TRIANGULATED SUBCATEGORIES OF EXTENSIONS AND TRIANGLES OF RECOLLEMENTS

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Abstract. Let T be a triangulated category with triangulated subcategories X and Y. We show that the subcategory of extensions $\mathsf{X} * \mathsf{Y}$ is triangulated if and only if every morphism from X to Y factors thorough $\mathsf{X} \cap \mathsf{Y}$.

In this situation, we show that there is a stable t-structure $\left(\frac{\mathsf{X}}{\mathsf{X}\cap\mathsf{Y}}, \frac{\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}\right)$ in $\frac{\mathsf{X}*\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}$. We use this to give a recipe for constructing triangles of recollements and recover some triangles of recollements from the literature.

This is joint work with Peter Jørgensen.

1. Introduction

Let T be a triangulated category. If X and Y are full subcategories of T, then the *subcategory of extensions* $\mathsf{X} * \mathsf{Y}$ is the full subcategory of objects e for which there is a distinguished triangle $x \to e \to y$ with $x \in \mathsf{X}$, $y \in \mathsf{Y}$. Subcategories of extensions have recently been of interest to a number of authors, see [1], [5], [6], [12].

We give necessary and sufficient conditions for $\mathsf{X} * \mathsf{Y}$ to be triangulated. It has been known that $\mathsf{X} * \mathsf{Y}$ is triangulated if there is no morphism from X to Y. Theorem 1 shows that this classical fact essentially gives the sufficient condition as well.

Theorem 1. *Let* X, Y *be triangulated subcategories of* T. *Then* $\mathsf{X}*\mathsf{Y}$ *is a triangulated subcategory of* $\mathsf{T} \Leftrightarrow \mathsf{Y} * \mathsf{X} \subseteq \mathsf{X} * \mathsf{Y} \Leftrightarrow$. $\mathrm{Hom}_{\mathsf{T}/\mathsf{X}\cap\mathsf{Y}}(\mathsf{X}/\mathsf{X}\cap\mathsf{Y}, \mathsf{Y}/\mathsf{X}\cap\mathsf{Y}) = 0$.

If this is the case, $\mathsf{X}/\mathsf{X}\cap\mathsf{Y}$ and $\mathsf{Y}/\mathsf{X}\cap\mathsf{Y}$ give a stable t-structure in $\mathsf{X} * \mathsf{Y}/\mathsf{X}\cap\mathsf{Y}$. Recall that a pair of triangulated subcategories (U, V) of T is called a stable t-structure if $\mathsf{U} * \mathsf{V} = \mathsf{T}$ and $\mathrm{Hom}_{\mathsf{T}}(\mathsf{U}, \mathsf{V}) = 0$, see [9, def. 9.14]. Indeed, for a given thick subcategory U of T, there is a one-to -one correspondence between stable t-structures of T/U and pairs of thick subcategories X, Y with $\mathsf{T} = \mathsf{X} * \mathsf{Y}$ and $\mathsf{X} \cap \mathsf{Y} = \mathsf{U}$, see [7] Lemma 4.6 .

Finally, under stronger assumptions, we show that a pair (or a triple) of triangulated subcategories of extensions induces a so-called (triangle of) recollements in a quotient category. A pair of stable t-structures (U, V), (V, W) is the equivalent notion to a recollement [8]. A triangle of recollements is a triple of stable t-structures (U, V), (V, W), (W, U). Triangles of recollements were introduced in [4, def. 0.3] and have a very high degree of symmetry; for instance, $\mathsf{U} \simeq \mathsf{V} \simeq \mathsf{W} \simeq \mathsf{T}/\mathsf{U} \simeq \mathsf{T}/\mathsf{V} \simeq \mathsf{T}/\mathsf{W}$. They have applications to the construction of triangle equivalences, see [4, prop. 1.16].

This is a preliminary report. The detailed version of this paper will be submitted for publication elsewhere.

