ON THE POSET OF PRE-PROJECTIVE TILTING MODULES OVER PATH ALGEBRAS

RYOICHI KASE

Abstract. We study posets of pre-projective tilting modules over path algebras. We give a criterion for Ext-vanishing and an equivalent condition for a poset of pre-projective tilting modules $\mathcal{T}_p(Q)$ to be a distributive lattice. Moreover we realize $\mathcal{T}_p(Q)$ as an ideal-poset.

Introduction

Tilting theory first appeared in an article by Brenner and Butler [2]. In that article the notion of a tilting module for finite dimensional algebras was introduced. Tilting theory now appears in many areas of mathematics, for example algebraic geometry, theory of algebraic groups and algebraic topology. Let T be a tilting module for a finite dimensional algebra A and let $B = \mathrm{End}_A(T)$. Then Happel showed that the two bounded derived categories $\mathcal{D}^{\mathrm{b}}(A)$ and $\mathcal{D}^{\mathrm{b}}(B)$ are equivalent as triangulated category [4]. Therefore, classifying tilting modules is an important problem.

Theory of tilting-mutation introduced by Riedtmann and Schofield is an approach to this problem. They introduced a tilting quiver whose vertices are (isomorphism classes of) basic tilting modules and arrows correspond to mutations [9]. Happel and Unger defined a partial order on the set of basic tilting modules and showed that the tilting quiver coincides with the Hasse quiver of this poset [5]. This poset is now studied by many authors.

Notations. Let Q be a finite connected quiver without loops or oriented cycles. We denote by Q_0 (resp. Q_1) the set of vertices (resp. arrows) of Q. For any arrow $\alpha \in Q_1$ we denote by $s(\alpha)$ its starting point and denote by $t(\alpha)$ its target point (i.e. α is an arrow from $s(\alpha)$ to $t(\alpha)$). We call a vertex $x \in Q_0$ a source (resp. sink) if there is an arrow starting at x (resp. ending at x) and there is no arrow ending at x (resp. starting at x). Let kQ be the path algebra of Q over an algebraically closed field k. Denote by mod-kQ the category of finite dimensional right kQ-modules and by ind-kQ the full subcategory of indecomposable modules. For any module $M \in$ mod-kQ we denote by $|M|$ the number of pairwise non isomorphic indecomposable direct summands of M. Let $P(i)$ be the indecomposable projective module in mod-kQ associated with vertex $i \in Q_0$.

Aim. If Q is a non-Dynkin quiver, kQ is a representation-infinite algebra. In this case, to determine rigid modules is nearly impossible. However the pre-projective component of the Auslander-Reiten quiver of mod-kQ is completely determined. For example, there

The detailed version of this paper will be submitted for publication elsewhere.

is a bijection between the set of (isomorphism classes of) indecomposable pre-projective modules over kQ and $\mathbb{Z}_{\geq 0} \times Q_0$.

In this paper, we consider the set $\mathcal{T}_{\mathrm{p}}(Q)$ of basic pre-projective tilting modules and study its combinatorial structure in the case when Q is a non-Dynkin quiver. For the purpose we have to answer to the following problem:

• When does the Ext^1_{kQ}-group between two indecomposable pre-projective modules vanish?

We introduce a function $l_Q : Q_0 \times Q_0 \to \mathbb{Z}_{\geq 0}$ and, by using this function, we give an answer to this question for any quiver satisfying the following condition (C):

$$\text{(C)} \qquad \delta(a) := \#\{\alpha \in Q_1 \mid s(\alpha) = a \text{ or } t(\alpha) = a\} \geq 2, \quad \forall a \in Q_0.$$

By applying this result we have the following.

