ONE POINT EXTENSION OF A QUIVER ALGEBRA DEFINED BY TWO CYCLES AND A QUANTUM-LIKE RELATION

DAIKI OBARA

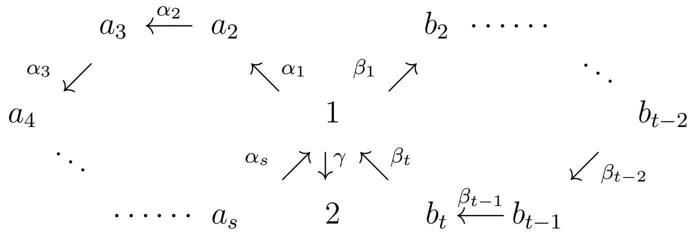
Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Tokyo University of Science, Japan, 12–14 October 2013.

In this paper, we consider a one point extension algebra B of a quiver algebra A_q over a field k defined by two cycles and a quantum-like relation depending on a nonzero element q in k. We determine the Hochschild cohomology ring of B modulo nilpotence and show that if q is a root of unity then B negates Snashall-Solberg's conjecture.

Introduction

Let A be an indecomposable finite dimensional algebra over a field k. We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

Let q be a non-zero element in k and s, t integers with $s, t \geq 1$. Let Γ be the quiver with $s + t$ vertices and $s + t + 1$ arrows as follows:



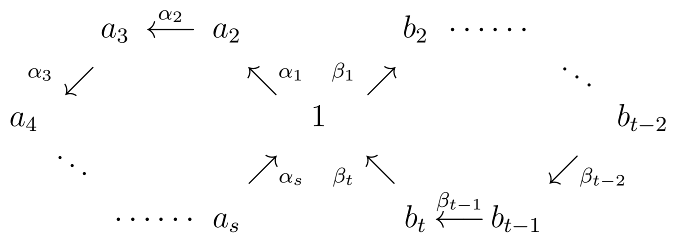
and $I_{q,v,u}$ the ideal of $k\Gamma$ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}, \gamma X^{sv+u}$$

for $a, b \geq 2$, $0 \leq v \leq a - 1$, $0 \leq u \leq s - 1$ and $(v, u) \neq (0, 0)$ where we set $X:= \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y:= \beta_1 + \beta_2 + \cdots + \beta_t$. Paths in Γ are written from right to left. In this paper, we consider the quiver algebra $B = k\Gamma/I_{q,v,u}$. We denote the trivial path at the vertex $a(i)$ and at the vertex $b(j)$ by $e_{a(i)}$ and by $e_{b(j)}$ respectively. We regard the numbers i in the subscripts of $e_{a(i)}$ modulo s and j in the subscripts of $e_{b(j)}$ modulo t.

In the case $s = t = 1$ and $a = b = 2$, B is a Koszul algebra. In this case, the Hochschild cohomology ring of B modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}$ is not finitely generated as a k-algebra (see [4]).

This algebra B is a one point extension of a quiver algebra $A_q = kQ/I_q$ where Q the following quiver:



and I_q is the ideal of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}.$$

This algebra A_q is the quiver algebra defined by two cycles and a quantum-like relation. In [2] and [3], we described the minimal projective bimodule resolution of A_q and showed that if q is a root of unity then $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables and that if q is not a root of unity then $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the field k.

The Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ was used in [5] to define a support variety for any finitely generated module over a finite dimensional algebra A. In [5], Snashall and Solberg conjectured that if A is an artin k-algebra then $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra.

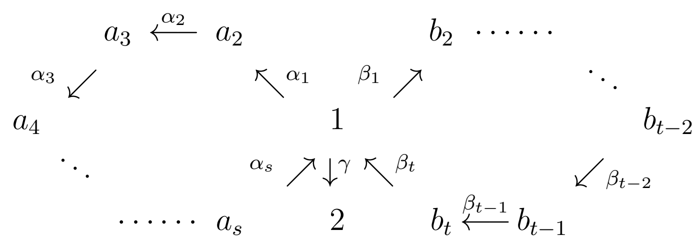
In this paper, we determine the Hochschild cohomology ring of B modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}$ and show that if q is a root of unity then $\mathrm{HH}^*(B)/\mathcal{N}$ is not finitely generated as an algebra. So B negates Snashall-Solberg's conjecture.

