The dimension formula of the cyclic homology of truncated quiver algebras over a field of positive characteristic

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Section 1
Introduction

Notation

- K: a commutative ring
- $\mathbb{N}$: the set of all natural numbers containing 0
- Δ: a finite quiver
- Δ_n: the set of all paths of length $n (n \geq 0)$ in Δ
- G_n: a cyclic group of order n generated by t_n
- $A^{\otimes n} := A \otimes_K A \otimes_K \cdots \otimes_K A$ for a K-algebra A (the n-fold tensor product of A)
- R^m_Δ: the two-sided ideal of $K\Delta$ generated by the all paths of length $m \geq 1$
- $A^e := A \otimes_K A^{\mathrm{op}}$: the enveloping algebra of A

Hochschild homology groups

Definition

The Hochschild complex is the following complex:

$$C(A) : \cdots \xrightarrow{b} A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \cdots \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A \longrightarrow 0,$$

where the K-homomorphisms $b : A^{\otimes n+1} \to A^{\otimes n}$ is given by

$$b(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i (x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n) + (-1)^n (x_n x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}).$$

$HH_n(A) := H_n(C(A))$: the n-th Hochschild homology group of A $(n \geq 0)$.

Hochschild homology groups

If a unital algebra A is projective as a module over K, then there is an isomorphism

$$HH_n(A) \cong \mathrm{Tor}_n^{A^e}(A, A).$$

For a unital K-algebra A, the bar resolution $C^{\mathrm{bar}}(A)$ of A is given by

$$C^{\mathrm{bar}}(A) : \cdots \xrightarrow{b'} A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} \cdots \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \longrightarrow 0,$$

where the differentials b' are given by

$$b'(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i (x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n).$$

Cyclic group action

Let n be a positive integer. The action of G_n on the module $A^{\otimes n}$ is given by

$$t_n \cdot (x_1 \otimes x_2, \otimes \cdots \otimes x_n) := (-1)^{n-1}(x_n \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}),$$

where $x_i \in A$.
Let $D := 1 - t_n$ and $N := 1 + t_n + t_n^2 + \cdots + t_n^{n-1}$ as elements of the group ring $\mathbb{Z}G_n$. Then

$$\begin{aligned} & D: A^{\otimes n} \longrightarrow A^{\otimes n} \\ \text{and} \quad & N: A^{\otimes n} \longrightarrow A^{\otimes n} \end{aligned}$$

are K-homomorphisms.

Cyclic homology groups

Definition

For a K-algebra A, the following is a first quadrant bicomplex $CC(A)$ of A:

$$\begin{array}{ccccccccc}
 & \downarrow b & & \downarrow -b' & & \downarrow b & \\
CC(A)_{*,2}: & A^{\otimes 3} & \xleftarrow{D} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{D} \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & \\
CC(A)_{*,1}: & A^{\otimes 2} & \xleftarrow{D} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{D} \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & \\
CC(A)_{*,0}: & A & \xleftarrow{D} & A & \xleftarrow{N} & A & \xleftarrow{D} \\
 & CC(A)_{0,*} & & CC(A)_{1,*} & & CC(A)_{2,*} &
\end{array}$$

$HC_n(A) := H_n(\mathrm{Tot}(CC(A)))$: the n-th cyclic homology group of $A (n \geq 0)$

Truncated quiver algebra $K\Delta/R_\Delta^m$

By adjoining the element $\perp$, we will consider the following set:

$$\hat{\Delta} = \{\perp\} \cup \bigcup_{i=0}^{\infty} \Delta_i.$$

This set is a semigroup with the multiplication defined by

$$\delta \cdot \gamma = \begin{cases} \delta\gamma & \text{if} \quad t(\delta) = s(\gamma), \\ \perp & \text{otherwise}, \end{cases} \qquad \delta, \gamma \in \bigcup_{i=0}^{\infty} \Delta_i,$$

and

$$\perp \cdot \gamma = \gamma \cdot \perp = \perp, \quad \gamma \in \hat{\Delta}.$$

$K\hat{\Delta}$ is a semigroup algebra and the path algebra $K\Delta$ is isomorphic to $K\hat{\Delta}/(\perp)$.

- $K\Delta$ is $\hat{\Delta}$-graded, that is, $K\Delta = \bigoplus_{\gamma \in \hat{\Delta}} (K\Delta)_\gamma$, where $(K\Delta)_\gamma = K\gamma$ for $\gamma \in \bigcup_{i=0}^{\infty} \Delta_i$ and $(K\Delta)_\perp = 0$.
- $K\Delta$ is $\mathbb{N}$-graded, that is, $K\Delta = \bigoplus_{i=0}^{\infty} K\Delta_i$.
- R_Δ^m is $\hat{\Delta}$-graded and $\mathbb{N}$-graded.