Theorem 1. *If Q satisfies the condition (C), then for any $T \in \mathcal{T}_{\mathrm{p}}$ there exists $(r_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}_{\geq 0}$ such that $T \simeq \bigoplus_{i \in Q_0} \tau_Q^{-r_i} P(i)$.*

Moreover, the map $\bigoplus_{i \in Q_0} \tau^{-r_i} P(i) \mapsto (r_i)_{i \in Q_0}$ induces a poset inclusion,

$$(\mathcal{T}_{\mathrm{p}}(Q), \leq) \to (\mathbb{Z}^{Q_0}, \leq^{\mathrm{op}}),$$

where $(r_i) \leq^{\mathrm{op}} (s_i) \overset{\mathrm{def}}{\Leftrightarrow} r_i \geq s_i$ for any $i \in Q_0$.

The above result says that if Q satisfies the condition (C), then study of the poset $\mathcal{T}_{\mathrm{p}}(Q)$ comes down to combinatorics on Z^{Q_0}.

As an application, we see a connection between the posets of tilting modules and distributive lattices. In particular we realize a poset of tilting modules as an ideal-poset.

1. Preliminary

1.1. **Tilting modules.** In this sub-section we will recall the definition of tilting modules and basic results for combinatorics of the set of tilting modules.

Definition 2. A module $T \in$ mod-kQ is tilting module if,
(1) $\mathrm{Ext}^1_{kQ}(T, T) = 0$,
(2) $|T| = \# Q_0$.

We denote by $\mathcal{T}(Q)$ the set of (isomorphism classes of) basic tilting modules in mod-kQ.

Proposition 3. [6, Lemma 2.1] *Let $T, T' \in \mathcal{T}(Q)$. Then the following relation $\leq$ define a partial order on $\mathcal{T}(Q)$,*

$$T \geq T' \overset{\mathrm{def}}{\Leftrightarrow} \mathrm{Ext}^1_{kQ}(T, T') = 0.$$

1.2. **Lattices and distributive lattices.** In this subsection we will recall definition of a lattice and a distributive lattice.

Definition 4. A poset $(L, \leq)$ is a lattice if for any $x, y \in L$ there is the minimum element of $\{z \in L \mid z \geq x, y\}$ and there is the maximum element of $\{z \in L \mid z \leq x, y\}$.

In this case we denote by $x \vee y$ the minimum element of $\{z \in L \mid z \geq x, y\}$ and call it join of x and y. We also denote by $x \wedge y$ the maximum element of $\{z \in L \mid z \leq x, y\}$ and call it meet of x and y.

Definition 5. A lattice L is a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ holds for any $x, y, z \in L$.

We note that any finite distributive lattice is realized as an ideal-poset of some finite poset. This fact is known as Birkhoff's representation theorem.

Theorem 6. (*Birkhoff's representation theorem,* cf. [1],[3]) *Let L be a finite distributive lattice and $J \subset L$ be the poset of join-irreducible elements of L. Then L is isomorphic to $\mathcal{I}(J)$.*

2. Pre-projective tilting modules

2.1. **Criterion for Ext-vanishing.**

For any vertex $x \in Q_0$, we set

$$\delta(x) := \#\{\alpha \in Q_1 \mid s(\alpha = x) \text{or } t(\alpha = x)\}.$$

Now we consider the following condition:

$$\text{(C)} \qquad \delta(a) := \#\{\alpha \in Q_1 \mid s(\alpha) = a \text{ or } t(\alpha) = a\} \geq 2, \quad \forall a \in Q_0.$$

For a walk $w : x = x_0 \overset{\alpha_1}{-} x_1 \overset{\alpha_2}{\rightarrow} \cdots \overset{\alpha_r}{\rightarrow} x_r = y$ from x to y on Q, we set

$$c^+(w) := \#\{t \mid \text{there is an arrow from } x_{t-1} \text{ to } x_t\}.$$

Then we define

$$l_Q(i,j) := \begin{cases} \min\{c^+(w) \mid w : \text{walk from } i \text{ to } j \text{ on } Q\} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Then we have a criterion for Ext-vanishing.

