A CLASSIFICATION OF CYCLOTOMIC KLR ALGEBRAS OF TYPE $A_n^{(1)}$

MASAHIDE KONISHI

Abstract. A Khovanov-Lauda-Rouquier algebra (KLR algebra for short) is defined by these two data, a quiver Γ and a weight ν on its vertices. Furthermore we obtain a cyclotomic KLR algebra by fixing another weight Λ on vertices. There exists idempotents called KLR idempotents in KLR algebras, but they are not primitive in general.

In past report, we fix a quiver Γ type $A_n^{(1)}$, ν and Λ some special case then we showed all the non-zero KLR idempotents are primitive in the cyclotomic KLR algebra.

In this report, we start from that and fix a quiver Γ type $A_n^{(1)}$, obtain ν and Λ such that non-zero KLR idempotents are all primitive in the cyclotomic KLR algebra.

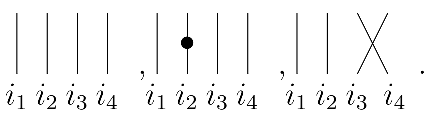
1. Definitions

At the beginning, we give definitions of KLR algebras and cyclotomic KLR algebras. Sometimes it is defined by using only generators and its relations, however the diagram interpretation of elements are quite useful, such as some statements are proved more simple. Because of that reason, we use diagrams in this report.

At first, we fix a quiver Γ without loops and multiple arrows. Each elements of its vertices set Γ_0 is used as colors of strands later, while the quiver are used for defining relations between diagrams.

Second, we fix a weight $\nu = \sum_{i \in \Gamma_0} a_i \nu_i (a_i \in \mathbb{Z}_{\geq 0})$ on vertices. This shows how many strands are there for each colors, furthermore the diagrams using $|\nu| = \sum_{i \in \Gamma_0} a_i$ strands are the generators of KLR algebras as a vector space.

We have not touched about what is the diagrams, roughly speaking, that is "colored braids with dots". There are some examples below;



We said "colored braid" just now, each i_k put below presents the color of the strand. Used colors are vertices of Γ (i.e. elements of Γ_0), furthermore the number of each colored strands is obtained from ν a weight on Γ_0. Those three diagrams are the main three kinds of diagrams, colored parallel strands, the dot and the crossing[1].

The detailed version of this paper will be submitted for publication elsewhere.

[1]The sum for colors is taken as dots and crossings.

Definition of the multiplication for two diagrams x and y is quite simple. We put the diadram y below the diagram x. If the colors of each strands then define the diagram xy as a concatenation, otherwise xy is 0. The leftmost diagram is an idempotent with this multiplication.

We put relations defined by quiver Γ to define a KLR algebra and take a quotient by an ideal defined from another weight Λ on Γ_0 to define a cyclotomic KLR algebra.

We use below notation for an information about colors. Set $m = |\nu|$,

$$\mathrm{Seq}(\nu) = \{(i_1, i_2, \cdots, i_m) \in (\Gamma_0)^m \mid \textit{each } i \in \Gamma_0 \textit{ appears } a_i \textit{ times}\}$$

For example, if $\Gamma_0 = \{0, 1\}$, $\nu = 2\nu_0 + \nu_1$, we get $\mathrm{Seq}(\nu) = \{(0,0,1), (0,1,0), (1,0,0)\}$.

We denote $\mathbf{e}(\mathbf{i})$ the diagram with parallel m strands colored $i_1, i_2, \cdots, i_m$ from left to right, $y_k\mathbf{e}(\mathbf{i})$ with a dot on kth strand, $\psi_l\mathbf{e}(\mathbf{i})$ with lth and $(l+1)$st strands crossed. We can obtain more complicated diagrams by fixing a shape with y_k and ψ_l and a color with $\mathbf{e}(\mathbf{i})$. y_k and ψ_l are the elements which took a sum about colors of strands, we can fix a color by multiplying $\mathbf{e}(\mathbf{i})$. In the definition below, there exists some cases we should divide relations by colors, so someimes put $\mathbf{e}(\mathbf{i})$.

Definition 1. KLR algebras $R_\Gamma(\nu)$ are defined by these generators and relations. Set $m = |\nu|$.