2. Triangulated subcategory of extensions

Theorem 1 ([7] Theorem 4.1)**.** *Let* X*,* Y *be triangulated subcategories of* T *and let* $Q : \mathsf{T} \to \mathsf{T}/\mathsf{X} \cap \mathsf{Y}$ *be the quotient functor. Then the following are equivalent.*

(1) $\mathsf{X} * \mathsf{Y}$ *is a triangulated subcategory of* T*.*
(2) $\mathsf{Y} * \mathsf{X} \subseteq \mathsf{X} * \mathsf{Y}$*.*
(3) *Each morphism* $f : x \to y$ *with* $x \in \mathsf{X}$*,* $y \in \mathsf{Y}$ *factors through some object of* $\mathsf{X} \cap \mathsf{Y}$*.*
(4) $\mathrm{Hom}_{Q(\mathsf{T})}(Q(\mathsf{X}), Q(\mathsf{Y})) = 0$*.*
(5) $\mathsf{X} * \mathsf{Y}'$ *is a triangulated subcategory of* T *for every triangulated subcategory* Y' *of* Y *containing* $\mathsf{X} \cap \mathsf{Y}$*.*
(6) $\mathsf{X}' * \mathsf{Y}$ *is a triangulated subcategory of* T *for every triangulated subcategory* X' *of* X *containing* $\mathsf{X} \cap \mathsf{Y}$*.*

If $\mathsf{X} \cap \mathsf{Y} = 0$ in particular, we recover the following. This fact is well known but we have been unable to locate a reference.

Corollary 2. *Let* X*,* Y *be triangulated subcategories of* T*. If* $\mathrm{Hom}_{\mathsf{T}}(\mathsf{X}, \mathsf{Y}) = 0$ *then* $\mathsf{X} * \mathsf{Y}$ *is a triangulated subcategory of* T*.*

Lemma 3 ([7] Lemma 4.6)**.** *Let* U *and* V *be triangulated subcategories of* T *and assume that* $\mathsf{S} = \mathsf{U} * \mathsf{V}$ *is triangulated. Let* $Q : \mathsf{T} \to \mathsf{T}/\mathsf{U} \cap \mathsf{V}$ *and* $Q' : \mathsf{S} \to \mathsf{S}/\mathsf{U} \cap \mathsf{V}$ *be the quotient functors. We have the following.*

(1) $(Q'(\mathsf{U}), Q'(\mathsf{V}))$ *is a stable t-structure of* $Q'(\mathsf{S})$*.*
(2) *If* $\mathsf{U} \cap \mathsf{V}$ *is thick, then* $(Q(\mathsf{U}), Q(\mathsf{V}))$ *is a stable t-structure of* $Q(\mathsf{S})$*. In particular,* $\mathsf{S} = \mathsf{T}$ *if and only if* $Q(\mathsf{S}) = Q(\mathsf{T})$*.*

Remark 4. Yoshizawa gives the following example in [12, cor. 3.3]: If R is a commutative noetherian ring and S is a Serre subcategory of $\mathsf{Mod}\, R$, then $(\mathsf{mod}\, R) * \mathsf{S}$ is a Serre subcategory of $\mathsf{Mod}\, R$. Here $\mathsf{Mod}\, R$ is the category of R-modules and $\mathsf{mod}\, R$ is the full subcategory of finitely generated R-modules.

One might suspect a triangulated analogue to say that if T is compactly generated and U is a triangulated subcategory of T, then so is $\mathsf{T}^c * \mathsf{U}$ where T^c denotes the triangulated subcategory of compact objects. See [10, defs. 1.6 and 1.7]. However, this is false:

Set $\mathsf{T} = \mathsf{D}(\mathbb{Z})$ and $\mathsf{U} = \mathsf{D}(\mathbb{Q})$. Then T is compactly generated by $\{\, \Sigma^i \mathbb{Z} \mid i \in \mathbb{Z} \,\}$. There is a homological epimorphism of rings $\mathbb{Z} \to \mathbb{Q}$ which induces an embedding of triangulated categories $\mathsf{U} \hookrightarrow \mathsf{T}$, see [2, def. 4.5]and [11, thm. 2.4]. Since $\mathbb{Q}$ is a field, each object of U has homology modules of the form $\coprod \mathbb{Q}$. This means that viewed in T, the only object of U which has finitely generated homology modules is 0. Hence 0 is the only object of U which is compact in T, see [10, cor. 2.3]. That is, $\mathsf{T}^c \cap \mathsf{U} = 0$.

