SUPPORT τ-TILTING MODULES AND PREPROJECTIVE ALGEBRAS

YUYA MIZUNO

Abstract. We study support τ-tilting modules over preprojective algebras of Dynkin type. We classify basic support τ-tilting modules by giving a bijection with elements in the corresponding Weyl groups. We also study g-matrices of support τ-tilting modules, which show terms of minimal projective presentations of indecomposable direct summands. We give an explicit description of g-matrices and prove that cones given by g-matrices coincide with chambers of the associated root systems.

1. Introduction

The preprojective algebra associated to a quiver was introduced by Gelfand-Ponomarev [GP] to study the preprojective representations of a quiver. Since then, they have been studied extensively not only from the viewpoint of representation theory of algebras (for example [BGL, DR1, DR2, Ri1]) but also in several contexts such as (algebraic, differential, symplectic) geometry and quantum groups.

In [BIRS] (also in [IR1]), the authors studied preprojective algebras via tilting theory for non-Dynkin quivers. By making heavy use of tilting theory, they succeed to give several important results such as a method for constructing a large class of 2-Calabi-Yau categories which have close connections with cluster algebras. On the other hand, in Dynkin cases (i.e. the underlying graph of a quiver is A_n $(n \geq 1)$, D_n $(n \geq 4)$ and E_n $(n = 6, 7, 8)$), the preprojective algebras are selfinjective, so that all tilting modules are trivial (i.e. projective). In this note, we will show that, instead of tilting modules, *support τ-tilting modules* play an important role in this case.

The notion of support τ-tilting modules was introduced in [AIR], which gives a generalization of tilting modules. They have several nice properties. For example, it is shown that there are deep connections between τ-tilting theory, torsion theory, silting theory, cluster theory and so on (refer to an introductory article [IR2]). Moreover, support τ-tilting modules have nicer mutation theory than tilting modules. Namely, any basic almost-complete support τ-tilting module is the direct summand of exactly two basic support τ-tilting modules. It implies that mutation of support τ-tilting modules is always possible and this property admits interesting combinatorial descriptions for support τ-tilting graphs. Furthermore, certain support τ-tilting modules over selfinjective algebras provide tilting complexes [M1]. It is therefore fruitful to investigate these remarkable modules for given algebras.

Conventions. Let K be an algebraically closed field and we denote by $D := \mathrm{Hom}_K(-, K)$. By a finite dimensional algebra Λ, we mean a basic finite dimensional algebra over K. By a module, we mean a right module unless stated otherwise. We denote by $\mathsf{mod}\Lambda$ the

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category of finitely generated Λ-modules and by $\mathsf{proj}\Lambda$ the category of finitely generated projective Λ-modules. For $X \in \mathsf{mod}\Lambda$, we denote by $\mathsf{Sub}X$ (respectively, $\mathsf{Fac}X$) the subcategory of $\mathsf{mod}\Lambda$ whose objects are submodules (respectively, factor modules) of finite direct sums of copies of X. We denote by $\mathsf{add}M$ the subcategory of $\mathsf{mod}\Lambda$ consisting of direct summands of finite direct sums of copies of $M \in \mathsf{mod}\Lambda$.

2. Preliminaries

2.1. **Preprojective algebras.**

In this subsection, we recall the definition of preprojective algebras and some properties of them.

Definition 1. Let Q be a finite connected acyclic quiver with vertices $Q_0 = \{1, \ldots, n\}$. The preprojective algebra associated to Q is the algebra

$$\Lambda_Q = \Lambda := K\overline{Q}/\langle \sum_{a \in Q_1} (aa^* - a^*a) \rangle$$

where $\overline{Q}$ is the double quiver of Q, which is obtained from Q by adding for each arrow $a : i \to j$ in Q_1 an arrow $a^* : i \leftarrow j$ pointing in the opposite direction.

Note that Λ_Q does not depend on the orientation of Q.

We collect some basic properties of preprojective algebras.