Proposition 7. [7] *If Q satisfies the condition* (C), *then*

$$\mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), \tau^{-s}P(j)) = 0 \Leftrightarrow r \leq s + l_Q(j,i)$$

Therefore $\mathcal{T}_{\mathrm{p}}(Q)$ may be embedded in $\mathbb{Z}$-lattice $\mathbb{Z}^{Q}_{\geq 0}$ as follows.

Proposition 8. [7] *If Q satisfies the condition* (C), *then for any $T \in \mathcal{T}_{\mathrm{p}}$ there exists $(r_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}_{\geq 0}$ such that $T \simeq \bigoplus_{i \in Q_0} \tau_Q^{-r_i} P(i)$.*

Moreover, the map $\bigoplus_{i \in Q_0} \tau^{-r_i} P(i) \mapsto (r_i)_{i \in Q_0}$ induces a poset inclusion,

$$(\mathcal{T}_{\mathrm{p}}(Q), \leq) \to (\mathbb{Z}^{Q_0}, \leq^{\mathrm{op}}),$$

where $(r_i) \leq^{\mathrm{op}} (s_i) \overset{\mathrm{def}}{\Leftrightarrow} r_i \geq s_i$ for any $i \in Q_0$.

2.2. **Lattice theoretical aspects.**

In this section we see a connection between posets of pre-projective tilting modules and distributive lattices.

Theorem 9. [8] *$\mathcal{T}_{\mathrm{p}}(Q)$ is an infinite distributive lattice if and only if Q satisfies the condition* (C).

Example 10. Let Q be the following quiver:

$$Q: \; 1 \rightrightarrows 2 \overset{\alpha}{\rightrightarrows} 3$$

Thus l_Q is given by the following table:

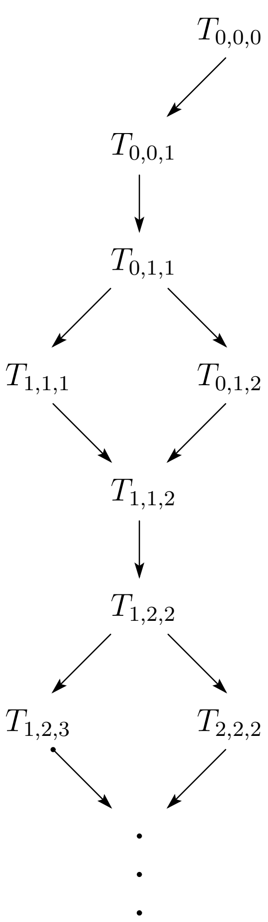
$l_Q(a,b)$	$b=1$	$b=2$	$b=3$
$a=1$	0	1	2
$a=2$	0	0	1
$a=3$	0	0	0

We put $Q' := Q \setminus \{\alpha\}$. Then we can check that

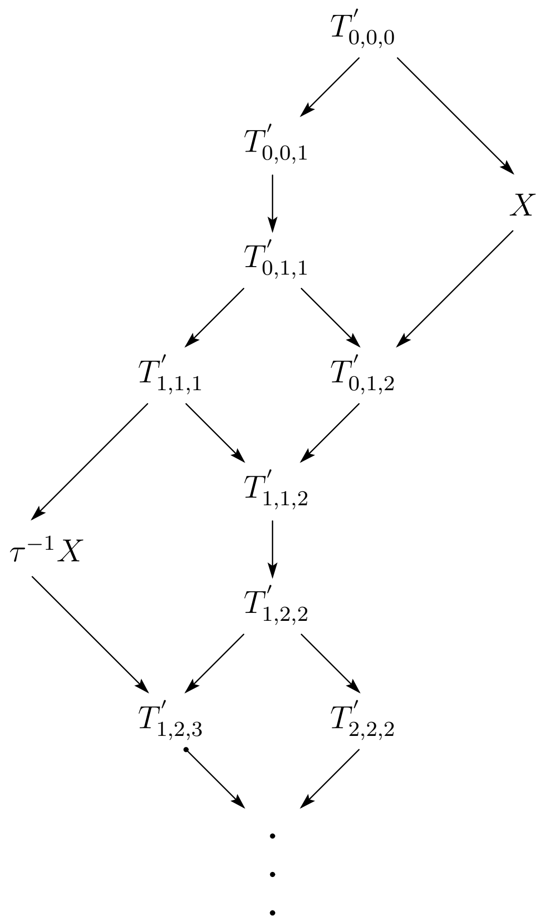
$$\mathrm{Ext}^1_{kQ'}(\tau^{-r}P(a), \tau^{-s}t(b)) = 0 \Leftrightarrow r \leq s + l_Q(b,a) \text{ or } (a=b=3,\ r=s+2).$$

