The dimension formula of the cyclic homology of truncated quiver algebras over a field of positive characteristic

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Section 1
Introduction

Hochschild homology groups

Definition

The Hochschild complex is the following complex:

$$C(A) : \cdots \to A^{n+1} \to A^n \to \cdots \to A^2 \to A \to 0,$$

where the $K$-homomorphisms $b : A^{n+1} \to A^n$ is given by

$$b(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i (x_0 \otimes \cdots \otimes x_{i+1} \otimes \cdots \otimes x_n) + (-1)^n (x_n x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}).$$

$HH_n(A) := H_n(C(A))$: the $n$-th Hochschild homology group of $A$ $(n \geq 0)$. 

Hochschild homology groups

If a unital algebra $A$ is projective as a module over $K$, then there is an isomorphism

$$HH_n(A) \cong \text{Tor}_{A^\text{op}}^n(A, A).$$

For a unital $K$-algebra $A$, the bar resolution $C^{\text{bar}}(A)$ of $A$ is given by

$$C^{\text{bar}}(A) : \cdots \to A^{n+1} \to A^n \to \cdots \to A^2 \to A \to 0,$$

where the differentials $b'$ are given by

$$b'(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i (x_0 \otimes \cdots \otimes x_{i+1} \otimes \cdots \otimes x_n).$$

Notation

- $K$: a commutative ring
- $\mathbb{N}$: the set of all natural numbers containing 0
- $\Delta$: a finite quiver
- $\Delta_n$: the set of all paths of length $n(n \geq 0)$ in $\Delta$
- $G_n$: a cyclic group of order $n$ generated by $t_n$
- $A^{\otimes n} := A \otimes_K A \otimes_K \cdots \otimes_K A$ for a $K$-algebra $A$ (the $n$-fold tensor product of $A$)
- $R_{\Delta_n}^K$: the two-sided ideal of $K\Delta$ generated by all the paths of length $m \geq 1$
- $A^{\circ} := A \otimes_K A^{\text{op}}$: the enveloping algebra of $A$
Cyclic group action

Let $n$ be a positive integer. The action of $G_n$ on the module $A^\otimes n$ is given by

$$t_n \cdot (x_1 \otimes x_2 \otimes \cdots \otimes x_n) := (-1)^{n-1} (x_n \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}),$$

where $x_i \in A$.

Let $D := 1 - t_n$ and $N := 1 + t_n + t_n^2 + \cdots + t_n^{n-1}$ as elements of the group ring $\mathbb{Z}G_n$. Then

$$D : A^\otimes n \to A^\otimes n$$

and

$$N : A^\otimes n \to A^\otimes n$$

are $K$-homomorphisms.

Cyclic homology groups

Definition

For a $K$-algebra $A$, the following is a first quadrant bicomplex $CC(A)$ of $A$:

$$CC(A)_{-m} := A^\otimes n \xrightarrow{D} A^\otimes n \xrightarrow{N} A^\otimes n \xrightarrow{D}.$$  

$\text{HC}_n(A) := H_n(Tot(CC(A)))$: the $n$-th cyclic homology group of $A$ ($n \geq 0$).

Truncated quiver algebra $K\Delta/R^m_{\Delta}$

By adjoining the element $\bot$, we will consider the following set:

$$\hat{\Delta} = \{ \bot \} \cup \bigcup_{i=0}^{\infty} \Delta_i.$$  

This set is a semigroup with the multiplication defined by

$$\delta \cdot \gamma = \begin{cases} \delta \gamma & \text{if } t(\delta) = s(\gamma), \delta, \gamma \in \bigcup_{i=0}^{\infty} \Delta_i, \\ \bot & \text{otherwise}, \end{cases}$$

and

$$\bot \cdot \gamma = \gamma \cdot \bot = \bot, \quad \gamma \in \hat{\Delta}.$$  

$K\hat{\Delta}$ is a semigroup algebra and the path algebra $K\Delta$ is isomorphic to $K\hat{\Delta}/(\bot)$.

$K\Delta$ is $\Delta$-graded, that is, $K\Delta = \bigoplus_{\gamma \in \hat{\Delta}} (K\Delta)_\gamma$, where $(K\Delta)_\gamma = K\gamma$ for $\gamma \in \bigcup_{i=0}^{\infty} \Delta_i$ and $(K\Delta)_\bot = 0$.

$K\Delta$ is $N$-graded, that is, $K\Delta = \bigoplus_{n=0}^{\infty} K\Delta_n$.

$R^m_{\Delta}$ is $\hat{\Delta}$-graded and $N$-graded.