Proposition 2. *Let Q be an acyclic quiver and Λ the preprojective algebra of Q. Then Q is a Dynkin quiver if and only if Λ is a finite dimensional algebra. Further, if these equivalent conditions hold, then Λ is selfinejctive.*

Note that, even if Q is a Dynkin quiver, Λ is infinite representation type (i.e. there exists infinitely many indecomposable Λ-modules) in general [DR2].

We define the *Coxeter group* W_Q associated to Q, which is defined by the generators $s_1, \ldots, s_n$ and relations

- $s_i^2 = 1$,
- $s_i s_j = s_j s_i$ if there is no arrow between i and j in Q,
- $s_i s_j s_i = s_j s_i s_j$ if there is precisely one arrow between i and j in Q.

Each element $w \in W_Q$ can be written in the form $w = s_{i_1} \cdots s_{i_k}$. If k is minimal among all such expressions for w, then k is called the *length* of w and we denote by $l(w) = k$. In this case, we call $s_{i_1} \cdots s_{i_k}$ a *reduced expression* of w.

2.2. **Support τ-tilting modules.**

In this subsection, we give definitions of support τ-tilting modules.

Definition 3. Let Λ be a finite dimensional algebra.

(a) We call X in $\mathsf{mod}\Lambda$ *τ-rigid* if $\mathrm{Hom}_\Lambda(X, \tau X) = 0$.

(b) We call X in $\mathsf{mod}\Lambda$ *τ-tilting* (respectively, *almost complete τ-tilting*) if X is τ-rigid and $|X| = |\Lambda|$ (respectively, $|X| = |\Lambda| - 1$), where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X.

(c) We call X in $\mathsf{mod}\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that X is a τ-tilting $(\Lambda/\langle e \rangle)$-module.

We also give the following definitions.

(d) We call a pair (X, P) of $X \in \mathsf{mod}\Lambda$ and $P \in \mathsf{proj}\Lambda$ *τ-rigid* if X is τ-rigid and $\mathrm{Hom}_\Lambda(P, X) = 0$.

(e) We call a τ-rigid pair (X, P) a *support τ-tilting* (respectively, *almost support τ-tilting*) pair if $|X| + |P| = |\Lambda|$ (respectively, $|X| + |P| = |\Lambda| - 1$).

We call (X, P) *basic* if X and P are basic, and we say that (X, P) is a *direct summand* of (X', P') if X is a direct summand of X' and P is a direct summand of P'. We denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules.

Note that (X, P) is a τ-rigid (respectively, support τ-tilting) pair for Λ if and only if X is a τ-rigid (respectively, τ-tilting) $(\Lambda/\langle e \rangle)$-module, where e is an idempotent of Λ such that $\mathsf{add}P = \mathsf{add}e\Lambda$ [AIR, Proposition 2.3]. Moreover, if (X, P) and (X, P') are support τ-tilting pairs for Λ, then we get $\mathsf{add}P = \mathsf{add}P'$. Thus, a basic support τ-tilting module X determines basic support τ-tilting pair (X, P) uniquely and we can identify basic support τ-tilting modules with basic support τ-tilting pairs.

Example 4. Let $\Lambda := KQ$ be the path algebra given by the following quiver

$$Q := \ (1 \longleftarrow 2)\ .$$

Then one can check that there exist support τ-tilting modules as follows

$$e_1\Lambda \oplus e_2\Lambda,\ S_2 \oplus e_2\Lambda,\ e_1\Lambda,\ S_2 \text{ and } 0.$$

They can be identified with support τ-tilting pairs

$$(e_1\Lambda \oplus e_2\Lambda, 0),\ (S_2 \oplus e_2\Lambda, 0),\ (e_1\Lambda, e_2\Lambda),\ (S_2, e_1\Lambda) \text{ and } (0, e_1\Lambda \oplus e_2\Lambda),$$

respectively.

One of the important properties of support τ-tilting modules is a partial order.