- Generators: $\{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \mathrm{Seq}(\nu)\} \cup \{y_1, \cdots, y_m\} \cup \{\psi_1, \cdots, \psi_{m-1}\}$
- Relations:

$$\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}\mathbf{e}(\mathbf{i}),$$

$$\sum_{\mathbf{i} \in \mathrm{Seq}(\nu)} \mathbf{e}(\mathbf{i}) = 1,$$

$$y_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})y_k,$$

$$\psi_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{s_k}\cdot\mathbf{i})\psi_k \quad (\mathbf{s_k}\cdot\mathbf{i} = (i_1, \cdots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \cdots, i_m)),$$

$$y_k y_l = y_l y_k,$$

$$\psi_k y_l = y_l \psi_k \ (l \neq k, k+1),$$

$$\psi_k \psi_l = \psi_l \psi_k \ (|k - l| > 1),$$

$$\psi_k y_{k+1}\mathbf{e}(\mathbf{i}) = \begin{cases} (y_k\psi_k + 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+1}) \\ y_k\psi_k\mathbf{e}(\mathbf{i}) & (\textit{otherwise}) \end{cases},$$

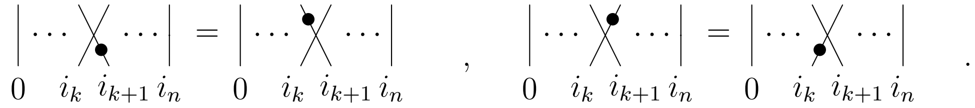
$$y_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \begin{cases} (\psi_k y_k + 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+1}) \\ \psi_k y_k\mathbf{e}(\mathbf{i}) & (\textit{otherwise}) \end{cases},$$

$$\psi_k^2\mathbf{e}(\mathbf{i}) = \begin{cases} 0 & (i_k = i_{k+1}) \\ \mathbf{e}(\mathbf{i}) & (\textit{no arrows between } i_k \textit{ and } i_{k+1}) \\ (y_{k+1} - y_k)\mathbf{e}(\mathbf{i}) & (i_k \to i_{k+1}) \\ (y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftarrow i_{k+1}) \\ (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftrightarrow i_{k+1}) \end{cases},$$

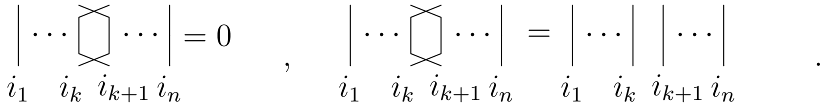
$$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \begin{cases} (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+2}\ ,\ i_k \to i_{k+1}) \\ (\psi_{k+1}\psi_k\psi_{k+1} - 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+2}\ ,\ i_k \leftarrow i_{k+1}) \\ (\psi_{k+1}\psi_k\psi_{k+1} - 2y_{k+1} + y_k + y_{k+2})\mathbf{e}(\mathbf{i}) & (i_k = i_{k+2}\ ,\ i_k \leftrightarrow i_{k+1}) \\ \psi_{k+1}\psi_k\psi_{k+1}\mathbf{e}(\mathbf{i}) & (\textit{otherwise}) \end{cases},$$

We describe relations after 8th with diagrams from easier one.

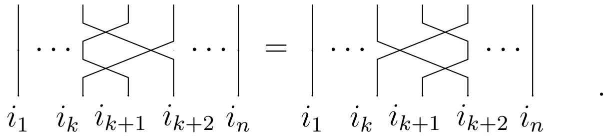
$\psi_k y_{k+1}\mathbf{e}(\mathbf{i}) = y_k\psi_k\mathbf{e}(\mathbf{i})$ $(i_k \neq i_{k+1})$, $y_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \psi_k y_k\mathbf{e}(\mathbf{i})$ $(i_k \neq i_{k+1})$.