Therefore $\mathcal{T}_{\mathrm{p}}(Q)$ and $\mathcal{T}_{\mathrm{p}}(Q')$ is given by the following:

$\mathcal{T}_{\mathrm{p}}(Q)$



$\mathcal{T}_{\mathrm{p}}(Q')$



where

$$\begin{aligned} T_{r,s,t} &:= \tau_Q^{-r}P(1) \oplus \tau_Q^{-s}P(2) \oplus \tau_Q^{-t}P(3), \\ T'_{r,s,t} &:= \tau_{Q'}^{-r}P'(1) \oplus \tau_{Q'}^{-s}P'(2) \oplus \tau_{Q'}^{-t}P'(3), \\ X &:= P(1) \oplus P(3) \oplus \tau_Q^{-2}P(3). \end{aligned}$$

In particular $\mathcal{T}_{\mathrm{p}}(Q)$ is a distributive lattice and $\mathcal{T}_{\mathrm{p}}(Q')$ is not a distributive lattice.

If Q satisfies the condition (C), then Proposition 7 implies that a module

$$T(a) := \bigoplus_{x \in Q_0} \tau^{-l_Q(a,x)} P(x)$$

is a pre-projective tilting module, for any vertex $a \in Q_0$. In fact, by the definition of l_Q, we have

$$l_Q(a,x) \leq l_Q(a,y) + l_Q(y,x) \text{ for any } x, y \in Q_0.$$

Moreover, $\tau^{-r}T(a)$ is a minimal element of $\{T \in \mathcal{T}_{\mathrm{p}}(Q) \mid \tau^{-r}P(a) \in \operatorname{add} T\}$. Therefore we obtain the following.

Lemma 11. [8] *Assume that* Q *satisfies the condition* (C)*. Then the set of join-irreducible elements of* $\mathcal{T}_{\mathrm{p}}(Q)$ *is* $\{\tau^{-r}T(a) \mid a \in Q_0,\ r \in \mathbb{Z}_{\geq 0}^{Q_0}\}$.

For any poset P, we denote by $\mathcal{I}(P)$ the ideal-poset of P. Now let $\mathcal{J}(Q) \subset \mathcal{T}_{\mathrm{p}}(Q)$ be the sub-poset of join-irreducible elements. Now we give Birkhoff's type result for the poset of pre-projective tilting modules $\mathcal{T}_{\mathrm{p}}(Q)$.

Proposition 12. [8] *Assume that* Q *satisfies the condition* (C)*. Then a map*

$$\rho : \mathcal{I}(\mathcal{J}(Q)) \setminus \{\emptyset\} \ni I \mapsto \bigvee_{\mathbf{i} \in I} T(\mathbf{i}) \in \mathcal{T}_{\mathrm{p}}(Q)$$

induces a poset isomorphism

$$\mathcal{I}(\mathcal{J}(Q)) \setminus \{\emptyset\} \simeq \mathcal{T}_{\mathrm{p}}(Q).$$

Let $\Gamma_{\mathrm{p}}(Q)$ be the pre-projective component of Auslander-Reiten quiver of mod-kQ. We define a poset $\mathcal{P}(Q)$ as follows:
- $\mathcal{P}(Q) = \{$indecomposable pre-projective modules over $kQ\}/\simeq$ as a set.
- $X \geq Y$ if there is a path from X to Y in $\Gamma_{\mathrm{p}}(Q)$.