Definition 5. Let Λ be a finite dimensional algebra. For $T, T' \in \mathrm{s}\tau\text{-tilt}\,\Lambda$, we write

$$T' \geq T$$

if $\mathsf{Fac}T' \supset \mathsf{Fac}T$, where $\mathsf{Fac}X$ the subcategory of $\mathsf{mod}\Lambda$ whose objects are factor modules of finite direct sums of copies of X. Then $\geq$ gives a partial order on $\mathrm{s}\tau\text{-tilt}\,\Lambda$ [AIR, Theorem 2.18]. Clearly, Λ is the unique maximal element and 0 is the unique minimal element.

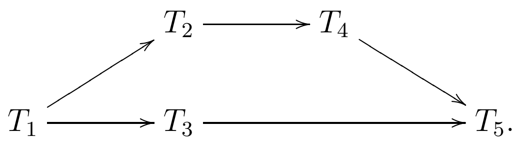
Example 6. Let $\Lambda := KQ$ be the path algebra given by the following quiver

$$Q := \ (1 \longleftarrow 2)\ .$$

Let $T_1 := e_1\Lambda \oplus e_2\Lambda,\ T_2 := S_2 \oplus e_2\Lambda,\ T_3 := e_1\Lambda,\ T_4 := S_2$ and $T_5 := 0$.
Then we have

$$\mathsf{Fac}T_1 = \mathsf{add}\{e_1\Lambda \oplus e_2\Lambda \oplus S_2\},\ \mathsf{Fac}T_2 = \mathsf{add}\{S_2 \oplus e_2\Lambda\},\ \mathsf{Fac}T_3 = \mathsf{add}\{e_1\Lambda\},\ \mathsf{Fac}T_4 = \mathsf{add}\{S_2\},$$

where $\mathsf{add}X$ denote the subcategory of $\mathsf{mod}\Lambda$ consisting of direct summands of finite direct sums of $X \in \mathsf{mod}\Lambda$. Then, from Definition 5, one can obtain the following Hasse quiver.



3. Support τ-tilting modules and the Weyl group

Our main aim is to give a complete description of all support τ-tilting modules over preprojective algebras of Dynkin type. For this purpose, we give the following bijection.

Theorem 7. *Let Q be a Dynkin quiver and Λ the preprojective algebra of Q. There exist bijections between the isomorphism classes of basic support τ-tilting Λ-modules and the elements of W_Q.*

In the following two subsections, we explain Theorem 7.

3.1. **Support τ-tilting ideals.** Let Q be a Dynkin quiver with $Q_0 = \{1, \ldots, n\}$ and Λ the preprojective algebra of Q. We denote by the two-sided ideal I_i of Λ generated by $1 - e_i$, where e_i is a primitive idempotent of Λ for $i \in Q_0$, that is, $I_i := \Lambda(1 - e_i)\Lambda$ for $i \in Q_0$. We see that the ideal has the following property.

Lemma 8. *Let $X \in \mathsf{mod}\Lambda$. Then XI_i is maximal amongst submodules Y of X such that any composition factor of X/Y is isomorphic to a simple module Λ/I_i.*

We denote by $\langle I_1, \ldots, I_n \rangle$ the set of ideals of Λ which can be written as

$$I_{i_1} I_{i_2} \cdots I_{i_k}$$

for some $k \geq 0$ and $i_1, \ldots, i_k \in Q_0$.

Our aim in this subsection is to show the following.

Theorem 9. *Any $T \in \langle I_1, \ldots, I_n \rangle$ is a basic support τ-tilting modules of Λ.*

For a proof, we use the following important property.

Definition 10. [AIR, Theorem 2.18 and 2.28] Let Λ be a finite dimensional algebra. Then

- $(*)$ any basic almost support τ-tilting pair (U, Q) is a direct summand of exactly two basic support τ-tilting pairs (T, P) and (T', P'). Moreover we have $T > T'$ or $T < T'$.

Under the above setting, let X be an indecomposable direct summand of T or P. We write $(T', P') = \mu_X(T, P)$ or simply $T' = \mu_X(T)$ and say that T' is a *mutation* of T. In particular, we write $T' = \mu_X^-(T)$ if $T > T'$ (respectively, $T' = \mu_X^+(T)$ if $T < T'$) and we say that T' is a *left mutation* (respectively, *right mutation*) of T. By $(*)$, exactly one of the left mutation or right mutation occurs.