The content of the paper is organized as follows. In Section 1, we determine the minimal projective bimodule resolution of B. In Section 2 we determine the ring structure of $\mathrm{HH}^*(B)/\mathcal{N}$.

1. A projective bimodule resolution of B

In this section, we determine a minimal projective bimodule resolution of B.

Let k be a field, $q \in k$ a nonzero element and s, t integers with $s, t \geq 1$. Let $B = k\Gamma/I_{q,v,u}$ where Γ is the following quiver:



and $I_{q,v,u}$ is the ideal of $k\Gamma$ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}, \gamma X^{sv+u}$$

for $a,b \geq 2, 0 \leq v \leq a-1$, $0 \leq u \leq s-1$ and $(v,u) \neq (0,0)$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$.

Then we note that the following elements in B form a k-basis of B.

$$
\begin{array}{ll}
X^{sl+l'}e_{a(i)} & \text{for } 2 \leq i \leq s, 0 \leq l \leq a-1, 0 \leq l' \leq s-1,\\
Y^{tl+l'}e_{b(j)} & \text{for } 1 \leq j \leq t, 0 \leq l \leq b-1, 0 \leq l' \leq t-1,\\
X^{si+l}Y^{tj+l'} & \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 0 \leq l' \leq t-1,\\
X^{si}Y^{tj+l'} & \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 0 \leq l' \leq t-1,\\
X^{si}Y^{tj}X^{l} & \text{for } 0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l \leq s-1,\\
Y^{l'}X^{si}Y^{tj} & \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l' \leq t-1,\\
Y^{l'}X^{si}Y^{tj}X^{l} & \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 1 \leq l' \leq t-1,\\
X^{si+l}Y^{tj}X^{l'} & \text{for } 0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l, l' \leq s-1,\\
Y^{l}X^{si}Y^{tj+l'} & \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l, l' \leq t-1.\\
e_2, & \\
\gamma X^{sl+l'} & \text{for } 0 \leq l \leq v-1, 0 \leq l' \leq s-1,\\
\gamma X^{sv+l'} & \text{for } 0 \leq l' \leq u-1 \text{ if } u \neq 0,\\
\gamma X^{sl}Y^{tl'+l''} & \text{for } \begin{cases} 0 \leq l \leq v, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u \neq 0,\\ 0 \leq l \leq v-1, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u = 0,\end{cases}\\
\gamma X^{sl}Y^{tl'}X^{l''} & \text{for } \begin{cases} 0 \leq l \leq v, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u \neq 0,\\ 0 \leq l \leq v-1, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u = 0.\end{cases}
\end{array}
$$

So we have the dimension of the algebra B as follows:

$$
\dim_k B = \begin{cases} ab(s+t-1)^2 + (v+1)b(s+t-1) - s + u + 1 & \text{if } u \neq 0,\\ ab(s+t-1)^2 + vb(s+t-1) + 1 & \text{if } u = 0.\end{cases}
$$

Let M be the right A_q-module with the following basis elements:

$$
\begin{array}{ll}
X^{sl+l'} & \text{for } 0 \leq l \leq v-1, 0 \leq l' \leq s-1,\\
X^{sv+l'} & \text{for } 0 \leq l' \leq u-1 \text{ if } u \neq 0,\\
X^{sl}Y^{tl'+l''} & \text{for } \begin{cases} 0 \leq l \leq v, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u \neq 0,\\ 0 \leq l \leq v-1, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u = 0,\end{cases}\\
X^{sl}Y^{tl'}X^{l''} & \text{for } \begin{cases} 0 \leq l \leq v, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u \neq 0,\\ 0 \leq l \leq v-1, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u = 0.\end{cases}
\end{array}
$$

Then we regard the algebra B as the one point extension $\begin{pmatrix} k & M \\ 0 & A_q \end{pmatrix}$ of A_q by the A_q-module M. Let $\mathcal{F}\colon \operatorname{Mod} A_q^e \to \operatorname{Mod} B^e$ and $\mathcal{G}\colon \operatorname{Mod} A_q \to \operatorname{Mod} B^e$ be the natural functors given by $\mathcal{F}(Q) = \begin{pmatrix} 0 & M \\ 0 & A_q \end{pmatrix} \otimes_{A_q} Q$ and $\mathcal{G}(L) = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$. We give an explicit projective bimodule resolution of a one point extension algebra by using the following Theorem in [1].

