STABLE SET POLYTOPES OF TRIVIALLY PERFECT GRAPHS

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Abstract. We give necessary and sufficient conditions for strong Koszulness of toric rings associated with stable set polytopes of graphs.

1. Introduction

Let G be a simple graph on the vertex set $V(G) = [n]$ with the edge set $E(G)$. $S \subset V(G)$ is said to be *stable* if $\{i, j\} \notin E(G)$ for all $i, j \in S$. Note that $\emptyset$ is stable. For each stable set S of G, we define $\rho(S) = \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^n$, where $\mathbf{e}_i$ is the i-th unit coordinate vector in $\mathbb{R}^n$.

The convex hull of $\{\rho(S) \mid S \text{ is a stable set of } G\}$ is called the *stable set polytope* of G (see [2]) , denoted by $\mathcal{Q}_G$. $\mathcal{Q}_G$ is a kind of $(0,1)$-polytope. For this polytope, we define the subring of $k[T, X_1, \ldots, X_n]$ as follows:

$$k[\mathcal{Q}_G] := k[T \cdot X_1^{a_1} \cdots X_n^{a_n} \mid (a_1, \ldots, a_n) \text{ is a vertex of } \mathcal{Q}_G],$$

where k is a field. $k[\mathcal{Q}_G]$ is called the *toric ring associated with the stable set polytope of* G. We can regard $k[\mathcal{Q}_G]$ as a graded k-algebra by setting $\deg T \cdot X_1^{a_1} \cdots X_n^{a_n} = 1$.

In the theory of graded algebras, the notion of Koszulness (introduced by Priddy [15]) plays an important role and is closely related to the Gröbner basis theory.

Let $\mathcal{P}$ be an integral convex polytope (i.e., a convex polytope each of whose vertices has integer coordinates) and $k[\mathcal{P}] := k[T \cdot X_1^{a_1} \cdots X_n^{a_n} \mid (a_1, \ldots, a_n) \text{ is a vertex of } \mathcal{P}]$ be the toric ring associated with $\mathcal{P}$. In general, it is known that

$$\begin{array}{c} \text{The defining ideal of } k[\mathcal{P}] \text{ possesses a quadratic Gröbner basis} \\ \Downarrow \\ k[\mathcal{P}] \text{ is Koszul} \\ \Downarrow \\ \text{The defining ideal of } k[\mathcal{P}] \text{ is generated by quadratic binomials} \end{array}$$

follows from general theory (for example, see [1]).

In this note, we study the notion of a *strongly Koszul* algebra. In [7], Herzog, Hibi, and Restuccia introduced this concept and discussed the basic properties of strongly Koszul algebras. Moreover, they proposed the conjecture that the strong Koszulness of R is at the top of the above hierarchy, that is,

Conjecture 1 (see [7]). *The defining ideal of a strongly Koszul algebra $k[\mathcal{P}]$ possesses a quadratic Gröbner basis.*

The final version of this paper has been submitted for publication elsewhere.

A ring R is *trivial* if R can be constructed by starting from polynomial rings and repeatedly applying tensor and Segre products. In this note, we propose the following conjecture.

Conjecture 2. *Let $\mathcal{P}$ be a $(0,1)$-polytope and $k[\mathcal{P}]$ be the toric ring generated by $\mathcal{P}$. If $k[\mathcal{P}]$ is strongly Koszul, then $k[\mathcal{P}]$ is trivial.*

In the case of a $(0,1)$-polytope, Conjecture 2 implies Conjecture 1. If $\mathcal{P}$ is an order polytope or an edge polytope of bipartite graphs, then Conjecture 2 holds true [7].

In this note, we prove Conjecture 2 for stable set polytopes. The main theorem of this note is the following:

Theorem 3 ([13])**.** *Let G be a graph. Then the following assertions are equivalent:*

(1) *$k[\mathcal{Q}_G]$ is strongly Koszul.*
(2) *G is a trivially perfect graph.*

In particular, if $k[\mathcal{Q}_G]$ is strongly Koszul, then $k[\mathcal{Q}_G]$ is trivial.

Throughout this note, we will use the standard terminologies of graph theory in [4].

2. Strongly Koszul algebra

Let k be a field, R be a graded k-algebra, and $\mathfrak{m} = R_+$ be the homogeneous maximal ideal of R.