Thus a truncated quiver algebra $A = K\Delta/R^m_{\Delta}$ is a $\hat{\Delta}$-graded and $N$-graded algebra.

The cyclic homology of algebra is investigated for classes of various algebras, for example,

- 2-nilpotent algebras (Cibils, 1990)
- truncated quiver algebras over a field of characteristic zero (Taillefer, 2001)
- quadratic monomial algebras (Sköldberg, 2001)
- monomial algebras over a field of characteristic zero (Han, 2006)

etc.

On the other hand, the cyclic homology is Morita invariant.

Aim

- Find the dimension formula of the cyclic homology of truncated quiver algebras $K\Delta/R^m_{\Delta}$ over a field $K$ of positive characteristic.
Section 2

Hochschild homology and cyclic homology of truncated quiver algebras

Theorem (Sköldberg, 1999)

The following is a projective $\hat{\Delta}$-graded resolution of a truncated quiver algebra $A = K\Delta/P_{\Delta}^2$ as a left $A^*$-module:

$$P_A : \cdots \xrightarrow{d_{i+1}} P_{i} \xrightarrow{d_{i}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{e} A \xrightarrow{} 0.$$  

Here the modules are defined by

$$P_i = A \otimes_{K\Delta_0} K\Gamma(i) \otimes_{K\Delta_0} A,$$

$$P_0 = A \otimes_{K\Delta_0} K\Delta_0 \otimes_{K\Delta_0} A \cong A \otimes_{K\Delta_0} A,$$

where $\Gamma(i)$ is given by

$$\Gamma(i) = \begin{cases} \Delta_{cm} & \text{if } i = 2c (c \geq 0), \\ \Delta_{cm+1} & \text{if } i = 2c + 1 (c \geq 0), \end{cases}$$

Cycle and Basic cycle

A cycle $\gamma$ is called a basic cycle if there does not exist a cycle $\delta$ such that $\gamma = \delta^j$ for some positive integer $j(\geq 2)$.  

$\Delta_{cm}^n$: the set of all cycles of length $n$.  

$\Delta_{cm}^n$: the set of all basic cycles of length $n$.  

Hochschild homology of truncated quiver algebras

Let a path $\alpha_1 \cdots \alpha_{n-1} \alpha_n$ be a cycle, where $\alpha_i$ is an arrow in $\Delta$. Then the action of $G_\Delta$ on $\Delta_{cm}^n$ is given by

$$t_n \cdot (\alpha_1 \cdots \alpha_{n-1} \alpha_n) := \alpha_n \alpha_1 \cdots \alpha_{n-1}.$$  

Similarly, $G_\Delta$ acts on $\Delta_{cm}^n$:

- $a_n$: the cardinal number of the set of all $G_\Delta$-orbits on $\Delta_{cm}^n$.
- $b_n$: the cardinal number of the set of all $G_\Delta$-orbits on $\Delta_{cm}^n$.
Hochschild homology and cyclic homology of truncated quiver algebras

**Theorem (Sköldberg, 1999)**

Let $K$ be a commutative ring, $A = K\Delta/R^2_\Delta$ and $q = cm + e$ for $0 \leq e \leq m - 1$. Then the degree $q$ part of the $p$-th Hochschild homology $HH^p_q(A)$ is given by

$$
\begin{align*}
&\dim_{K^e} \text{ if } 1 \leq e \leq m - 1 \text{ and } 2e \leq p \leq 2e + 1,
&\dim_{K^e} \left( \frac{m}{\gcd(m, r)} : K \to K \right)_{b_r} \text{ if } e = 0 \text{ and } 0 < 2e - 1 = p,
&\dim_{K^e} \left( \frac{m}{\gcd(m, r)} : K \to K \right)_{b_r} \text{ if } e = 0 \text{ and } 0 < 2e = p,
&K^e \Delta_0 \text{ if } p = q = 0,
&0 \text{ otherwise}.
\end{align*}
$$

Cyclic homology of truncated quiver algebras

**Theorem (Taillefer, 2001)**

Let $K$ be a field of characteristic zero. The cyclic homology group of a truncated quiver algebra $A$ is given by

$$
\dim_K HC_{2c}(A) = \# \Delta_0 + \sum_{s=1}^{m-1} a_{cm+s}
\dim_K HC_{2c=1}(A) = \sum_{s=1}^{m-1} \sum_{r > 0} \frac{(\gcd(m, r) - 1)b_r}{s.t. \ r|c+1)m}.
$$

