DEFINING RELATIONS OF 3-DIMENSIONAL QUADRATIC AS-REGULAR ALGEBRAS

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ABSTRACT. Classification of AS-regular algebras is one of the main interests in noncommutative algebraic geometry. In this article, we focus on the 3-dimensional quadratic case. We find defining relations of 3-dimensional quadratic AS-regular algebras. Also, we classify these algebras up to isomorphism and up to graded Morita equivalence in terms of their defining relations.

1. INTRODUCTION

Classification of AS-regular algebras is one of the main interests in noncommutative algebraic geometry. In fact, the geometric classification of 3-dimensional AS-regular algebras due to Artin, Tate and Van den Bergh [2] is regarded as one of the starting points of the field. In this article, we focus on 3-dimensional quadratic AS-regular algebras. By restricting to these algebras, they are in one-to-one correspondence with geometric pairs (E, σ) classified by A-T-V [2]. In this article, we try to find defining relations of 3-dimensional quadratic AS-regular algebras and answer the question when algebras given by defining relations are isomorphic or graded Morita equivalent.

2. PRELIMINARIES

Throughout this article, we fix an algebraically closed field k of characteristic zero. Let A be a graded k-algebra. We denote by $\mathrm{GrMod}A$ the category of graded right A-modules. We say that two graded algebras A and A' are graded Morita equivalent if the categories $\mathrm{GrMod}A$ and $\mathrm{GrMod}A'$ are equivalent.

The definition of an AS-regular algebra below is stronger than its original definition [**1**].

Definition 1. [1] A Noetherian connected graded algebra A is an AS-regular algebra of dimension d if

- $\mathrm{gldim}A = d < \infty$, and
- $\mathrm{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

Example 2. Let $A = k\langle x, y, z\rangle/(yx - \alpha z^2, zy - \beta x^2, xz - \gamma y^2)$, where $\alpha\beta\gamma \neq 0$. Then A is a 3-dimensional AS-regular algebra.

Artin, Tate and Van den Bergh classified 3-dimensional AS-regular algebras by using geometric techniques. In this article, we will focus on the quadratic case and classify 3-dimensional quadratic AS-regular algebras algebraically.

The detailed version of this paper will be submitted for publication elsewhere.

Let A be a graded algebra finitely generated in degree 1 over k. Note that A can be presented as $A = T(V)/I$ where V is a finite dimensional vector space over k, $T(V)$ is the tensor algebra on V, and I is a homogeneous two-sided ideal of $T(V)$. We say $A = T(V)/(R)$ is a quadratic algebra when $R \subset V \otimes_k V$ is a subspace and (R) is the two-sided ideal of $T(V)$ generated by R . By choosing a basis $\{x_1, \cdots, x_n\}$ for V over k , a quadratic algebra A is also presented as $A = k\langle x_1, \cdots, x_n\rangle/(f_1, \cdots, f_m)$ where deg $x_i = 1$ for all i and f_j are homogeneous noncommutative polynomials of degree two for all j. For a quadratic algebra $A = T(V)/(R)$, we define

$$\mathcal{V}(R) := \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid \ f(p, q) = 0 \ \text{ for all } f \in R\}.$$

Definition 3. [3] A quadratic algebra $A = T(V)/(R)$ is called geometric if there exists a geometric pair (E, σ) where $E \subset \mathbb{P}(V^*)$ is a closed k-subscheme and σ is a k-automorphism of E such that

(G1) $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$,
(G2) $R = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0, \forall p \in E\}$.

If A satisfies the condition (G1), A determines a geometric pair (E, σ). Conversely, A is determined by a geometric pair (E, σ) if A satisfies the condition (G2) and we write $A = \mathcal{A}(E, \sigma)$ in this case.

By [2], every 3-dimensional quadratic AS-regular algebra A is geometric. Moreover, E is either $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$, that is, E is $\mathbb{P}^2$, a union of three lines which make a triangle, a union of three lines meeting at one point, a union of a line and a conic meeting at two point, a union of a line and a conic meeting at one point, a nodal curve, a cuspidal curve, a union of a line and a double line, a triple line , or an elliptic curve.

The types of (E, σ) of 3-dimensional quadratic AS-regular algebras are defined in [4] which are slightly modified from the original types defined in [1] and [2]. We follow the types defined in [4].