Definition 4 ([7])**.** A graded k-algebra R is said to be *strongly Koszul* if $\mathfrak{m}$ admits a minimal system of generators $\{u_1, \ldots, u_t\}$ which satisfies the following condition:

> For all subsequences $u_{i_1}, \ldots, u_{i_r}$ of $\{u_1, \ldots, u_t\}$ $(i_1 \leq \cdots \leq i_r)$ and for all $j = 1, \ldots, r-1$, $(u_{i_1}, \ldots, u_{i_{j-1}}) : u_{i_j}$ is generated by a subset of elements of $\{u_1, \ldots, u_t\}$.

A graded k-algebra R is called Koszul if $k = R/\mathfrak{m}$ has a linear resolution. By the following theorem, we can see that a strongly Koszul algebra is Koszul.

Proposition 5 ([7, Theorem 1.2])**.** *If R is strongly Koszul with respect to the minimal homogeneous generators $\{u_1, \ldots, u_t\}$ of $\mathfrak{m} = R_+$, then for all subsequences $\{u_{i_1}, \ldots, u_{i_r}\}$ of $\{u_1, \ldots, u_t\}$, $R/(u_{i_1}, \ldots, u_{i_r})$ has a linear resolution.*

The following proposition plays an important role in the proof of the main theorem.

Theorem 6 ([7, Proposition 2.1])**.** *Let S be a semigroup and $R = k[S]$ be the semigroup ring generated by S. Let $\{u_1, \ldots, u_t\}$ be the generators of $\mathfrak{m} = R_+$ which correspond to the generators of S. Then, if R is strongly Koszul, then for all subsequences $\{u_{i_1}, \ldots, u_{i_r}\}$ of $\{u_1, \ldots, u_t\}$, $R/(u_{i_1}, \ldots, u_{i_r})$ is also strongly Koszul.*

By this theorem, we have

Corollary 7 (see [14])**.** *If $k[\mathcal{Q}_G]$ is strongly Koszul, then $k[\mathcal{Q}_{G_W}]$ is strongly Koszul for all induced subgraphs G_W of G.*

3. Hibi ring and comparability graph

In this section, we introduce the concepts of a Hibi ring and a comparability graph. Both are defined with respect to a partially ordered set.

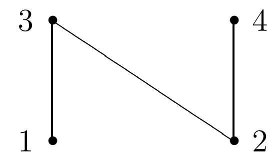
Let $P = \{p_1, \ldots, p_n\}$ be a finite partially ordered set consisting of n elements, which is referred to as a *poset*. Let $J(P)$ be the set of all poset ideals of P, where a poset ideal of P is a subset I of P such that if $x \in I$, $y \in P$, and $y \leq x$, then $y \in I$. Note that $\emptyset \in J(P)$.

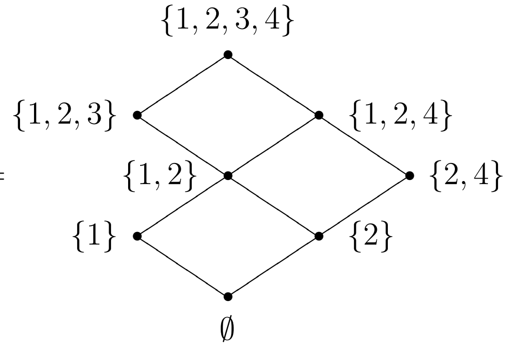
First, we give the definition of the Hibi ring introduced by Hibi.

Definition 8 ([8])**.** For a poset $P = \{p_1, \ldots, p_n\}$, the *Hibi ring* $\mathcal{R}_k[P]$ is defined as follows:

$$\mathcal{R}_k[P] := k[T \cdot \prod_{i \in I} X_i \mid I \in J(P)] \subset k[T, X_1, \ldots, X_n]$$

Example 9. Consider the following poset $P = (1 \leq 3, 2 \leq 3 \text{ and } 2 \leq 4)$.