Using mutations, we give the following key proposition.

Proposition 11. *Let $T \in \langle I_1, \ldots, I_n \rangle$ and assume that T is a basic support τ-tilting Λ-module. If $I_iT \neq T$, then there is a left mutation of T associated to e_iT and $\mu_{e_iT}^-(T) \cong I_iT$. In particular, I_iT is a basic support τ-tilting Λ-module.*

Namely, a multiplication by I_i gives a left mutation of T if $I_iT \neq T$. Using this property inductively, we can obtain Theorem 9.

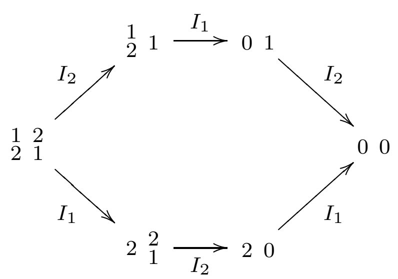
There exists a close relationship between mutations and partial orders.

Definition 12. [AIR, Corollary 2.34] Let Λ be a finite dimensional algebra. We define the *support τ-tilting quiver* $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ as follows.
• The set of vertices is $\mathrm{s}\tau\text{-tilt}\,\Lambda$.
• Draw an arrow from T to T' if T' is a left mutation of T.
Then $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ coincides with the Hasse quiver of the partially ordered set $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

Hence, the Hasse quiver of Example 6 gives behavior of mutations of support τ-tilting modules.

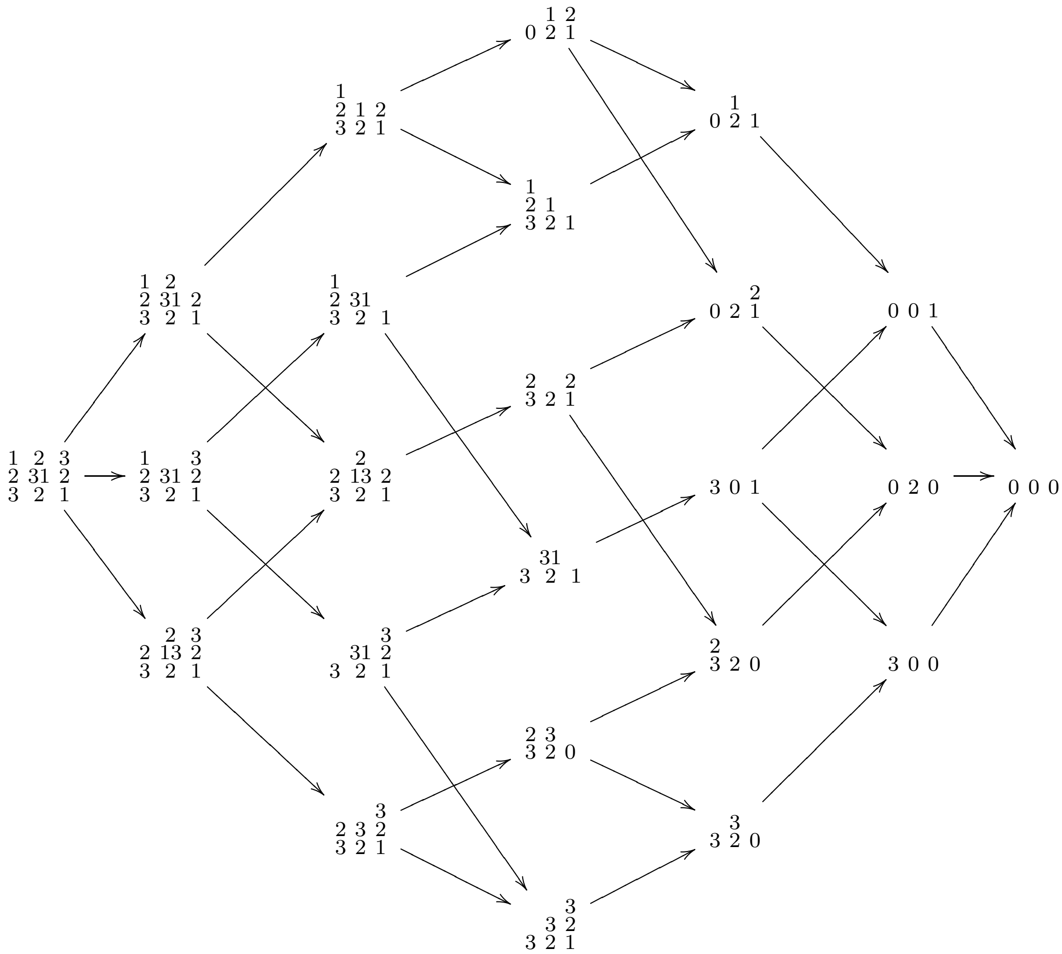
Now using support τ-tilting quiver, we describe support τ-tilting modules of preprojective algebras.