Then we have the following.

Theorem 13. [8] *Assume that* Q *satisfies the condition* (C)*. Then there is a poset isomorphism*

$$\mathcal{J}(Q) \simeq \mathcal{P}(Q).$$

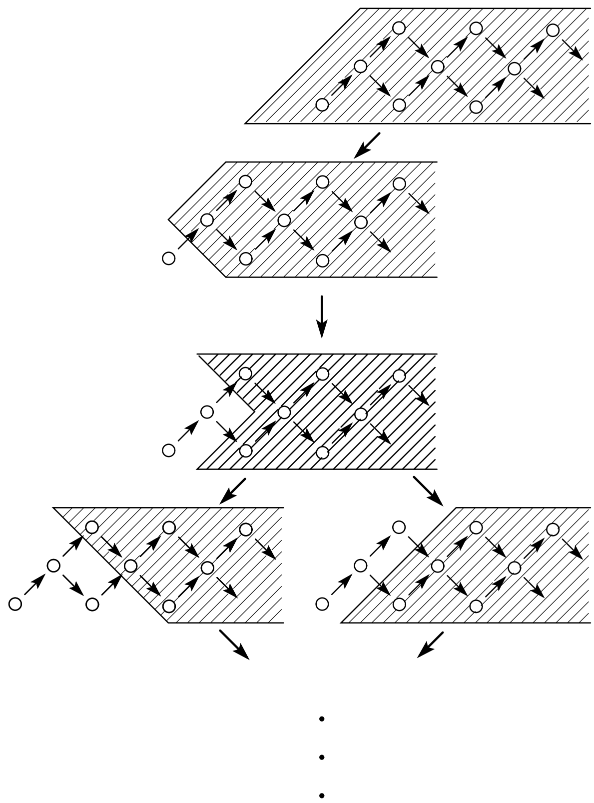
In particular, we have a poset isomorphism

$$\mathcal{T}_{\mathrm{p}}(Q) \simeq \mathcal{I}(\mathcal{P}(Q)) \setminus \{\emptyset\}.$$

Example 14. Let Q be the following quiver:

$$Q:\ 1 \rightrightarrows 2 \overset{\alpha}{\rightrightarrows} 3$$

Then $\mathcal{I}(\mathcal{P}(Q)) \setminus \{\emptyset\}$ is given by the following:



References

[1] G. Birkhoff, *Lattice Theory* , 3rd ed. Providence, RI: Amer. Math. Soc., 1967.

[2] S. Brenner, M.C.R Butler, *Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp.103-169, Lecture Notes in Math., **832**, Springer, Berlin-New York, 1980.

[3] G. Gratzer, *Lattice Theory: First Concepts and Distributive Lattices* , San Francisco, CA: W. H. Freeman, 1971.

[4] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Mathematical Society Lecture Note Series **119**, Cambridge University Press, 1988.

[5] D. Happel and L. Unger, *On a partial order of tilting modules*, Algebr. Represent. Theory **8** (2005), no.2, 147-156.

[6] ———, *On the quiver of tilting modules* , J. Algebra **284** (2005), no.2, 857-868.

[7] R. Kase, *A pre-projective part of tilting quivers over certain path algebras* , Comm. Algebra, to appear.

[8] ———, *Distributive lattices and the poset of pre-projective tilting modules* , arXiv: 1307.8247.

[9] C. Riedtmann and A. Schofield, *On a simplicial complex associated with tilting modules* , Comment. Math. Helv **66** (1991), no.1, 70-78.

Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University
Toyonaka, Osaka 560-0043, Japan

E-mail address: r-kase@cr.math.sci.osaka-u.ac.jp