Theorem 1. *Let* $\cdots \to Q_n \xrightarrow{\delta_n} \cdots \to Q_1 \xrightarrow{\delta_1} Q_0 \xrightarrow{\delta_0} A \to 0$ *be an* A^e*-projective resolution of* A *and* $\cdots \to L_n \xrightarrow{r_n} \cdots \to L_2 \xrightarrow{r_2} L_1 \xrightarrow{r_1} L_0 \xrightarrow{r_0} M \to 0$ *a right* A*-projective resolution of* M*. Then we have a* B^e*-projective resolution of* $B = \begin{pmatrix} k & M \\ 0 & A \end{pmatrix}$:

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0,$$

where $P_0 = \mathcal{F}(Q_0) \oplus (Be' \otimes e'B)$, $d_0 = (\mathcal{F}(\delta_0), id_{Be' \otimes e'B})$, $P_n = \mathcal{F}(Q_n) \oplus \mathcal{G}(L_{n-1})$ *and* $d_n = \begin{pmatrix} \mathcal{F}(\delta_n) & \sigma_n \\ 0 & -\mathcal{G}(r_{n-1}) \end{pmatrix}$ *for* $n \geq 1$, *where* $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in B$ *is new vertex and* σ_n: $\mathcal{G}(L_{n-1}) \to \mathcal{F}(Q_{n-1})$ *is a* B^e*-homomorphism such that* $\mathcal{F}(\delta_n) \circ \sigma_{n+1} = \sigma_n \circ \mathcal{G}(\eta_n)$, *where* σ_0 *is the natural monomorphism.*

Remark 2. The following sequence is a minimal projective resolution of M.

$$\cdots \to L_{2n} \xrightarrow{r_{2n}} L_{2n-1} \xrightarrow{r_{2n-1}} \cdots \to L_1 \xrightarrow{r_1} L_0 \xrightarrow{r_0} M \to 0$$

where $L_{2n} = e_1 A_q$, $L_{2n+1} = e_{a(s+1-u)} A_q$ for $n \geq 0$, r_0 is a natural epimorphism and for $n \geq 1$,

$$r_{2n-1}(e_{a(s+1-u)}) = X^{sv+u} e_{a(s+1-u)},$$
$$r_{2n}(e_1) = X^{s(a-v-1)+s-u} e_1.$$

In [2], we gave the minimal projective bimodule resolution of A_q. Then, by Theorem 1 we have the minimal projective bimodule resolution of B.

For $n \geq 0$, we define left B^e-modules, equivalently B-bimodules

$$P_0 = Be_1 \otimes e_1 B \oplus \coprod_{i=2}^{s} Be_{a(i)} \otimes e_{a(i)} B \oplus \coprod_{j=2}^{t} Be_{b(j)} \otimes e_{b(j)} B \oplus Be_2 \otimes e_2 B,$$

$$P_{2n} = \coprod_{l=0}^{2n} Be_1 \otimes e_1 B \oplus \coprod_{i=2}^{s} Be_{a(i)} \otimes e_{a(i)} B \oplus \coprod_{j=2}^{t} Be_{b(j)} \otimes e_{b(j)} B \oplus Be_2 \otimes e_{a(s+1-u)} B,$$

$$P_{2n+1} = \coprod_{l=1}^{2n} Be_1 \otimes e_1 B \oplus \coprod_{i=1}^{s} Be_{a(i+1)} \otimes e_{a(i)} B \oplus \coprod_{j=1}^{t} Be_{b(j+1)} \otimes e_{b(j)} B \oplus Be_2 \otimes e_1 B.$$