- Type $\mathbb{P}^2$: E is $\mathbb{P}^2$, and $\sigma \in \mathrm{Aut}_k\mathbb{P}^2 = \mathrm{PGL}_3(k)$.
- Type S_1 : E is a triangle, and σ stabilizes each component.
- Type S_2 : E is a triangle, and σ interchanges two of its components.
- Type S_3 : E is a triangle, and σ circulates three components.
- Type S_1' : E is a union of a line and a conic meeting at two points, and σ stabilizes each component and two intersection points.
- Type S_2' : E is a union of a line and a conic meeting at two points, and σ stabilizes each component and interchanges two intersection points.
- Type T_1 : E is a union of three lines meeting at one point, and σ stabilizes each component.
- Type T_2 : E is a union of three lines meeting at one point, and σ interchanges two of its components.
- Type T_3 : E is a union of three lines meeting at one point, and σ circulates three components.
- Type T_1' : E is a union of a line and a conic meeting at one point, and σ stabilizes each component.
- Type N : E is a nodal cubic curve.
- Type C : E is a cuspidal cubic curve.

We introduce some Lemmas which are used for classification.

Lemma 4. *Let A and A' be geometric algebras with $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$. Then $A \cong A'$ as graded algebras if and only if there exists $\tau \in \mathrm{Aut}_k \mathbb{P}(V^*)$ which restricts to an isomorphism $\tau : E \to E'$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E' \\ \sigma \downarrow & & \downarrow \sigma' \\ E & \xrightarrow{\tau} & E' \end{array}$$

commutes.

Lemma 5. [3] *Let A and A' be geometric algebras with $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$. Then* $\mathrm{GrMod}A \cong \mathrm{GrMod}A'$ *if and only if there exists a sequence of automorphisms $\tau_n \in$* $\mathrm{Aut}_k \mathbb{P}(V^*)$ *which restricts to a sequence of isomorphisms $\tau_n : E \to E'$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\tau_n} & E' \\ \sigma \downarrow & & \downarrow \sigma' \\ E & \xrightarrow{\tau_{n+1}} & E' \end{array}$$

commute for all $n \in \mathbb{Z}$.

In general, classifying quadratic algebras up to isomorphism is easier than classifying them up to graded Morita equivalence. Our method is to define a new graded algebra $\overline{A}$ and classify original algebra A up to graded Morita equivalence by classifying $\overline{A}$ up to isomorphism.

Remark 6. Let $A = T(V)/(R)$ be a 3-dimensional quadratic AS-regular algebra. Then A is Koszul and $A^! = T(V^*)/(R^\perp)$ is Frobenius. It follows that we can take the Nakayama automorphism $\nu \in \mathrm{Aut}_k A^!$ of $A^!$. It was shown that ν naturally induces $\nu \in \mathrm{Aut}_k E$ by abuse of notation.

Using the automorphism $\nu \in \mathrm{Aut}_k E$, we define a new graded algebra $\overline{A}$ from A.

Definition 7. [4] Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quadratic AS-regular algebra and $\nu \in \mathrm{Aut}_k E$ the automorphism induced by the Nakayama automorphism of $A^!$. Define a new graded algebra by $\overline{A} := \mathcal{A}(E, \nu\sigma^3)$.

Theorem 8. [4] *Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be 3-dimensional quadratic AS-regular algebras. Suppose that (E, σ) and (E', σ') are the same Type $\mathbb{P}^2, S_i, S'_i$, T_i or T'_i. Then* $\mathrm{GrMod}A$ *is equivalent to* $\mathrm{GrMod}A'$ *if and only if $\overline{A}$ is isomorphic to $\overline{A'}$ as graded algebras.*

3. Main results

We completed classification of 3-dimensional quadratic AS-regular algebras in the cases of Type $\mathbb{P}^2$, S_i, S'_i, T_i, T'_i, N, C. We will explain our method using Thoerem 8 in some details for simplest case here. The remaining cases are also proved by using Lemma 4, Lemma 5, Theorem 8 and [5].