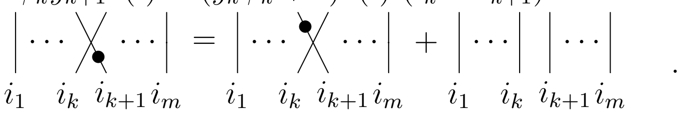
$\psi_k^2\mathbf{e}(\mathbf{i}) = 0$ $(i_k = i_{k+1})$, $\psi_k^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ (*no arrows between* i_k *and* i_{k+1}) .



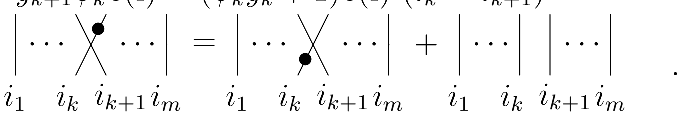
$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \psi_{k+1}\psi_k\psi_{k+1}\mathbf{e}(\mathbf{i})$ $(i_k \neq i_{k+2}$, *or no arrows between* i_k *and* $i_{k+1})$.



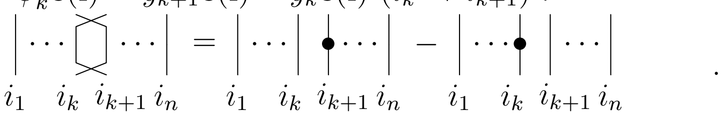
$\psi_k y_{k+1}\mathbf{e}(\mathbf{i}) = (y_k\psi_k + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+1})$



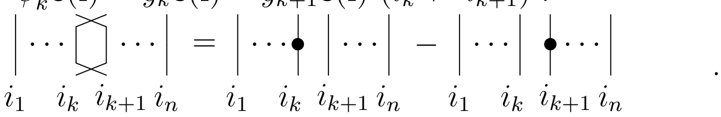
$y_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_k y_k + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+1})$



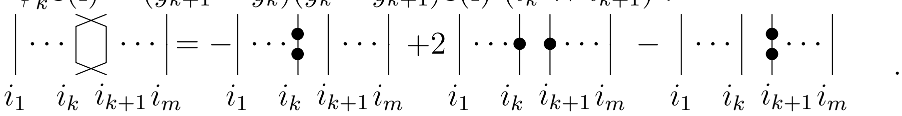
$\psi_k^2\mathbf{e}(\mathbf{i}) = y_{k+1}\mathbf{e}(\mathbf{i}) - y_k\mathbf{e}(\mathbf{i})$ $(i_k \to i_{k+1})$.



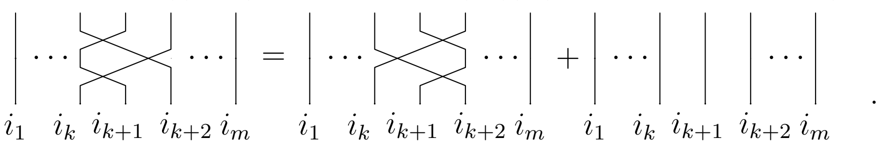
$\psi_k^2\mathbf{e}(\mathbf{i}) = y_k\mathbf{e}(\mathbf{i}) - y_{k+1}\mathbf{e}(\mathbf{i})$ $(i_k \leftarrow i_{k+1})$.



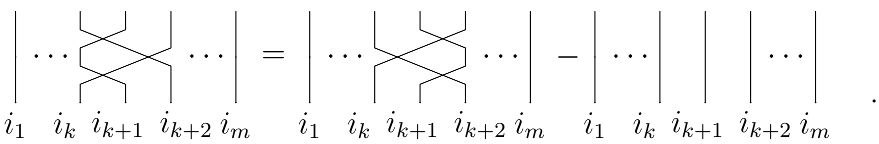
$\psi_k^2\mathbf{e}(\mathbf{i}) = (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i})$ $(i_k \leftrightarrow i_{k+1})$.



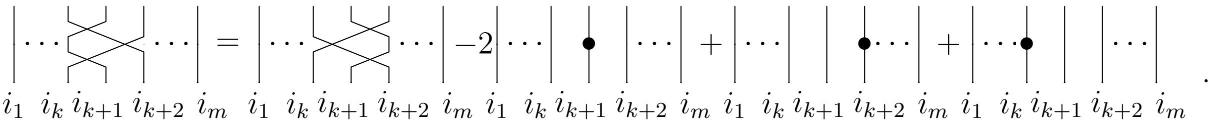
$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+2}$, $i_k \to i_{k+1})$.