Example 13. (a) Let Λ be the preprojective algebra of type A_2. In this case, $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is given as follows.



Here we represent modules by their radical filtrations and we write a direct sum $X \oplus Y$ by XY.

(b) Let Λ be the preprojective algebra of type A_3. In this case, $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is given as follows



Remark 14. In these examples, $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ consists of a finite connected component. We will show that this is always true for preprojective algebras of Dynkin type in the next subsection. Thus, all support τ-tilting modules can be obtained by mutations from Λ.

3.2. **A connection with the Weyl group.** Let Q be a Dynkin quiver with $Q_0 = \{1, \ldots, n\}$ and Λ the preprojective algebra of Q. To give an explicit description of support τ-tilting Λ-modules, we provide a connection with the Weyl group.

We use the following result (see [BIRS, M2]).

Theorem 15. *There exists a bijection* $W_Q \to \langle I_1, \ldots, I_n \rangle$*. It is given by* $w \mapsto I_w = I_{i_1} I_{i_2} \cdots I_{i_k}$ *for any reduced expression* $w = s_{i_1} \cdots s_{i_k}$*.*

From this theorem and Theorem 9, we obtain one finite connected component in $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$. Then we can apply the following result.

Proposition 16. [AIR, Corollary 2.38] *If* $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ *has a finite connected component* C*, then* $C = \mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$*.*

As a conclusion, we can obtain the following statement.

Theorem 17. *Any basic support τ-tilting Λ-module is isomorphic to an element of $\langle I_1, \ldots, I_n \rangle$.*

We also use the following lemma.

Lemma 18. *If right ideals T and U are isomorphic as Λ-modules, then $T = U$.*

Then, combining the above results, we get the desired statement.

Proof of Theorem 7. We will give a bijection between sτ-tiltΛ and W_Q. A bijection – and W_Q can be given similarly.

By Theorem 9 and 15, we have a map $W_Q \ni w \mapsto I_w \in$ sτ-tiltΛ. This map is surjective since any support τ-tilting Λ-module is isomorphic to I_w for some $w \in W_Q$ by Theorem 17. Moreover it is injective by Theorem 15 and Lemma 18. Thus we get the conclusion. $\square$

At the end of this subsection, we briefly give a relationship between a partial order of support τ-tilting modules and that of W_Q.

Definition 19. Let $u, w \in W_Q$. We write $u \leq_L w$ if there exist $s_{i_k}, \ldots, s_{i_1}$ such that $w = s_{i_k} \ldots s_{i_1} u$ and $l(s_{i_j} \ldots s_{i_1} u) = l(u) + j$ for $0 \leq j \leq k$. Clearly $\leq_L$ gives a partial order on W_Q, and we call $\leq_L$ the *left order* (it is also called *weak order*). We denote by $\mathcal{H}(W_Q, \leq_L)$ the Hasse quiver of left order on W_Q.

Then we have the following results.