Example 9. Let (E,σ) be of Type S_1. Then we may assume that $E = \mathcal{V}(x)\cup\mathcal{V}(y)\cup\mathcal{V}(z)$ and $\sigma \in \mathrm{Aut}_k E$ is given by

$$\begin{aligned}\sigma|_{\mathcal{V}(x)}(0,b,c) &= (0,b,\alpha c),\\ \sigma|_{\mathcal{V}(y)}(a,0,c) &= (\beta a,0,c),\\ \sigma|_{\mathcal{V}(z)}(a,b,0) &= (a,\gamma b,0).\end{aligned}$$

where $\alpha\beta\gamma \neq 0,1$. We can determine $A = \mathcal{A}(E,\sigma)$ from the property (G2) of geometric algebra. In this case, $A = \mathcal{A}(E,\sigma)$ is given by

$$A = k\langle x,y,z\rangle/(yz-\alpha zy, zx-\beta xz, xy-\gamma yx) =: A_{\alpha,\beta,\gamma}.$$

By using Lemma 4 above, $A_{\alpha,\beta,\gamma}$ and $A_{\alpha',\beta',\gamma'}$ are isomorphic as graded algebras if and only if

$$(\alpha',\beta',\gamma') = \begin{cases}(\alpha,\beta,\gamma),(\beta,\gamma,\alpha),(\gamma,\alpha,\beta),\\ (\alpha^{-1},\gamma^{-1},\beta^{-1}),(\beta^{-1},\alpha^{-1},\gamma^{-1}),(\gamma^{-1},\beta^{-1},\alpha^{-1}).\end{cases}$$

Next, we define $\overline{A_{\alpha,\beta,\gamma}}$ to classify $A_{\alpha,\beta,\gamma}$ up to graded Morita equivalence. In this case, $\nu \in \mathrm{Aut}_k E$ is given by

$$\nu(a,b,c) = ((\gamma/\beta)a,(\alpha/\gamma)b,(\beta/\alpha)c).$$

It follows that

$$\begin{aligned}\overline{A_{\alpha,\beta,\gamma}} &= \mathcal{A}(E,\nu\sigma^3)\\ &= k\langle x,y,z\rangle/(yz-\alpha\beta\gamma zy, zx-\alpha\beta\gamma xz, xy-\alpha\beta\gamma yx)\\ &= A_{\alpha\beta\gamma,\alpha\beta\gamma,\alpha\beta\gamma}\ .\end{aligned}$$

By Theorem 8,

$$\begin{aligned}\mathrm{GrMod}A_{\alpha,\beta,\gamma} \cong \mathrm{GrMod}A_{\alpha',\beta',\gamma'} &\Leftrightarrow \overline{A_{\alpha,\beta,\gamma}} \cong \overline{A_{\alpha',\beta',\gamma'}}\\ &\Leftrightarrow A_{\alpha\beta\gamma,\alpha\beta\gamma,\alpha\beta\gamma} \cong A_{\alpha'\beta'\gamma',\alpha'\beta'\gamma',\alpha'\beta'\gamma'}\\ &\Leftrightarrow \alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}\ .\end{aligned}$$

We write down our results.

Theorem 10. *Let $A = \mathcal{A}(E,\sigma)$ be a 3-dimensional quadratic AS-regular algebra. In each type, we list the defining relations, when they are isomorphic and when they are graded Morita equivalent in terms of parameters in the defining relations as in Example 9.*

Type	*defining relations / isomorphism / graded Morita equivalent*
$\mathbb{P}^2$	*(case 1)*$xy-(\beta/\alpha)yx, yz-(\gamma/\beta)zy, zx-(\alpha/\gamma)xz$, *where* $\alpha\beta\gamma\neq 0$ *(case 2)*$xy-yx+y^2, xz-\alpha zx+\alpha zy, yz-\alpha zy$, *where* $\alpha\neq 0$ *(case 3)*$xy-yx+y^2-zx, xz+yz-zx, zy-yz-z^2$
	There are no isomorphic relations between different cases. *(case1)*$(\alpha',\beta',\gamma') = \begin{cases}(\alpha,\beta,\gamma),(\alpha,\gamma,\beta),(\beta,\alpha,\gamma),\\ (\beta,\gamma,\alpha),(\gamma,\alpha,\beta),(\gamma,\beta,\alpha)\end{cases}$ *in* $\mathbb{P}^2$ *(case2)*$A_\alpha \cong A_{\alpha'}$ *if and only if* $\alpha=\alpha'$
	$\mathrm{GrMod}A \cong \mathrm{GrMod}A'$ *for any* A, A' *of Type* $\mathbb{P}^2$