Thus a truncated quiver algebra $A = K\Delta/R_\Delta^m$ is a $\hat{\Delta}$-graded and $\mathbb{N}$-graded algebra.

The cyclic homology of algebra is investigated for classes of various algebras, for example,

- 2-nilpotent algebras (Cibils, 1990)
- truncated quiver algebras over a field of characteristic zero (Taillefer, 2001)
- quadratic monomial algebras (Sköldberg, 2001)
- monomial algebras over a field of characteristic zero (Han, 2006)

etc.
On the other hand, the cyclic homology is Morita invariant.

Aim

Aim

- Find the dimension formula of the cyclic homology of truncated quiver algebras $K\Delta/R_\Delta^m$ over a field K of positive characteristic.

Section 2

Hochschild homology and cyclic homology of truncated quiver algebras

Theorem(Sköldberg, 1999)

The following is a projective $\hat{\Delta}$-graded resolution of a truncated quiver algebra $A = K\Delta/R_\Delta^m$ as a left A^e-module:

$$P_A : \cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0.$$

Here the modules are defined by

$$P_i = A \otimes_{K\Delta_0} K\Gamma^{(i)} \otimes_{K\Delta_0} A,$$
$$P_0 = A \otimes_{K\Delta_0} K\Delta_0 \otimes_{K\Delta_0} A \cong A \otimes_{K\Delta_0} A,$$

where $\Gamma^{(i)}$ is given by

$$\Gamma^{(i)} = \begin{cases} \Delta_{cm} & \text{if } i = 2c \ (c \geq 0), \\ \Delta_{cm+1} & \text{if } i = 2c+1 \ (c \geq 0), \end{cases}$$

Theorem(Sköldberg, 1999)

and the differentials are defined by

$$d_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta) = \sum_{j=0}^{m-1} \alpha\alpha_1 \cdots \alpha_j \otimes \alpha_{1+j} \cdots \alpha_{(c-1)m+1+j} \otimes \alpha_{(c-1)m+2+j} \cdots \alpha_{cm}\beta,$$

$$d_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta) = \alpha\alpha_1 \otimes \alpha_2 \cdots \alpha_{cm+1} \otimes \beta - \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1}\beta,$$

where $\alpha_i \in \Delta_1$ for $i \geq 1$ and $\alpha, \beta \in A$ and ε is the multiplication.

Cycle and Basic cycle

A cycle γ is called a basic cycle if there does not exist a cycle δ such that $\gamma = \delta^j$ for some positive integer $j (\geq 2)$.

- Δ_n^{c}: the set of all cycles of length n.
- Δ_n^{b}: the set of all basic cycles of length n.

Cycle and Basic cycle

Let a path $\alpha_1 \cdots \alpha_{n-1}\alpha_n$ be a cycle, where α_i is an arrow in Δ. Then the action of G_n on Δ_n^{c} is given by

$$t_n \cdot (\alpha_1 \cdots \alpha_{n-1}\alpha_n) := \alpha_n\alpha_1 \cdots \alpha_{n-1}.$$

Similarly, G_n acts on Δ_n^{b}.

- a_n: the cardinal number of the set of all G_n-orbits on Δ_n^{c}.
- b_n: the cardinal number of the set of all G_n-orbits on Δ_n^{b}.

Hcohschild homology of truncated quiver algebras

L denotes the complex $A \otimes_{K\Delta_0^e} K\Gamma^{(*)}$ which is isomorphic to $A \otimes_{A^e} P_A$ by the following isomorphism φ:

$$\varphi : A \otimes_{A^e} P_A \xrightarrow{\sim} A \otimes_{A^e} A^e \otimes_{K\Delta_0^e} K\Gamma^{(*)} \xrightarrow{\sim} A \otimes_{K\Delta_0^e} K\Gamma^{(*)}.$$

Since the complex L is decomposed into subcomplexes $L_{\overline{\gamma}}$ by

$$L \cong \bigoplus_{q=0}^{\infty} \bigoplus_{\overline{\gamma} \in \Delta_q^{\mathrm{c}}/G_q} L_{\overline{\gamma}},$$

the p-th Hochschild homology group of A is $\mathbb{N}$-graded for $p \geq 0$, that is, the p-th Hochschild homology of A is given by

$$HH_p(A) = \bigoplus_{q=0}^{\infty} HH_{p,q}(A).$$

Hochschild homology of truncated quiver algebras

Theorem (Sköldberg, 1999)