Theorem 20. *The bijection in $W_Q \to$ sτ-tiltΛ in Theorem 7 gives an isomorphism of partially ordered sets $(W_Q, \leq_L)$ and $(\text{s}\tau\text{-tilt}\Lambda, \leq)^{op}$.*

We remark that the *Bruhat order* on W_Q coincides with the reverse inclusion relation on $\langle I_1, \cdots I_n \rangle$ [ORT, Lemma 6.5].

4. g-matrices and cones

In this last section, we introduce the notion of *g-vectors* [DK] (which is also called *index* [AR, P]) and *g-matrices* of support τ-tilting modules. We refer to [AIR, section 5] for a background of this notion.

Definition 21. Let Λ be a finite dimensional algebra and $K_0(\mathsf{proj}\Lambda)$ the Grothendieck group of the additive category $\mathsf{proj}\Lambda$. Then the isomorphism classes $e_1\Lambda, \ldots, e_n\Lambda$ of indecomposable projective Λ-modules form a basis of $K_0(\mathsf{proj}\Lambda)$. Consider X in $\mathsf{mod}\Lambda$ and let

$$P_1^X \longrightarrow P_0^X \longrightarrow X \longrightarrow 0$$

be its minimal projective presentation in $\mathsf{mod}\Lambda$. Then we define the *g-vector* of X as the element of the Grothendieck group given by

$$g(X) := [P_0^X] - [P_1^X] = \sum_{i=1}^{n} g_i(X) e_i \Lambda.$$

Let (X, P) be a support τ-tilting pair for Λ with $X = \bigoplus_{i=1}^{\ell} X_i$ and $P = \bigoplus_{i=\ell+1}^{n} P_i$, where X_i and P_i are indecomposable. Then define $g(X_i)$ as above and $g(P_i) := -[P_i]$. We define the *g-matrix* of (X, P) by

$$g(X, P) := (g(X_1), \cdots, g(X_\ell), g(P_{\ell+1}), \cdots, g(P_n)) \in \mathrm{GL}(\mathbb{Z}^n).$$

Note that it forms a basis of $K_0(\mathsf{proj}\Lambda)$ [AIR, Theorem 5.1].

Moreover, define its *cone* by

$$C(X, P) := \{a_1 g(X_1) + \cdots + a_\ell g(X_\ell) + a_{\ell+1} g(P_{\ell+1}) + \cdots + a_n g(P_n) \mid a_i \in \mathbb{R}_{>0}\}.$$

Now we give an example.

Example 22. Let $\Lambda := KQ$ be the path algebra given by the following quiver.

$$Q := (1 \longleftarrow 2).$$

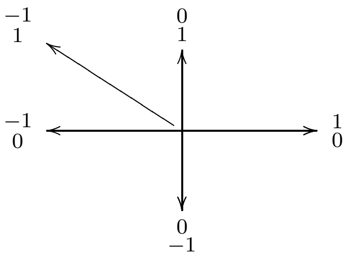
As we have seen before, we have support τ-tilting pairs as follows

$$(e_1\Lambda \oplus e_2\Lambda, 0),\ (S_2 \oplus e_2\Lambda, 0),\ (e_1\Lambda, e_2\Lambda),\ (S_2, e_1\Lambda) \text{ and } (0, e_1\Lambda \oplus e_2\Lambda).$$

Then we have their g-matrices as follows

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and their cones can be described as follows.



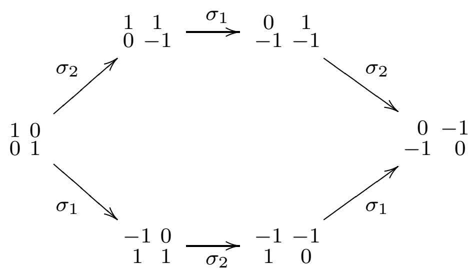
It is quite interesting to investigate behavior of cones of support τ-tilting modules for given algebras (cf. cones of tilting modules [H]).

At the end of this note, we give a description of cones of preprojective algebras. Let Q be a Dynkin quiver with vertices $Q_0 = \{1, \ldots, n\}$ and Λ the preprojective algebra of Q. Then we have the following result.

