IDEALS AND TORSION THEORIES

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Abstract. We introduce ideal theoretic conditions on an ideal $\mathcal{I}$ of an abelian category $\mathcal{A}$, which are show to be equivalent to the condition that the ideal is associated to a torsion class (resp. pre-torsion class, Serre subcategory) of $\mathcal{A}$. We also discuss an ideal which is associated to a radical (a sub functor of the identity functor $\mathrm{id}_{\mathcal{A}}$ which has a special property) of $\mathcal{A}$.

This work came out from an attempt to obtain a formalism of the argument which is given in the proof of the following theorem on 2-representation infinite (2-RI) algebras ([1]): Let Λ be a 2-RI algebra and τ_2, τ_2^- be 2-Auslander-Reiten translations. A Λ-module M is called θ-minimal if the canonical morphism $M \to \tau_2\tau_2^- M$ is injective.

1. Motivation: Auslander-Reiten component of a 2-representation infinite algebra.

This work came out from an attempt to obtain a formalism of the argument which is given in the proof of the following theorem on 2-representation infinite algebra.

First we recall the definition of n-representation infinite algebra introduced by Herschend-Iyama-Oppermann [1]. Let A be a finite dimensional algebra over a field k of finite global dimension. Recall that the Nakayama functor ν is the derived tensor product $- \otimes_A^A D(A)$ of the k-dual $D(A) := \mathrm{Hom}_k(A, k)$, which gives a triangle autoequivalence of the bounded derived category $\mathcal{D}_A := \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$. For a integer $n \in \mathbb{Z}$ we set $\nu_n := \nu \circ [-n]$.

Definition 1. A finite dimensional algebra A is called *n-representation infinite (n-RI) algebra* if it is of global dimension n and the complex $\nu_n^{-p}(A)$ belongs to the standard heart mod-A for $p \in \mathbb{N}$.

Let A be n-RI algebra. The Hom-$\otimes$ adjunction induced by the A-A bi-module $\theta := \mathrm{Ext}_A^n(D(A), A)$ is called n-Auslander-Reiten translations. More precisely, we set $\tau_n(-) := \mathrm{Hom}_A(\theta, -)$ and $\tau_n^- := - \otimes_A \theta$.

$$\tau_n^- := - \otimes_A \theta : \text{mod-}A \rightleftarrows \text{mod-}A : \tau_n(-) := \mathrm{Hom}_A(\theta, -).$$

We remark that τ_n and τ_n^- dose not give equivalences and, in particular, are not inverse to each other. Hence, the adjoint unit morphism $M \to \tau_n\tau_n^- M$ is possibly not an isomorphism.

To state our motivating theorem, we introduce one terminology. An A-module M is called *θ-minimal* if the adjoint unit morphism $M \to \tau_n\tau_n^- M$ is injective.

Theorem 2. *Let A be a 2-RI algebra and $M \neq 0$ be a θ-minimal indecomposable A-module. Assume that* $\mathrm{Hom}_A(M, A) = 0$. *(e.g., M is a non-projective* 2*-preprojective module or* 2*-regular module.) Then the Auslander-Reiten component Γ_M containing M is*

The detailed version of this paper will be submitted for publication elsewhere.

of type $\mathbb{Z}\mathrm{A}_\infty$ unless it contains projective module or injective modules and M is placed in the bottom of Γ_M.

The proof goes as follows: from the property $\mathrm{Hom}_A(M, A) = 0$, we can deduce that $(\tau_1 M) \otimes_A \theta = 0$ where τ_1 is the usual Auslander-Reiten translation. Together with the θ-minimality of M, the latter property implies that the middle term M_1 of the Auslander-Reiten sequence is indecomposable.

$$0 \to \tau_1 M \to M_1 \to M \to 0.$$

This part of argument is quite formal and leads us to the theory of quasi torsion ideals, which will be discussed in the main body of this note. Using similar argument, we conclude the results.

2. QUASI TORSION IDEALS

Let $\mathcal{A}$ be an ableian category. Recall that a sub functor of the Hom-functor $\mathcal{A}(-,+) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$ is called an ideal of $\mathcal{A}$. Here we view $\mathcal{A}$ as a ring with several objects.

Definition 3. Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ be an ideal of $\mathcal{A}$.

(1) An object $M \in \mathcal{A}$ is called *$\mathcal{I}$-minimal* if $\mathcal{I}(-, M) = 0$;

(2) An exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called a *quasi-torsion sequence*(qt sequence) with respect to $\mathcal{I}$, if N is $\mathcal{I}$-minimal and f belongs to the ideal $\mathcal{I}$.

(3) An ideal $\mathcal{I}$ is called *quasi-torsion ideal*(qt ideal), if there exists a qt sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

for all objects M of $\mathcal{A}$.