The generators $e_1 \otimes e_1$, $e_{a(i)} \otimes e_{a(i)}$, $e_{b(j)} \otimes e_{b(j)}$, $e_2 \otimes e_{a(s+1-u)}$ and $e_2 \otimes e_2$ of P_{2n} are labeled ε_l^{2n} for $0 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n}$ for $2 \leq i \leq s$, and $\varepsilon_{b(j)}^{2n}$ for $2 \leq j \leq t$, $\varepsilon^{2n\prime}$ and $\varepsilon^{0\prime}$ respectively. Similarly, we denote the generators $e_1 \otimes e_1$, $e_{a(i+1)} \otimes e_{a(i)}$, $e_{b(j+1)} \otimes e_{b(j)}$ and $e_2 \otimes e_1$ of P_{2n+1} by ε_l^{2n+1} for $1 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n+1}$ for $1 \leq i \leq s$, $\varepsilon_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ and $\varepsilon^{2n+1\prime}$ respectively.

Then we have the B^e-homomorphisms $\sigma_n\colon \mathcal{G}(L_{n-1}) \to \mathcal{F}(Q_{n-1})$ as follows:

$$\begin{aligned}
&\sigma_0(\varepsilon^{0\prime}\gamma) = \gamma,\\
&\sigma_{2n-1}(\varepsilon^{2n-1\prime}) = \gamma\varepsilon^{2n-2}_{2n-2},\\
&\sigma_{2n}(\varepsilon^{2n\prime}) = \sum_{l=0}^{v-1}\sum_{l'=0}^{s-1}\gamma X^{sl+l'}\varepsilon^{2n-1}_{a(s-l')}X^{s(v-l)+u-1-l'} + \sum_{l'=0}^{u-1}\gamma X^{sv+l'}\varepsilon^{2n-1}_{a(s-l')}X^{u-1-l'}.
\end{aligned}$$

So we have the minimal projective bimodule resolution of B as follows.

Theorem 3. *The following sequence $\mathbb{P}$ is a minimal projective resolution of the left B^e-module B:*

$$\mathbb{P}: \cdots \to P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} P_{2n-1} \to \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0.$$

where $d_0\colon P_0 \to B$ is the multiplication map, and left B^e-homomorphisms d_{2n} and d_{2n+1} are defined by

d_1 :

$$\begin{cases}
\varepsilon^1_{b(j)} \mapsto \varepsilon^0_{b(j+1)}Y - Y\varepsilon^0_{b(j)} & \text{for } 1 \le j \le t,\\
\varepsilon^1_{a(i)} \mapsto \varepsilon^0_{a(i+1)}X - X\varepsilon^0_{a(i)} & \text{for } 1 \le i \le s,\\
\varepsilon^{1\prime} \mapsto \gamma\varepsilon^0_0 - \varepsilon^{0\prime}\gamma,
\end{cases}$$

d_{2n} :