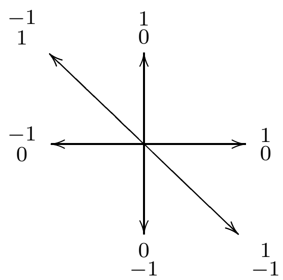
Theorem 23. *The set of g-matrices of support τ-tilting Λ-modules coincides with the subgroup $\langle \sigma_1, \ldots, \sigma_n \rangle$ of $GL(\mathbb{Z}^n)$ generated by σ_i for all $i \in Q_0$, where σ_i is the contragradient of the geometric representation* [BB]. *In particular, cones of basic support τ-tilting Λ-modules give chambers of the associated root system of Q.*

Thus, cones of preprojective algebras are completely determined by simple calculations.

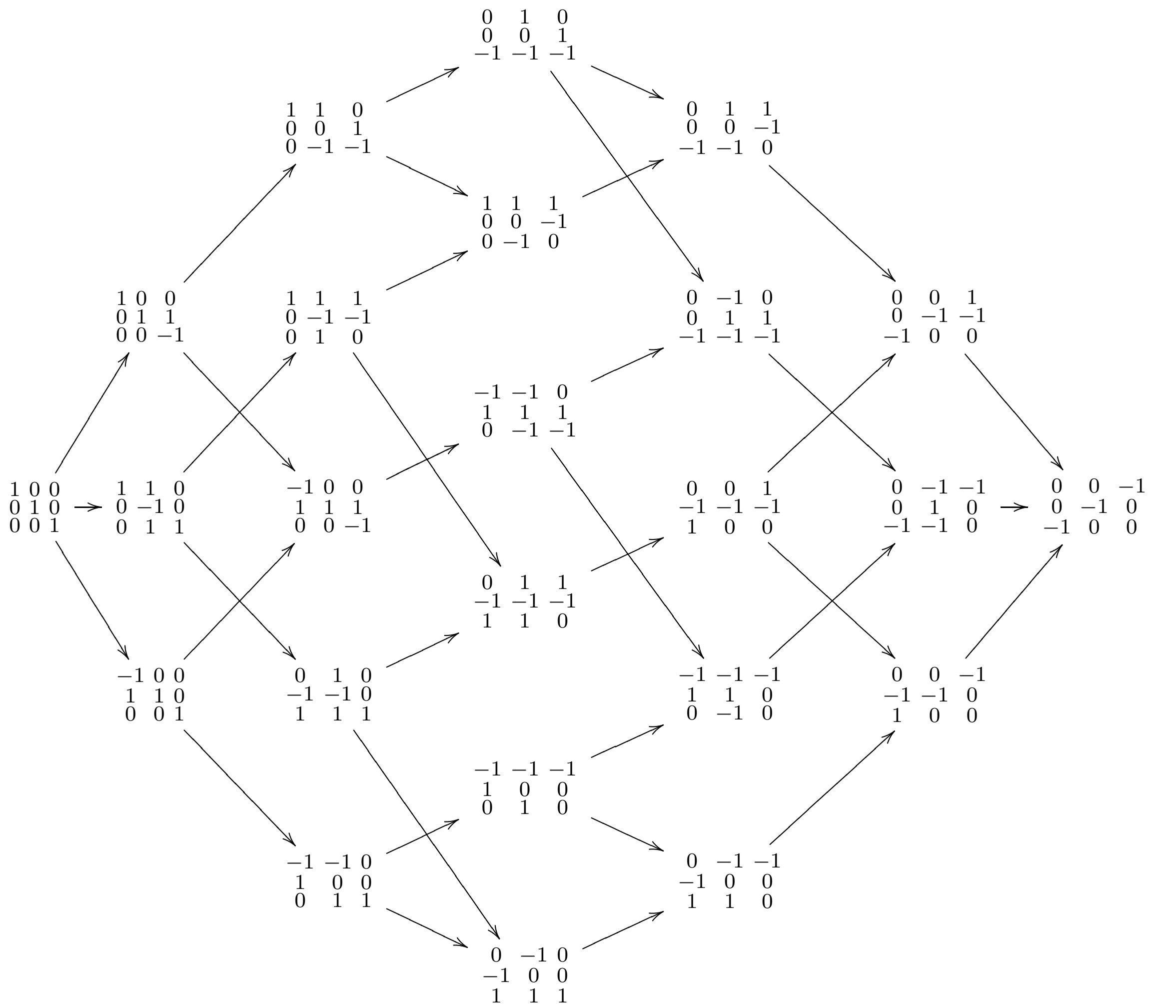
Example 24. (a) Let Λ be the preprojective algebra of type A_2. In this case, the g-matrices of Example 13 (a) are given as follows, where $\sigma_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.