$P =$ 

$J(P) =$ 

Then we have

$$\mathcal{R}_k[P] = k[T, TX_1, TX_2, TX_1X_2, TX_2X_4, TX_1X_2X_3, TX_1X_2X_4, TX_1X_2X_3X_4].$$

Hibi showed that a Hibi ring is always normal. Moreover, a Hibi ring can be represented as a factor ring of a polynomial ring: if we let

$$I_P := (X_IX_J - X_{I\cap J}X_{I\cup J} \mid I, J \in J(P), I \not\subseteq J \text{ and } J \not\subseteq I)$$

be the binomial ideal in the polynomial ring $k[X_I \mid I \in J(P)]$ defined by a poset P, then $\mathcal{R}_k[P] \cong k[X_I \mid I \in J(P)]/I_P$. Hibi also showed that I_P has a quadratic Gröbner basis for any term order which satisfies the following condition: the initial term of $X_IX_J - X_{I\cap J}X_{I\cup J}$ is X_IX_J. Hence a Hibi ring is always Koszul from general theory.

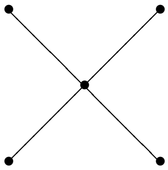
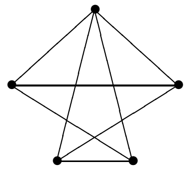
Next, we introduce the concept of a comparability graph.

Definition 10. A graph G is called a *comparability graph* if there exists a poset P which satisfies the following condition:

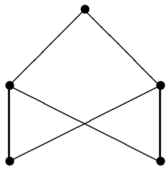
$$\{i, j\} \in E(G) \iff i \geq j \text{ or } i \leq j \text{ in } P.$$

We denote the comparability graph of P by $G(P)$.

Example 11. The lower-left poset P defines the comparability graph $G(P)$.

$P =$  $G(P) =$ 

Remark 12. It is possible that $P \neq P'$ but $G(P) = G(P')$. Indeed, for the following poset P', $G(P')$ is identical to $G(P)$ in the above example.

$P' =$ 

Complete graphs are comparability graphs of totally ordered sets. Bipartite graphs and trivially perfect graphs (see the next section) are also comparability graphs. Moreover, if G is a comparability graph, then the suspension (e.g., see [11, p.4]) of G is also a comparability graph.

Recall the following definitions of two types of polytope which are defined by a poset.

Definition 13 (see [16])**.** Let $P = \{p_1, \ldots, p_n\}$ be a finite poset.

(1) The *order polytope* $\mathcal{O}(P)$ of P is the convex polytope which consists of $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $0 \leq a_i \leq 1$ with $a_i \geq a_j$ if $p_i \leq p_j$ in P.

(2) The *chain polytope* $\mathcal{C}(P)$ of P is the convex polytope which consists of $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $0 \leq a_i \leq 1$ with $a_{i_1} + \cdots + a_{i_k} \leq 1$ for all maximal chain $p_{i_1} < \cdots < p_{i_k}$ of P.

Let $\mathcal{C}(P)$ and $\mathcal{O}(P)$ be the chain polytope and order polytope of a finite poset P, respectively. In [16], Stanley proved that

$$\{\text{The vertices of } \mathcal{O}(P)\} = \{\rho(I) \mid I \text{ is a poset ideal of } P\},$$
$$\{\text{The vertices of } \mathcal{C}(P)\} = \{\rho(A) \mid A \text{ is an anti-chain of } P\},$$

where $A = \{p_{i_1}, \ldots, p_{i_k}\}$ is an anti-chain of P if $p_{i_s} \not\leq p_{i_t}$ and $p_{i_s} \not\geq p_{i_t}$ for all $s \neq t$. Hence we have $\mathcal{Q}_{G(P)} = \mathcal{C}(P)$.

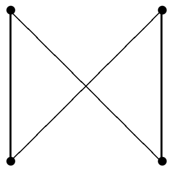
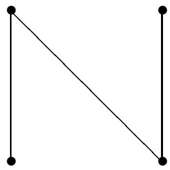
In [9], Hibi and Li answered the question of when $\mathcal{C}(P)$ and $\mathcal{O}(P)$ are unimodularly equivalent. From their study, we have the following theorem.

Theorem 14 ([9, Theorem 2.1])**.** *Let P be a poset and $G(P)$ be the comparability graph of P. Then the following are equivalent:*

(1) *The X-poset in Example 3.4 does not appear as a subposet* (*refer to* [17, *Chapter* 3]) *of P.*

(2) $\mathcal{R}_k[P] \cong k[\mathcal{Q}_{G(P)}]$.

Example 15. The cycle of length 4 C_4 and the path of length 3 P_4 are comparability graphs of Q_1 and Q_2, respectively.