Let K be a commutative ring, $A = K\Delta/R_\Delta^m$ and $q = cm + e$ for $0 \le e \le m-1$. Then the degree q part of the p-th Hochschild homology $HH_{p,q}(A)$ is given by

$$
\begin{cases}
K^{a_q} & \text{if } 1 \le e \le m-1 \text{ and } 2c \le p \le 2c+1, \\
\bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \operatorname{Ker}\left(\cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \\
 & \text{if } e = 0 \text{ and } 0 < 2c-1 = p, \\
\bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \operatorname{Coker}\left(\cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \\
 & \text{if } e = 0 \text{ and } 0 < 2c = p, \\
K^{\#\Delta_0} & \text{if } p = q = 0, \\
0 & \text{otherwise.}
\end{cases}
$$

Cyclic homology of truncated quiver algebras

Theorem (Taillefer, 2001)

Let K be a field of charactaristic zero. The cyclic homology group of a truncated quiver algebra A is given by

$$
\dim_K HC_{2c}(A) = \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e},
$$

$$
\dim_K HC_{2c+1}(A) = \sum_{\substack{r>0 \\ \text{s.t. } r|(c+1)m}} (\gcd(m,r)-1)b_r.
$$

Section 3

Main result

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Let K be a field of characteristic ζ and $A = K\Delta/R_\Delta^m$. Then the dimension formula of the cyclic homology of A is given by, for $c \ge 0$,

$$
\begin{aligned}
\dim_K HC_{2c}(A) = \#\Delta_0 &+ \sum_{e=1}^{m-1} a_{cm+e} \\
&+ \sum_{c'=0}^{c-1} \sum_{e=1}^{m-1} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta | c'm+e}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r|c'm, \\ \gcd(m,r)\zeta | m}} b_r \\
&+ \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta | \gcd(m,r)c'}} (\gcd(m,r)-1)b_r,
\end{aligned}
$$

Main result

Main result

$$
\begin{aligned}
\dim_K HC_{2c+1}(A) = &\sum_{\substack{r>0 \\ \text{s.t. } r|(c+1)m}} (\gcd(m,r)-1)b_r \\
&+ \sum_{c'=0}^{c} \sum_{e=1}^{m-1} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta | c'm+e}} b_r + \sum_{c'=1}^{c+1} \sum_{\substack{r>0 \\ \text{s.t. } r|c'm, \\ \gcd(m,r)\zeta | m}} b_r \\
&+ \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta | \gcd(m,r)c'}} (\gcd(m,r)-1)b_r.
\end{aligned}
$$

Cibils' projective resolution

Lemma (Cibils, 1989)

Let I be an admissible ideal of $K\Delta$. E denotes the subalgebra of $A = K\Delta/I$ generated by Δ_0. The following is a projective resolution of A as left A^e-module:

$$
Q_A : \cdots \to Q_n \xrightarrow{d_n} Q_{n-1} \to \cdots \to Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} A \to 0
$$

where $Q_n = A \otimes_E (\operatorname{rad} A)^{\otimes_E^n} \otimes_E A$ and differntials are given by

$$
d_n(x_0 \otimes \cdots \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i (x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1})
$$

for $x_0, x_{n+1} \in A$, $x_1, \ldots, x_n \in \operatorname{rad} A$.

Cibils' mixed complex

For the algebra A with a decomposition $A = E \oplus r$, where E is a separable subalgebra and r is a two-sided ideal, Cibils(1990) gives the following mixed complex $(A \otimes_{E^e} r^{\otimes_E^n}, b, B)$:
$b : A \otimes_{E^e} r^{\otimes_E^{n+1}} \to A \otimes_{E^e} r^{\otimes_E^n}$ is the Hochschild boundary and
$B : A \otimes_{E^e} r^{\otimes_E^n} \to A \otimes_{E^e} r^{\otimes_E^{n+1}}$ is given by the formula

$$B(x_0 \otimes_{E^e} x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n} (-1)^{in} (1 \otimes_{E^e} x_i \otimes \cdots \otimes (x_0)_r \otimes \cdots \otimes x_{i-1}),$$

where $x_0 = (x_0)_E + (x_0)_r \in E \oplus r$.
The cyclic homology groups of the mixed complex and the cyclic homology groups of A are isomorphic.

The cyclic homology groups of Cibils' mixed complex is isomorphic to the total homology groups of the following first quadrant bicomplex:

$$\begin{array}{ccccccc}
\downarrow b & & \downarrow b & & \downarrow b & & \\
A \otimes_{E^e} r^{\otimes_E^2} & \xleftarrow{B} & A \otimes_{E^e} r & \xleftarrow{B} & A & \longleftarrow & \\
\downarrow b & & \downarrow b & & \downarrow & & \\
A \otimes_{E^e} r & \xleftarrow{B} & A & \longleftarrow & 0 & \longleftarrow & \\
\downarrow b & & \downarrow & & \downarrow & & \\
A & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow &
\end{array}$$

Denote the above bicomplex by $\overline{C}_E(A)$. For a truncated quiver algebra $A = K\Delta/R_\Delta^m$, there exists the decomposition $K\Delta/R_\Delta^m = K\Delta_0 \oplus \mathrm{rad}\, A$.