Hence, their cones are given as follows.



(b) Let Λ be the preprojective algebra of type A_3. In this case, the g-matrices of Example 13 (b) are given as follows.



5. Further connections

In this section, we explain connections with other works. Here, let $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ be the bounded derived category of $\mathsf{mod}\Lambda$ and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ the bounded homotopy category of $\mathsf{proj}\Lambda$. Then we have the following bijections

Theorem 25. *Let Q be a Dynkin quiver with vertices $Q_0 = \{1, \ldots, n\}$ and Λ the preprojective algebra of Q. There are bijections between the following objects.*

(a) *The elements of the Weyl group W_Q.*
(b) *The set $\langle I_1, \ldots, I_n\rangle$.*
(c) *The set of isomorphism classes of basic support τ-tilting Λ-modules.*
(d) *The set of isomorphism classes of basic support τ-tilting Λ^{op}-modules.*
(e) *The set of torsion classes in $\mathsf{mod}\Lambda$.*
(f) *The set of torsion-free classes in $\mathsf{mod}\Lambda$.*
(g) *The set of isomorphism classes of basic two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.*
(h) *The set of intermediate bounded co-t-structures in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ with respect to the standard co-t-structure.*

(i) *The set of intermediate bounded t-structures in* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ *with length heart with respect to the standard t-structure.*
(j) *The set of isomorphism classes of two-term simple-minded collections in* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$.
(k) *The set of quotient closed subcategories in* $\mathsf{mod}KQ$.
(l) *The set of subclosed subcategories in* $\mathsf{mod}KQ$.

We have given bijections between (a), (b), (c) and (d) in the previous section. Bijections between (g), (h), (i) and (j) are the restriction of [KY] and it is given in [BY, Corollary 4.3] (it is stated for Jacobian algebras, but the statement holds for any finite dimensional algebra). Bijections between (a), (k) and (l) are given by [ORT] (note that a bijection (a) and (k) holds for any acyclic quiver with a slight modification, see [ORT]).

A bijection between (c) and (g) is shown by [AIR, Theorem 3.2] for any finite dimensional algebra.

Bijections between (a), (e) and (f) follow from the next statement, which provides complete descriptions of torsion classes and torsion-free classes in $\mathsf{mod}\Lambda$.

Proposition 26. (i) *For any torsion class* $\mathcal{T}$ *in* $\mathsf{mod}\Lambda$, *there exists* $w \in W_Q$ *such that* $\mathcal{T} = \mathsf{Fac}I_w$.
(ii) *For any torsion-free class* $\mathcal{F}$ *in* $\mathsf{mod}\Lambda$, *there exists* $w \in W_Q$ *such that* $\mathcal{F} = \mathsf{Sub}\Lambda/I_w$.

Remark 27. It is shown that objects $\mathsf{Fac}I_w$ and $\mathsf{Sub}\Lambda/I_w$ have several nice properties. For example, $\mathsf{Fac}I_w$ and $\mathsf{Sub}\Lambda/I_w$ are *Frobenius* categories and, moreover, *stable 2-CY* categories which have cluster-tilting objects. They also play important roles in the study of cluster algebra structures for a coordinate ring of the unipotent cell associated with w (see [BIRS, GLS]).

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Graduate School of Mathematics
Nagoya University
Frocho, Chikusaku, Nagoya 464-8602 Japan
E-mail address: yuya.mizuno@math.nagoya-u.ac.jp