$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+2}$, $i_k \leftarrow i_{k+1})$.



$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} - 2y_{k+1} + y_k + y_{k+2})\mathbf{e}(\mathbf{i})\ (i_k = i_{k+2}\ ,\ i_k \leftrightarrow i_{k+1})$.



We fix $\Lambda = \sum_{i\in\Gamma_0} b_i\Lambda_i (b_i \in \mathbb{Z}_{\geq 0})$ and let I^Λ be an ideal of $R_\Gamma(\nu)$ generated by $\left\{y_1^d\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in \mathrm{Seq}(\nu), d = b_{i_1}\right\}$. We call $R_\Gamma(\nu)^\Lambda = R_\Gamma(\nu)/I^\Lambda$ a cyclotomic KLR algebra.

Generators of the ideal are diagrams with b_{i_1} dots on the leftmost strand of each $\mathbf{e}(\mathbf{i})$. We see i_1 for $\mathbf{i}$ and then refer b_{i_1}, it is the place easy to confuse, be careful.

2. Problem

Those two relations break an idempotency of a KLR idenpotent $\mathbf{e}(\mathbf{i})$;

$\mathbf{e}(\mathbf{i}) = \psi_k y_{k+1}\mathbf{e}(\mathbf{i}) - y_k\psi_k\mathbf{e}(\mathbf{i})\ (i_k = i_{k+1})$,

$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i})\ (i_k = i_{k+2}\ \ i_k \to i_{k+1})$.

Say inversely, only those two relations can break an idempotency. We note both relations can appear only if there exists a color used twice or more.

We can easily conclude that, on KLR algebras, the existence of a non-primitive KLR idempotent and a color used twice or more are equivalent.

However, on cyclotomic KLR algebras, sometimes there exists zero term in above relations and the above equivalence can be broken. There is one natural question, when all non-zero KLR idempotents are primitive on $R_\Gamma(\nu)$? (Characterize such ν and Γ!) To try to give an answer, we notice that depends on the shape of Γ [2].

We restrict the problem to the case "essentially type $A_n^{(1)}$" and obtain the answer.

Let a quiver $A_n^{(1)}$ $(n \geq 1)$ with vertices $\{0, \cdots, n\}$ and arrows from each k to $k+1$ $(0 \leq k \leq n-1)$ and from n to 0.

Moreover, assume $a_i > 0 (0 \leq i \leq n)$ in ν to reflect "essentially" the structure of KLR algebras.

We omit Γ from $R_\Gamma(\nu)^\Lambda$.

Theorem 2. *For a cyclotomic KLR algebraR_ν^Λ, all non-zero* $\mathbf{e}(\mathbf{i})$ *are primitive and* ν *and* Λ *satisfy one of followings are equivalent.*

(a) $R_\nu^\Lambda = 0$.

(b) $\nu = \sum_{0\leq i\leq n} \nu_i, \Lambda$ *is arbitrary.*

(c) $\nu = \sum_{0\leq i\leq n} \nu_i + \nu_k, \Lambda = \Lambda_k (0 \leq k \leq n)$.

[2]Roughly, the underlying graph of Γ is tree or not.

2.1. **Sketch of Proof.** Proof is done as following steps.
(i) Check for the case (b), (c).
(ii) Construct counterexample (non-zero non-primitive $\mathbf{e}(\mathbf{i})$) in "minimal case" about ν, Λ.
(iii) Check for induction on ν.
(iv) Check for induction on Λ.

We do (i) later and start with (ii) to (iv). Since the case $\Lambda = 0$ is included in case (a), we may assume $\Lambda \neq 0$. Since $A_n^{(1)}$ is rotation symmetry, we may assume $b_0 > 0$ in Λ.

In this situation we may take those two cases as (ii) minimal cases about ν,Λ:
(I) $\nu = \sum_{0 \leq i \leq n} \nu_i + \nu_k, \Lambda = \Lambda_1 \ (k \neq 1)$.
(II) $\nu = \sum_{0 \leq i \leq n} \nu_i + \nu_1, \Lambda = 2\Lambda_1$.