We consider the spectral sequence associated with the filtration of $F\mathrm{Tot}(\overline{C}_E(A))$ given by

$$(F^p\mathrm{Tot}(\overline{C}_E(A)))_n = \bigoplus_{i \leq p} \overline{C}_E(A)_{i,n-i}.$$

The cyclic homology groups $HC_n(A)$ is given by

$$HC_n(A) \cong H_n(\mathrm{Tot}(\overline{C}_E(A))) \cong \bigoplus_{p+q=n} E_{p,q}^2.$$

E^1-page of the spectral sequence is drawn by the following illustration:

$$\begin{array}{ccccccc}
HH_2(A) & \xleftarrow{B} & HH_1(A) & \xleftarrow{B} & HH_0(A) & \xleftarrow{0} & \\
HH_1(A) & \xleftarrow{B} & HH_0(A) & \xleftarrow{0} & 0 & \xleftarrow{0} & \\
HH_0(A) & \xleftarrow{0} & 0 & \xleftarrow{0} & 0 & \xleftarrow{0} &
\end{array}$$

$B : HH_n(A) \to HH_{n+1}(A)$

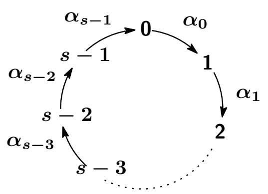
We construct chain maps $\iota : P_A \to Q_A$, and $\pi : Q_A \to P_A$ which induce isomorphism $HH_n(A) \cong HH_n(A)$ for $n \geq 0$. We consider $B : HH_n(A) \to HH_{n+1}(A)$ as follows:

$$\begin{array}{lccc}
\text{Sköldberg:} & A \otimes_{K\Delta_0^e} K\Gamma^{(n)} & & A \otimes_{K\Delta_0^e} K\Gamma^{(n+1)} \\
 & \downarrow\wr & & \uparrow\wr \\
 & A \otimes_{A^e} P_n & & A \otimes_{A^e} P_{n+1} \\
 & \downarrow \mathrm{id}_A \otimes \iota & & \uparrow \mathrm{id}_A \otimes \pi \\
 & A \otimes_{A^e} Q_n & & A \otimes_{A^e} Q_{n+1} \\
 & \downarrow\wr & & \uparrow\wr \\
\text{Cibils:} & A \otimes_{K\Delta_0^e} \mathrm{rad}\, A^{\otimes_{K\Delta_0}^n} & \xrightarrow{B} & A \otimes_{K\Delta_0^e} \mathrm{rad}\, A^{\otimes_{K\Delta_0}^{n+1}}
\end{array}$$

Section 4

Example

Let K be a field of characteristic ζ. We consider the cyclic homoology of a truncated quiver algebra $A = K\Delta/R_\Delta^m$, where Δ is the following quiver:



The truncated algebra A is called a truncated cycle algebra.
Note that a_r and b_r is given by

$$a_r = \begin{cases} 1 & \text{if } \ s \mid r, \\ 0 & \text{otherwise}, \end{cases} \qquad b_r = \begin{cases} 1 & \text{if } \ s = r, \\ 0 & \text{otherwise}. \end{cases}$$

For $x \in \mathbb{R}$, we denote the largest integer i satisfying $i \leq x$ by $[x]$. The dimension formula of the cyclic homology of a truncated cycle algebra A is given by

$$\begin{aligned}
&\dim_K HC_{2c}(A) \\
&= s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right] + \sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right) \\
&\quad + \left(\left[\frac{m}{\gcd(m,s)\zeta}\right] - \left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c}\left(\left[\frac{c'm}{s}\right] - \left[\frac{c'm-1}{s}\right]\right) \\
&\quad + (\gcd(m,s)-1)\left[\frac{\gcd(m,s)c}{s\zeta}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\dim_K HC_{2c+1}(A) \\
&= (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right] - \left[\frac{(c+1)m-1}{s}\right] + \left[\frac{\gcd(m,s)c}{s\zeta}\right]\right) \\
&\quad + \left(\left[\frac{m}{\gcd(m,s)\zeta}\right] - \left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c+1}\left(\left[\frac{c'm}{s}\right] - \left[\frac{c'm-1}{s}\right]\right) \\
&\quad + \sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right).
\end{aligned}$$

When $\zeta = 0$, the dimension of the cyclic homology of A is as follows:

$$\dim_K HC_{2c}(A) = s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right],$$

$$\begin{aligned}
&\dim_K HC_{2c+1}(A) \\
&\quad = (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right] - \left[\frac{(c+1)m-1}{s}\right]\right).
\end{aligned}$$

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