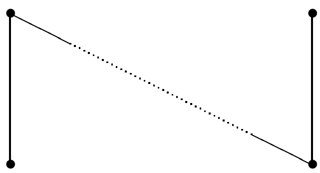
$Q_1 =$  $Q_2 =$ 

Hence $k[\mathcal{Q}_{C_4}] \cong \mathcal{R}_k[Q_1]$ and $k[\mathcal{Q}_{P_4}] \cong \mathcal{R}_k[Q_2]$.

A ring R is *trivial* if R can be constructed by starting from polynomial rings and repeatedly applying tensor and Segre products. Herzog, Hibi and Restuccia gave an answer for the question of when is a Hibi ring strongly Koszul.

Theorem 16 (see [7, Theorem 3.2])**.** *Let P be a poset and $R = \mathcal{R}_k[P]$ be the Hibi ring constructed from P. Then the following assertions are equivalent:*

(1) *R is strongly Koszul.*
(2) *R is trivial.*
(3) *The N-poset as described below does not appear as a subposet of P.*



By this theorem, Corollary 7, and Example 15, we have

Corollary 17. *If G contains C_4 or P_4 as an induced subgraph, then $k[\mathcal{Q}_G]$ is not strongly Koszul.*

4. TRIVIALLY PERFECT GRAPH

In this section, we introduce the concept of a trivially perfect graph. As its name suggests, a trivially perfect graph is a kind of perfect graph; it is also a kind of comparability graph, as described below.

Definition 18. For a graph G, we set

$$\alpha(G) := \max\{\#S \mid S \text{ is a stable set of } G\},$$

$$m(G) := \#\{\text{the set of maximal cliques of } G\}.$$

We call $\alpha(G)$ the *stability number* (or *independence number*) of G.

In general, $\alpha(G) \leq m(G)$. Moreover, if G is chordal, then $m(G) \leq n$ by Dirac's theorem [5]. In [6], Golumbic introduced the concept of a trivially perfect graph.

Definition 19 ([6]). We say that a graph G is *trivially perfect* if $\alpha(G_W) = m(G_W)$ for any induced subgraph G_W of G.

For example, complete graphs and star graphs (i.e., the complete bipartite graph $K_{1,r}$) are trivially perfect.

We define some additional concepts related to perfect graphs. Let C_G be the set of all cliques of G. Then we define

$$\omega(G) := \max\{\#C \mid C \in C_G\},$$
$$\theta(G) := \min\{s \mid C_1 \coprod \cdots \coprod C_s = V(G), C_i \in C_G\},$$
$$\chi(G) := \theta(\overline{G}),$$

where $\overline{G}$ is the complement of G. These invariants are called the *clique number*, *clique covering number*, and *chromatic number* of G, respectively.

In general, $\alpha(G) = \omega(\overline{G})$, $\theta(G) \leq m(G)$ and $\omega(G) \leq \chi(G)$. The definition of a perfect graph is as follows.

Definition 20. We say that a graph G is *perfect* if $\omega(G_W) = \chi(G_W)$ for any induced subgraph G_W of G.

Lovász proved that G is perfect if and only if $\overline{G}$ is perfect [12]. The theorem is now called the weak perfect graph theorem. With it, it is easy to show that a trivially perfect graph is perfect.

Proposition 21. *Trivially perfect graphs are perfect.*

Proof. Assume that G is trivially perfect. By [12], it is enough to show that $\overline{G}$ is perfect. For all induced subgraphs $\overline{G}_W$ of $\overline{G}$, we have

$$m(G_W) = \alpha(G_W) = \omega(\overline{G_W}) \leq \chi(\overline{G_W}) = \theta(G_W) \leq m(G_W)$$

by general theory (note that $\overline{G}_W = \overline{G_W}$). □

Golumbic gave a characterization of trivially perfect graphs.

Theorem 22 ([6, Theorem 2]). *The following assertions are equivalent:*

(1) *G is trivially perfect.*

(2) *G is C_4, P_4-free, that is, G contains neither C_4 nor P_4 as an induced subgraph.*

Proof. (1) ⇒ (2): It is clear since $\alpha(C_4) = 2$, $m(C_4) = 4$, and $\alpha(P_4) = 2$, $m(P_4) = 3$.