$$\begin{cases}
\varepsilon^{2n}_0 \mapsto \sum_{l=0}^{b-1}\sum_{l'=0}^{t-1} Y^{tl+l'}\varepsilon^{2n-1}_{b(t-l')}Y^{t(b-1-l)+t-l'-1},\\
\varepsilon^{2n}_{b(j)} \mapsto \sum_{l=0}^{b-1}\sum_{l'=0}^{t-1} Y^{tl+l'}\varepsilon^{2n-1}_{b(j-1-l')}Y^{t(b-1-l)+t-l'-1} \quad \text{for } 2 \le j \le t,\\
\varepsilon^{2n}_1 \mapsto\\
qY^t\varepsilon^{2n-1}_1 - \varepsilon^{2n-1}_1 Y^t - \sum_{j=1}^{t} X^sY^{t-j}\varepsilon^{2n-1}_{b(j)}Y^{j-1} + q^{bn+1}\sum_{j=1}^{t} Y^{t-j}\varepsilon^{2n-1}_{b(j)}Y^{j-1}X^s,\\
\varepsilon^{2n}_{l''} \mapsto\\
\begin{cases}
\sum_{j=0}^{b-1} q^{alj}Y^{tj}\varepsilon^{2n-1}_{2l}Y^{t(b-1-j)} + \sum_{i=0}^{a-1} q^{ib(n-l+1)}X^{s(a-1-i)}\varepsilon^{2n-1}_{2l-1}X^{si}\\
\qquad\qquad \text{if } l'' = 2l \text{ for } 1 \le l \le n-1,\\
q^{al'+1}Y^t\varepsilon^{2n-1}_{2l'+1} - \varepsilon^{2n-1}_{2l'+1}Y^t - X^s\varepsilon^{2n-1}_{2l'} + q^{b(n-l')+1}\varepsilon^{2n-1}_{2l'}X^s\\
\qquad\qquad \text{if } l'' = 2l'+1 \text{ for } 1 \le l' \le n-2,
\end{cases}\\
\varepsilon^{2n}_{2n-1} \mapsto\\
q^{an+1}\sum_{i=1}^{s} Y^tX^{s-i}\varepsilon^{2n-1}_{a(i)}X^{i-1} - \sum_{i=1}^{s} X^{s-i}\varepsilon^{2n-1}_{a(i)}X^{i-1}Y^t - X^s\varepsilon^{2n-1}_{2n} + q\varepsilon^{2n-1}_{2n}X^s,\\
\varepsilon^{2n}_{a(i)} \mapsto \sum_{l=0}^{a-1}\sum_{l'=0}^{s-1} X^{sl+l'}\varepsilon^{2n-1}_{a(i-1-l')}X^{s(a-1-l)+s-l'-1} \quad \text{for } 2 \le i \le s,\\
\varepsilon^{2n}_{2n} \mapsto \sum_{l=0}^{a-1}\sum_{l'=0}^{s-1} X^{sl+l'}\varepsilon^{2n-1}_{a(s-l')}X^{s(a-1-l)+s-l'-1},\\
\varepsilon^{2n\prime} \mapsto \sum_{l=0}^{v-1}\sum_{l'=0}^{s-1}\gamma X^{sl+l'}\varepsilon^{2n-1}_{a(s-l')}X^{s(v-l)+u-1-l'}\\
\qquad + \sum_{l'=0}^{u-1}\gamma X^{sv+l'}\varepsilon^{2n-1}_{a(s-l')}X^{u-1-l'} - \varepsilon^{2n-1\prime}X^{sv+u},
\end{cases}$$

d_{2n+1} :

$$
\begin{cases}
\varepsilon^{2n+1}_{b(j)} \mapsto \varepsilon^{2n}_{b(j+1)}Y - Y\varepsilon^{2n}_{b(j)} \text{ for } 1 \le j \le t, \\
\varepsilon^{2n+1}_{l''} \mapsto \\
\quad \begin{cases}
-\sum_{j=0}^{b-1} q^{j(al+1)}Y^{tj}\varepsilon^{2n}_{2l+1}Y^{t(b-1-j)} - X^s\varepsilon^{2n}_{2l} + q^{b(n-l)}\varepsilon^{2n}_{2l}X^s \\
\qquad\qquad\qquad\qquad\qquad \text{if } l'' = 2l+1 \text{ for } 0 \le l \le n-1, \\
-q^{al'}Y^t\varepsilon^{2n}_{2l'} + \varepsilon^{2n}_{2l'}Y^t + \sum_{i=0}^{a-1} q^{i(b(n-l')+1)}X^{s(a-1-i)}\varepsilon^{2n}_{2l'-1}X^{si} \\
\qquad\qquad\qquad\qquad\qquad \text{if } l'' = 2l' \text{ for } 1 \le l' \le n,
\end{cases} \\
\varepsilon^{2n+1}_{a(i)} \mapsto \varepsilon^{2n}_{a(i+1)}X - X\varepsilon^{2n}_{a(i)} \text{ for } 1 \le i \le s, \\
\varepsilon^{2n+1\prime} \mapsto \gamma\varepsilon^{2n}_{2n} - \varepsilon^{2n\prime}X^{s(a-1-v)+s-u},
\end{cases}
$$

for $n \ge 1$, where in the case $n = 1$, $\varepsilon^2_{l''}$ vanish, and the image of ε^2_1 by d_2 is

$$
-\sum_{j=1}^{t} X^sY^{t-j}\varepsilon^1_{b(j)}Y^{j-1} + q\sum_{j=1}^{t} Y^{t-j}\varepsilon^1_{b(j)}Y^{j-1}X^s + q\sum_{i=1}^{s} Y^tX^{s-i}\varepsilon^1_{a(i)}X^{i-1} - \sum_{i=1}^{s} X^{s-i}\varepsilon^1_{a(i)}X^{i-1}Y^t.
$$

2. The Hochschild cohomology ring of B modulo nilpotence

In this section, we consider the case that q is a root of unity. We set $\bar{z}$ is the remainder when we divide z by r for any integer z. Then we have $0 \le \bar{z} \le r-1$. Using the projective resolution in Theorem 3, we have a basis and Yoneda product of the Hochschild cohomology ring $\mathrm{HH}^*(B)$ of B. Then we determine the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}$ as follows.