About (I), for example $k = n$, we set $\mathbf{i} = (0, 1, \cdots, n-1, n, n)$, then since $y_{n+1}\mathbf{e}(\mathbf{i}) \neq 0$ [3] and $y_{n+2}\mathbf{e}(\mathbf{i}) \neq 0$ [4], $\mathbf{e}(\mathbf{i})$ can be decomposed by the relation above.

About (II), we set $\mathbf{i} = (0, 0, 1, \cdots, n-1, n)$, then since $y_1\mathbf{e}(\mathbf{i}) \neq 0$ and $y_2\mathbf{e}(\mathbf{i}) \neq 0$, $\mathbf{e}(\mathbf{i})$ can be decomposed by the relation above.

(iii) Induction on ν. It's not in the case (b) hence there exists k satisfying $a_k \geq 2$. Moreover, it's not in the case (c) hence one of the followings is satisfied:
(O) There exists $l \neq k$ such that $b_l > 0$.
(T) $b_k \geq 2$.

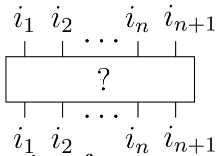
In both cases, since not in the case (a) there exists $\mathbf{i}$ with $\mathbf{e}(\mathbf{i}) \neq 0$. We try to construct $\mathbf{e}(\mathbf{i}') \neq 0$ like (I) or (II) by using $\mathbf{i}$. However to certify that we can't avoid using Graham-Lehrer conjecture[5] now, it is the most difficult part of the proof.

(iv) In the case (I), since I^Λ is maximal when $\Lambda = \Lambda_0$, non-zero non-primitive $\mathbf{e}(\mathbf{i})$ in $H_\nu^{\Lambda_0}$ is also in $H_\nu^{\Lambda'}$ where Λ' is another weight. For the case (II), we set $\Lambda = 2\Lambda_0$ and the same thing holds. [6]

We now back to (i). We can check it easily with following lemma.

Lemma 3. *Let A be associative algebra with unit, e be an idempotent in A. Then e is primitive and idempotents in eAe are only trivial two (0 and e) are equivalent.*

For (b), the elements in $\mathbf{e}(\mathbf{i})H_\nu^\Lambda\mathbf{e}(\mathbf{i})$ where $\mathbf{e}(\mathbf{i}) \neq 0$ are the linear combination of diagrams such as:



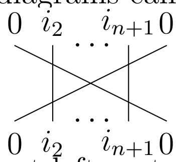
To fill up the "?" part, we use following fact:
"Every diagrams can be presented as linear combination of diagrams in which each strands cross at most once. "

[3] Refer [3].
[4] If the same color continues then "the number of dots we can put" is the same [1].
[5] solved.
[6] Comparing the case (I), we miss only the cases $\Lambda = c\Lambda_0$ $(c > 2)$ in (II).

In this case, since each strands has different color we get diagrams in which no strands cross, in the other words, parallel strands and dots. We can only do "vanish the diagram with some dots" or "swap the dots".

Then $\mathbf{e}(\mathbf{i})H_\nu^\Lambda\mathbf{e}(\mathbf{i})$ is isomorphic to a quotient of polynomial algebra with m indeterminants by some homogeneous polynomials. Moreover, idempotents in that algebra is only 0 or 1.

For (c) we can apply the same method but be careful about $i_1 = 0$ and there are two strands colored 0. In this case, these diagrams can appear:



However in this case $i_2 \neq 0$ appears at leftmost then it is 0. Crossing can be appeared at leftmost, but in that case there is two 0 strands at leftmost then it is 0. Hence the same as case (b), there are only diagrams with parallel strands and dots.

That is the sketch of the proof.

References

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[3] M. Konishi, *On level 1 cyclotomic KLR algebras of type* $A_n^{(1)}$, arXiv:math.RT/1305.3722 .

Graduate School of Mathematics
Nagoya University
Frocho, Chikusaku, Nagoya 464-8602 JAPAN
E-mail address: m10021t@math.nagoya-u.ac.jp