(2) ⇒ (1): Assume that G contains neither C_4 nor P_4 as an induced subgraph. If G is not trivially perfect, then there exists an induced subgraph G_W of G such that $\alpha(G_W) < m(G_W)$. For this G_W, there exists a maximal stable set S_W of G_W which satisfies the following:

There exists $s \in S_W$ such that $s \in C_1 \cap C_2$ for some distinct pair of cliques $C_1, C_2 \in C_{G_W}$.

Note that $\#S_W > 1$ since G_W is not complete. Then there exist $x \in C_1$ and $y \in C_2$ such that $\{x, s\}$, $\{y, s\} \in E(G_W)$ and $\{x, y\} \notin E(G_W)$.

Let $u \in S_W \setminus \{s\}$. If $\{x, u\} \in E(G_W)$ or $\{y, u\} \in E(G_W)$, then the induced graph $G_{\{x,y,s,u\}}$ is C_4 or P_4, a contradiction. Hence $\{x, u\} \notin E(G_W)$ and $\{y, u\} \notin E(G_W)$. Then

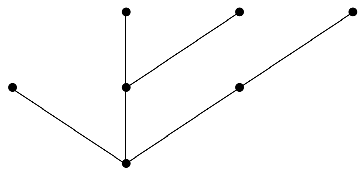
$\{x, y\} \cup \{S \setminus \{s\}\}$ is a stable set of G_W, which contradicts that S is maximal. Therefore, G is trivially perfect. ☐

Next, we show that a trivially perfect graph is a kind of comparability graph. First, we define the notion of a tree poset.

Definition 23 (see [18]). A poset P is a *tree* if it satisfies the following conditions:

(1) Each of the connected components of P has a minimal element.
(2) For all p, $p' \in P$, the following assertion holds: if there exists $q \in P$ such that $p, p' \leq q$, then $p \leq p'$ or $p \geq p'$.

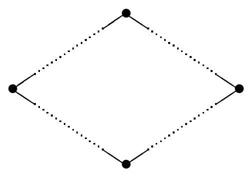
Example 24. The following poset is a tree:



Tree posets can be characterized as follows.

Proposition 25. *Let P be a poset. Then the following assertions are equivalent:*

(1) *P is a tree.*
(2) *Neither the X-poset in Example 11, the N-poset in Theorem 16, nor the diamond poset as described below appears as a subposet of P.*



In [18], Wolk discussed the properties of the comparability graphs of a tree poset and showed that such graphs are exactly the graphs that satisfy the "diagonal condition". This condition is equivalent to being C_4, P_4-free, and hence we have

Corollary 26. *Let G be a graph. Then the following assertions are equivalent:*

(1) *G is trivially perfect.*
(2) *G is a comparability graph of a tree poset.*
(3) *G is C_4, P_4-free.*

Remark 27. A graph G is a *threshold graph* if it can be constructed from a one-vertex graph by repeated applications of the following two operations:

(1) Add a single isolated vertex to the graph.
(2) Take a suspension of the graph.

The concept of a threshold graph was introduced by Chvátal and Hammer [3]. They proved that G is a threshold graph if and only if G is C_4, P_4, $2K_2$-free. Hence a trivially perfect graph is also called a *quasi-threshold graph*.

5. Proof of Main theorem

In this section, we prove the main theorem.

Theorem 28 ([13]). *Let G be a graph. Then the following assertions are equivalent:*

(1) *$k[\mathcal{Q}_G]$ is strongly Koszul.*
(2) *G is trivially perfect.*

Proof. We assume that G is trivially perfect. Then there exists a tree poset P such that $G = G(P)$ from Corollary 26. This implies that neither the X-poset in Example 11 nor the N-poset in Theorem 16 appears as a subposet of P by Proposition 25, and hence $k[\mathcal{Q}_{G(P)}] \cong \mathcal{R}_k[P]$ is strongly Koszul by Theorems 14 and 16.

Conversely, if G is not trivially perfect, G contains C_4 or P_4 as an induced subgraph by Corollary 26. Therefore, we have that $k[\mathcal{Q}_G]$ is not strongly Koszul by Corollary 17. □

Remark 29. On the recent work with Hibi and Ohsugi, we have that the Conjecture 2 is false [10]. We proved that there exist infinite many non-trivial strongly Koszul edge rings.

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