S_1	$yz-\alpha zy,\ zx-\beta xz,\ xy-\gamma yx,$ *where* $\alpha\beta\gamma\neq 0,1$
	$(\alpha',\beta',\gamma')=\begin{cases}(\alpha,\beta,\gamma),(\beta,\gamma,\alpha),(\gamma,\alpha,\beta),\\ (\alpha^{-1},\gamma^{-1},\beta^{-1}),(\beta^{-1},\alpha^{-1},\gamma^{-1}),(\gamma^{-1},\beta^{-1},\alpha^{-1})\end{cases}$
	$\alpha'\beta'\gamma'=(\alpha\beta\gamma)^{\pm 1}$
S_2	$zx-\alpha yz, zy-\beta xz, x^2+\alpha\beta y^2,$ *where* $\alpha\beta\neq 0$
	$(\alpha',\beta')=(\alpha,\beta)$ *in* $\mathbb{P}^1$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A, A' *of Type* S_2
S_3	$yx-\alpha z^2, zy-\beta x^2, xz-\gamma y^2,$ *where* $\alpha\beta\gamma\neq 0,1$
	$\alpha'\beta'\gamma'=\alpha\beta\gamma$
	$\alpha'\beta'\gamma'=(\alpha\beta\gamma)^{\pm 1}$
S_1'	$xy-\beta yx, x^2+yz-\alpha zy, zx-\beta xz,$ *where* $\alpha\beta^2\neq 0,1$
	$(\alpha',\beta')=(\alpha,\beta),(\alpha^{-1},\beta^{-1})$
	$\alpha'\beta'^2=(\alpha\beta^2)^{\pm 1}$
S_2'	*Every algebra is isomorphic to* $k\langle x,y,z\rangle/(xy-zx, yx-xz, x^2+y^2+z^2)$
T_1	$\begin{cases}xy-yx,\\ xz-zx-\beta x^2+(\beta+\gamma)yx, \\ yz-zy-\alpha y^2+(\alpha+\gamma)xy\end{cases}$ *where* $\alpha+\beta+\gamma\neq 0$
	$(\alpha',\beta',\gamma')=\begin{cases}(\alpha,\beta,\gamma),(\alpha,\gamma,\beta),(\beta,\alpha,\gamma),\\ (\beta,\gamma,\alpha),(\gamma,\alpha,\beta),(\gamma,\beta,\alpha)\end{cases}$ *in* $\mathbb{P}^2$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A, A' *of Type* T_1
T_2	$\begin{cases}x^2-y^2,\\ xz-zy-\beta xy+(\beta+\gamma)y^2, \\ yz-zx-\alpha yx+(\alpha+\gamma)x^2\end{cases}$ *where* $\alpha+\beta+\gamma\neq 0$
	$(\alpha'+\beta',\gamma')=(\alpha+\beta,\gamma)$ *in* $\mathbb{P}^1$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A, A' *of Type* T_2
T_3	$\begin{cases}x^2-xy+y^2,\\ xz+zy+\beta xy-(\beta+\gamma)y^2, \\ \alpha yx+\gamma y^2-yz+zx-zy\end{cases}$ *where* $\alpha+\beta+\gamma\neq 0$
	$(\alpha',\beta',\gamma')=(\alpha,\beta,\gamma),(\beta,\gamma,\alpha),(\gamma,\alpha,\beta)$ *in* $\mathbb{P}^2$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A, A' *of Type* T_3
T_1'	$\begin{cases}\alpha x^2+\beta(\alpha+\beta)xy-xz+zx-(\alpha+\beta)zy,\\ xy-yx-\beta y^2, \\ 2\beta xy-\beta^2y^2-yz+zy\end{cases}$ *where* $\alpha+2\beta\neq 0$
	$(\alpha',\beta')=(\alpha,\beta)$ *in* $\mathbb{P}^1$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A, A' *of Type* T_1'

N	$\alpha xy - yx, \alpha yz - zy + x^2, \alpha zx - xz + y^2$, *where* $\alpha(\alpha^3 - 1) \neq 0$
	$\alpha' = \alpha^{\pm 1}$
	$\alpha'^3 = \alpha^{\pm 3}$
C	*Every algebra is isomorphic to* $k\langle x, y, z\rangle / \begin{pmatrix} xy - yx - x^2, \\ zx - xz - x^2 - 3y^2, \\ zy - yz + 3y^2 + 2xz + 2xy \end{pmatrix}$

The classification of the cases when E is a union of a line and a double line, a triple line, or an elliptic curve is not finished yet and now in progress.

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