If $\mathsf{T}^c * \mathsf{U}$ were a triangulated subcategory of T, then Theorem B would give that $(\mathsf{T}^c, \mathsf{U})$ was a stable t-structure in $\mathsf{T}^c * \mathsf{U}$, but this is false since the canonical map $\mathbb{Z} \to \mathbb{Q}$ is a non-zero morphism from an object of T^c to an object of U.

3. Recollements

In the previous section we see that a pair of triangulated subcategories induces a stable t-structure if the category of their extensions is triangulated. It is natural to ask whether a (triangle of) recollement(s) is induced by a triple of triangulated subcategories $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ with $\mathsf{X} * \mathsf{Y}$, $\mathsf{Y} * \mathsf{Z}$ (and $\mathsf{Z} * \mathsf{X}$) triangulated. Apparently we don't know which category the recollement lives in. However using "enlargement" and "restriction" of categories, we construct a subquotient category with desired recollement. Throughout this section, $\langle \mathsf{X}_1, \cdots, \mathsf{X}_n \rangle$ is the smallest triangulated subcategory containing $\mathsf{X}_1, \cdots, \mathsf{X}_n$.

Lemma 5 (restriction. [7] Lemma 6.1)**.** *Let* U*,* V *and* W *be triangulated subcategories of* T*.*

(1) *Assume both* $\mathsf{U} * \mathsf{V}$ *and* $\mathsf{V} * \mathsf{W}$ *are triangulated. Then* $\mathsf{S} = (\mathsf{U} * \mathsf{V}) \cap (\mathsf{V} * \mathsf{W})$ *is represented as* $\mathsf{S} = \mathsf{U}_1 * \mathsf{V} = \mathsf{V} * \mathsf{W}_1$ *where* $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$ *and* $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$*.*
(2) *Assume each of* $\mathsf{U} * \mathsf{V}$*,* $\mathsf{V} * \mathsf{W}$ *and* $\mathsf{W} * \mathsf{U}$ *is triangulated. Then* $\mathsf{S} = (\mathsf{U} * \mathsf{V}) \cap (\mathsf{V} * \mathsf{W}) \cap (\mathsf{W} * \mathsf{U})$ *is represented as* $\mathsf{S} = \mathsf{U}_1 * \mathsf{V}_1 = \mathsf{V}_1 * \mathsf{W}_1 = \mathsf{W}_1 * \mathsf{U}_1$ *where* $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$*,* $\mathsf{V}_1 = \mathsf{V} \cap \mathsf{S}$ *and* $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$*.*

Lemma 6 (enlargement. [7] Lemma5.1)**.** *Let* U *and* V *be triangulated subcategories of* T*. Assume* $\mathsf{U} * \mathsf{V}$ *is triangulated. For each triangulated subcategories* $\mathsf{U}' \subset \mathsf{U}$ *and* $\mathsf{V}' \subset \mathsf{V}$*, we have the following.*

(1) $\mathsf{U} * \mathsf{V} = \mathsf{U} * \langle \mathsf{V}, \mathsf{U}' \rangle$*.*
(2) $\langle \mathsf{V}, \mathsf{U}' \rangle \cap \mathsf{U} = \langle \mathsf{U} \cap \mathsf{V}, \mathsf{U}' \rangle$*.*
(3) $\mathsf{U} * \mathsf{V} = \langle \mathsf{U}, \mathsf{V}' \rangle * \mathsf{V}$*.*
(4) $\langle \mathsf{U}, \mathsf{V}' \rangle \cap \mathsf{V} = \langle \mathsf{U} \cap \mathsf{V}, \mathsf{V}' \rangle$*.*

Lemma 7. *Let* U, V *and* W *be triangulated subcategory of* T*.*

(1) *Assume* $\mathsf{U} * \mathsf{V} = \mathsf{V} * \mathsf{W}$ *and is triangulated. Set* $\mathsf{S} = \mathsf{U} * \mathsf{V}$ *and let* $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle$ *be the canonical quotient functor. Then both* $(Q(\mathsf{U}), Q(\mathsf{V}))$ *and* $(Q(\mathsf{V}), Q(\mathsf{W}))$ *are stable t-structures of* $\mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle$*.*
(2) *Assume* $\mathsf{U} * \mathsf{V} = \mathsf{V} * \mathsf{W} = \mathsf{W} * \mathsf{U}$ *and is triangulated. Set* $\mathsf{S} = \mathsf{U} * \mathsf{V}$ *and let* $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W}, \mathsf{W} \cap \mathsf{U} \rangle$ *be the canonical quotient functor. Then* $(Q(\mathsf{U}), Q(\mathsf{V}))$*,* $(Q(\mathsf{V}), Q(\mathsf{W}))$ *and* $(Q(\mathsf{W}), Q(\mathsf{U}))$*are stable t-structures of* $\mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W}, \mathsf{W} \cap \mathsf{U} \rangle$*.*