Main result

**Main result**

Let $K$ be a field of characteristic $\zeta$ and $A = K\Delta/R^2_\Delta$. Then the dimension formula of the cyclic homology of $A$ is given by, for $c \geq 0$,

$$
\begin{align*}
\dim_K HC_{2c}(A) &= \# \Delta_0 + \sum_{s=1}^{m-1} a_{cm+s},
+ \sum_{s=0}^{c} \sum_{s=1}^{m-1} \sum_{r > 0} b_r + \sum_{s=1}^{c} \sum_{r > 0} a_{s.t. \ r|c+1)m,}
\sum_{s=1}^{c} \sum_{r > 0} \frac{(\gcd(m, r) - 1)b_r}{s.t. \ r|c+1)m}.
\end{align*}
$$

Cibils’ projective resolution

**Lemma (Cibils, 1989)**

Let $I$ be an admissible ideal of $K\Delta$. $E$ denotes the subalgebra of $A = K\Delta/I$ generated by $\Delta_0$. The following is a projective resolution of $A$ as left $A^e$-module:

$$
Q_A : \cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to A \to 0
$$

where $Q_n = A \otimes (rad A)^n \otimes E$ and differentials are given by

$$
d_n(x_0 \otimes \cdots \otimes x_n+1) = \sum_{i=0}^{n} (-1)^i (x_0 \otimes \cdots \otimes \hat{x_i} \otimes \cdots \otimes x_n+1)
$$

for $x_0, x_1, \ldots, x_n \in rad A$.
Hochschild homology and cyclic homology of truncated quiver algebras

Example

The cyclic homology groups of Cibils’ mixed complex is isomorphic to the total homology groups of the following first quadrant bicomplex:

\[ \begin{array}{ccc}
A & \otimes & E^r \\
\oplus & & \oplus \\
\bullet & \otimes & \bullet \\
\end{array} \]

Denote the above bicomplex by \( \overline{C}(E) \). For a truncated quiver algebra \( A = K\Delta/R_{s,n}^2 \), there exists the decomposition \( K\Delta/R_{s,n}^2 = K\Delta_{s,n} \oplus rad A \).

Section 4

Example

Let \( K \) be a field of characteristic \( \mathbb{F}_p \). We consider the cyclic homology of a truncated quiver algebra \( A = K\Delta/R_{s,n}^2 \), where \( \Delta \) is the following quiver:

\[
\begin{array}{ccc}
\alpha_0 & \rightarrow & \alpha_1 \\
\downarrow & & \downarrow \\
\alpha_2 & \rightarrow & \alpha_3 \\
\end{array}
\]

The truncated algebra \( A \) is called a truncated cycle algebra. Note that \( \alpha_s \) and \( \beta_s \) is given by

\[
\alpha_s = \begin{cases} 1 & \text{if } s|r, \\ 0 & \text{otherwise} \end{cases}, \quad \beta_s = \begin{cases} 1 & \text{if } s = r, \\ 0 & \text{otherwise}. \end{cases}
\]
For $x \in \mathbb{R}$, we denote the largest integer $i$ satisfying $i \leq x$ by $[x]$.

The dimension formula of the cyclic homology of a truncated cycle algebra $A$ is given by

$$\dim_K HC_2c(A) = s + \left(\frac{(c+1)m}{s} - \frac{cm}{s} \right) + \sum_{c' \geq 1} \left(\frac{(c'+1)m-1}{s\zeta} - \frac{c'm}{s\zeta} \right),$$

and

$$\dim_K HC_{2c+1}(A) = (\gcd(m,s) - 1) \left(\left\lfloor \frac{(c+1)m}{s} \right\rfloor - \left\lfloor \frac{(c+1)m-1}{s} \right\rfloor + \left\lfloor \frac{\gcd(m,s)c}{s\zeta} \right\rfloor \right) + \left(\left\lfloor \frac{m}{\gcd(m,s)\zeta} \right\rfloor - \left\lfloor \frac{m-1}{\gcd(m,s)\zeta} \right\rfloor \right) \sum_{c' \geq 1} \left(\frac{(c'+1)m-1}{s\zeta} - \frac{c'+1m-1}{s\zeta} \right),$$

When $\zeta = 0$, the dimension of the cyclic homology of $A$ is as follows:

$$\dim_K HC_2c(A) = s + \left(\frac{(c+1)m}{s} - \frac{cm}{s} \right),$$

$$\dim_K HC_{2c+1}(A) = (\gcd(m,s) - 1) \left(\left\lfloor \frac{(c+1)m}{s} \right\rfloor - \left\lfloor \frac{(c+1)m-1}{s} \right\rfloor \right).$$