Lemma 4. *For an object M, a qt sequence the middle term of which is M, is unique up to isomorphism (if it exists).*

If an ideal $\mathcal{I}$ is quasi-torsion. Then for an object $M \in \mathcal{A}$, an object $L \in \mathcal{A}$ which appearers in the left term of a qt-sequence $0 \to L \to M \to N \to 0$ whose middle term is M is uniquely determined up to isomorphisms. We denote such an object by $\mathsf{com}M$ and call *cominimal pert of M*.

Corollary 5. *Let $\mathcal{I}$ be a qt ideal. Let $0 \to L \to M \to N \to 0$ be a non-split qt sequence. If L and N are indecomposable, then the middle term M is also indecomposable.*

Example 6. Let A be a Noetherian ring and ρ be an A-A-bi-module. For A-modules M and N we set

$$\mathcal{I}(M, N) := \{f : M \to N \mid f \otimes_A \rho = 0\}.$$

It is easy to see that $\mathcal{I}$ is an ideal of the category mod-A of finite A-modules. Moreover, we can prove that $\mathcal{I}$ is a qt-ideal and $\mathcal{I}$-minimal objects are precisely modules M such that the unite morphism $M \to \mathrm{Hom}_A(\rho, M \otimes \rho)$ is injective.

Let $0 \to L \to M \to N \to 0$ be an exact sequence of A-modules. Assume that N is $\mathcal{I}$-minimal and $L \otimes_A \rho = 0$, then the above sequence is quasi-torsion. Assume moreover that

L and N are indecomposable. The by the above corollary, we conclude that the middle term M is also indecomposable. This is the way that we prove the indecomposability of the middle term M_1 in the proof of Theorem 2.

Recall that a *preradical* $\mathbf{r}$ of an abelian category $\mathcal{A}$ is a sub functor of the identity functor $\mathsf{id}_{\mathcal{A}}$. (Note that we can easily see that a preradical $\mathbf{r}$ is an additive functor.) A preradical $\mathbf{r}$ is called *radical* if it satisfies $\mathbf{r}(M/\mathbf{r}(M)) = 0$ for all $M \in \mathcal{A}$. (For details, see, e.g. [2]). For a radical $\mathbf{r}$, we define an ideal $\mathcal{I}_{\mathbf{r}}$ to be

$$\mathcal{I}_{\mathbf{r}}(M, N) := \{f \in \mathcal{A}(M, N) \mid f \text{ factors through } \mathbf{r}(M).\}.$$

Lemma 7. *A preradical $\mathbf{r}$ is a radical if and only if the ideal $\mathcal{I}_{\mathbf{r}}$ is quasi-torsion. If this is the case we have $\mathbf{r}M = \mathsf{com}M$.*

Theorem 8. *The following gives a one to one correspondence between quasi-torsion ideals and radicals on $\mathcal{A}$.*

$$\mathcal{I} \mapsto \mathsf{com}_{\mathcal{I}}, \ \mathbf{r} \mapsto \mathcal{I}_{\mathbf{r}}.$$

Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ be a quasi-torsion ideals.

Lemma 9. *The factor category $\mathcal{A}/\mathcal{I}$ has pseudo-kernels. Hence the category* $\mathrm{mod}\mathcal{A}/\mathcal{I}$ *of coherent functors is abelian.*

We discuss the global dimension of the category $\mathrm{mod}\mathcal{A}/\mathcal{I}$. When we view an abelian category $\mathcal{A}$ as a ring with several objects, this category is the category of finitely presented modules. We show that the global dimension has a relationship with nilpotensy of the qt-ideal $\mathcal{I}$.

Theorem 10. *Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ be a quasi-torsion ideals.*

(1) *If $\mathcal{A}$ is Artinian, then every coherent $\mathcal{A}/\mathcal{I}$-module has finite projective dimension.*
(2) *If $\mathcal{I}^n = \mathcal{I}^{n+1}$, then* $\mathrm{gldim}(\mathrm{mod}\mathcal{A}/\mathcal{I}) \leq 2n$.
(3) *If* $\mathrm{gldim}(\mathrm{mod}\mathcal{A}/\mathcal{I}^2) \leq n - 1$, *then $\mathcal{I}^n = \mathcal{I}^{n+1}$ and* $\mathrm{gldim}(\mathrm{mod}\mathcal{A}/\mathcal{I}) \leq 2n$.

As far as the author knows, such a phenomena rarely happen for a ring with single object, i.e., a ring in the usual sense. Therefore we can expect that, even from ring theoretical view point, a ring with several objects has special features which are not possessed by a ring with single objects.

We end this note by proposing a question: Can we get more strong result, like

$$\mathrm{gldimmod}\mathcal{A}/\mathcal{I} \leq 2n \quad \Longleftrightarrow \quad \mathcal{I}^n = \mathcal{I}^{n+1}????$$

References

[1] M. Herschend, O. Iyama, S. Oppermann, *n-representation infnite algebras*, math.12051272.
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