Proof. (i). We have $\mathsf{S} = \langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle * \mathsf{V} = \mathsf{V} * \langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle$ and $\langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle \cap \mathsf{V} = \langle \mathsf{U} \cap \mathsf{V}, \mathsf{W} \cap \mathsf{V} \rangle = \mathsf{V} \cap \langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle$ from Lemma 6. Lemma 3 gives two stable t-structures $(Q(\langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle), Q(\mathsf{V}))$ and $(Q(\mathsf{V}), Q(\langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle))$ of $Q(\mathsf{S})$, but $(Q(\langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle) = Q(\mathsf{U})$ and $Q(\langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle) = Q(\mathsf{W})$ hence we are done.

(ii). From Lemma 6, we have $\mathsf{S} = \langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle * \mathsf{V} = \langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle * \langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle$ and $\langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle \cap \langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle = \langle \langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle, \mathsf{W} \cap \mathsf{U} \rangle$. Lemma 3 gives a stable t-structure $(Q(\langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle), Q(\langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle))$ but $(Q(\langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle) = Q(\mathsf{U})$ and $Q(\langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle) = Q(\mathsf{V})$. Analogously we obtain other stable t-structures. □

Theorem 8. *Let* U*,* V *and* W *be triangulated subcategories of* T*.*

(1) *Assume both* $\mathsf{U} * \mathsf{V}$ *and* $\mathsf{V} * \mathsf{W}$ *are triangulated. Set* $\mathsf{S} = \mathsf{U} * \mathsf{V} \cap \mathsf{V} * \mathsf{W}$ *and let* $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle$ *be the canonical quotient functor. Then* $(Q(\mathsf{U}_1), Q(\mathsf{V}))$*, and* $(Q(\mathsf{V}), Q(\mathsf{W}_1))$ *are stable t-structures of* $Q(\mathsf{S})$ *where* $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$ *and* $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$*.*
(2) *Assume each of* $\mathsf{U} * \mathsf{V}$*,* $\mathsf{V} * \mathsf{W}$ *and* $\mathsf{W} * \mathsf{U}$ *is triangulated. Set* $\mathsf{S} = \mathsf{U} * \mathsf{V} \cap \mathsf{V} * \mathsf{W} \cap \mathsf{W} * \mathsf{U}$ *and let* $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W}, \mathsf{W} \cap \mathsf{U} \rangle$ *be the canonical quotient functor. Then* $(Q(\mathsf{U}_1), Q(\mathsf{V}_1))$*,* $(Q(\mathsf{V}_1), Q(\mathsf{W}_1))$ *and* $(Q(\mathsf{W}_1), Q(\mathsf{U}_1))$ *are stable t-structures of* $Q(\mathsf{S})$ *where* $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$*,* $\mathsf{V}_1 = \mathsf{V} \cap \mathsf{S}$ *and* $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$*.*

Example 9 (The homotopy category of projective modules)**.** Let R be an Iwanaga-Gorenstein ring, that is, a noetherian ring which has finite injective dimension from either side as a module over itself. Let $\mathsf{T} = \mathsf{K}_{(b)}(\mathrm{Prj}\ R)$ be the homotopy category of complexes of projective right-R-modules with bounded homology. Define subcategories of T by

$$\mathsf{X} = \mathsf{K}^-_{(b)}(\mathrm{Prj}\ R)\ \ ,\ \ \mathsf{Y} = \mathsf{K}_{ac}(\mathrm{Prj}\ R)\ \ ,\ \ \mathsf{Z} = \mathsf{K}^+_{(b)}(\mathrm{Prj}\ R)$$

where $\mathsf{K}^-_{(b)}(\mathrm{Prj}\ R)$ is the isomorphism closure of the class of complexes P with $P^i = 0$ for $i \gg 0$ and $\mathsf{K}^+_{(b)}(\mathrm{Prj}\ R)$ is defined analogously, while $\mathsf{K}_{ac}(\mathrm{Prj}\ R)$ is the subcategory of acyclic (that is, exact) complexes.

