee \\ &= \text{quot-ext}_{\hat{R_\mathfrak{m}}}(N) \\ &= \text{quot-ext}_R(N). \end{aligned}$$

□

- For $\mathcal{X} \subseteq \mathrm{Art}(R)$, we define

$$\Psi(\mathcal{X}) = \mathrm{Att}\mathcal{X} = \cup_{M \in \mathcal{X}} \mathrm{Att}_{\hat{R}} M.$$

- For a subset S of $\coprod_{\mathfrak{n} \in \max(R)} \mathrm{Spec}\ \hat{R}_\mathfrak{n}$, we define

$$\Phi(S) = \{M \in \mathrm{Art}(R)\ | \mathrm{Att}_{\hat{R}} M \subseteq S\}.$$

Note that

- $\Psi(\mathcal{X})$ is a subset of $\coprod_{\mathfrak{n} \in \max(R)} \mathrm{Spec}\ \hat{R}_\mathfrak{n}$.
- $\Phi(S)$ is closed under quotient modules and extensions.

Theorem 14

Let R be a noetherian ring. Then Ψ and Φ induce an inclusion preserving bijection:

$$\{\text{subcategories of } \mathrm{Art}(R) \text{ closed under quotient modules and extensions}\} \cong \{\text{subsets of the set consisting of closed prime ideals of } \hat{R}\}.$$

Moreover this induces the bijection:

$$\{\text{Serre subcategories of } \mathrm{Art}(R)\} \cong \left\{ \begin{array}{c} \text{specialization closed subsets of} \\ \text{the set consisting of closed prime ideals of } \hat{R} \end{array} \right\}.$$

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Thank you for your attention.

ご清聴ありがとうございました.