Note that Y is equal to $\mathsf{K}_{tac}(\mathrm{Prj}\ R)$, the subcategory of totally acyclic complexes, that is, acyclic complexes which stay acyclic under the functor $\mathrm{Hom}_R(-, Q)$ when Q is projective, see [3, cor. 5.5 and par. 5.12].

By [4, prop. 2.3(1), lem. 5.6(1), and rmk. 5.14] there are stable t-structures (X, Y), (Y, Z) in T.

If $P \in \mathsf{T}$ is given, then there is a distinguished triangle $P^{\geq 0} \to P \to P^{<0}$ where $P^{\geq 0}$ and $P^{<0}$ are hard truncations. Since $P^{\geq 0} \in \mathsf{Z}$ and $P^{<0} \in \mathsf{X}$, we have $\mathsf{T} = \mathsf{Z} * \mathsf{X}$.

We can hence apply Lemma 7. The intersection

$$\mathsf{X} \cap \mathsf{Z} \ \ = \ \ \mathsf{K}^-_{(b)}(\mathrm{Prj}\ R) \cap \mathsf{K}^+_{(b)}(\mathrm{Prj}\ R) \ \ = \ \ \mathsf{K}^b(\mathrm{Prj}\ R)$$

is the isomorphism closure of the class of bounded complexes. If we use an obvious shorthand for quotient categories, Lemma 7 (ii) therefore provides a triangle of recollements

$$\big(\mathsf{K}^-_{(b)}/\mathsf{K}^b(\mathrm{Prj}\ R)\ ,\ \mathsf{K}_{ac}(\mathrm{Prj}\ R)\ ,\ \mathsf{K}^+_{(b)}/\mathsf{K}^b(\mathrm{Prj}\ R)\big)$$

in $\mathsf{K}_{(b)}/\mathsf{K}^b(\mathrm{Prj}\ R)$. Note that $\mathsf{K}_{ac}(\mathrm{Prj}\ R)$ is equivalent to its projection to $\mathsf{K}_{(b)}/\mathsf{K}^b(\mathrm{Prj}\ R)$ by [4, prop. 1.5], so we can write $\mathsf{K}_{ac}(\mathrm{Prj}\ R)$ instead of the projection.

This example and its finite analogue were first obtained in [4, thms. 2.8 and 5.8] and motivated the definition of triangles of recollements.

References

[1] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), 1–36.

[2] W. Geigle and H. Lenzing, *Perpendicular categories and applications to representations and sheaves*, J. Algebra **144** (1991), 273–343.

[3] S. Iyengar and H. Krause, *Acyclicity versus total acyclicity for complexes over noetherian rings*, Doc. Math. **11** (2006), 207–240.

[4] O. Iyama, K. Kato, and J. Miyachi, *Recollement of homotopy categories and Cohen-Macaulay modules*, J. K-Theory **8** (2011), 507–542.

[5] O. Iyama and Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008), 117–168.

[6] P. Jørgensen and K. Kato, *Symmetric Auslander and Bass categories*, Math. Proc. Cambridge Phil. Soc. **250** (2011), 227–240.

[7] P. Jørgensen and K. Kato, *Triangulated subcategories of extensions and triangles of recollements*, preprint 2014.

[8] J. Miyachi, Localization of Triangulated Categories and Derived Categories, J. Algebra **141** (1991), 463-483.

[9] J. Miyachi, Derived Categories with applications to representations of algebras, Chiba University, 2000. Available at the author's webpage: `http://www.u-gakugei.ac.jp/~miyachi/seminar.html`.

[10] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown Representability*, J. Amer. Math. Soc. **9** (1996), 205–236.

[11] D. Pauksztello, *Homological epimorphisms of Differential Graded algebras*, Comm. Algebra **37** (2009), 2337–2350.

[12] T. Yoshizawa, *Subcategories of extension modules by Serre subcategories*, Proc. Amer. Math. Soc. **140** (2012), 2293-2305.

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