Theorem 4. *If $s, t \ge 2$ and q is an r-th root of unity then*

$$
\mathrm{HH}^*(B)/\mathcal{N} \cong
\begin{cases}
k \oplus k[x^{2r}, y^{2r}]x^{2r} & \text{if } \bar{a} \ne 0, \bar{b} \ne 0, \\
k \oplus k[x^2, y^{2r}]x^2 & \text{if } \bar{a} \ne 0, \bar{b} = 0, \\
k \oplus k[x^{2r}, y^2]x^{2r} & \text{if } \bar{a} = 0, \bar{b} \ne 0, \\
k \oplus k[x^2, y^2]x^2 & \text{if } \bar{a} = \bar{b} = 0.
\end{cases}
$$

where $x^l = e_{1,0} + \sum_{j=2}^{t} e_{b(j)}$ in $\mathrm{HH}^l(B)$ and $x^my^n = e_{1,n}$ in $\mathrm{HH}^{m+n}(B)$ for $l > 0$, $m, n > 0$.

Theorem 5. *If $s \ge 2$, $t = 1$ and q is an r-th root of unity then*

$$
\mathrm{HH}^*(B)/\mathcal{N} \cong
\begin{cases}
\begin{cases}
k \oplus k[x^r, y^{2r}]x^r & \text{if } \bar{a} \ne 0, \bar{b} \ne 0, \\
k \oplus k[x^r, y^2]x^r & \text{if } \bar{a} = 0, \bar{b} \ne 0, \\
k \oplus k[x, y^2]x & \text{if } \bar{a} = \bar{b} = 0,
\end{cases} & \text{if } \mathrm{char}\, k = 2, b = 2, r \text{ is odd}, \\
\begin{cases}
k \oplus k[x^{2r}, y^{2r}]x^{2r} & \text{if } \bar{a} \ne 0, \bar{b} \ne 0, \\
k \oplus k[x^2, y^{2r}]x^2 & \text{if } \bar{a} \ne 0, \bar{b} = 0, \\
k \oplus k[x^{2r}, y^2]x^{2r} & \text{if } \bar{a} = 0, \bar{b} \ne 0, \\
k \oplus k[x^2, y^2]x^2 & \text{if } \bar{a} = \bar{b} = 0,
\end{cases} & \text{others.}
\end{cases}
$$

where $x^my^n = e_{1,n}$ in $\mathrm{HH}^{m+n}(B)$ for $m > 0$ and $n \ge 0$.

Theorem 6. *If $s = 1$, $t \geq 2$ and q is an r-th root of unity then*

$$\mathrm{HH}^*(B)/\mathcal{N} \cong$$
$$\begin{cases} \begin{cases} k \oplus k[x^{2r}, y^r]x^{2r} & \text{if } \bar{a} \neq 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^r]x^2 & \text{if } \bar{a} \neq 0, \bar{b} = 0, \\ k \oplus k[x^2, y]x^2 & \text{if } \bar{a} = \bar{b} = 0, \end{cases} & \text{if } \operatorname{char} k = 2, a = 2, r \text{ is odd}, \\ \\ \begin{cases} k \oplus k[x^{2r}, y^{2r}]x^{2r} & \text{if } \bar{a} \neq 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^{2r}]x^2 & \text{if } \bar{a} \neq 0, \bar{b} = 0, \\ k \oplus k[x^{2r}, y^2]x^{2r} & \text{if } \bar{a} = 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^2]x^2 & \text{if } \bar{a} = \bar{b} = 0, \end{cases} & \text{others.} \end{cases}$$

where $x^l = e_{1,0} + \sum_{j=2}^{t} e_{b(j)}$ *in* $\mathrm{HH}^l(B)$ *and* $x^m y^n = e_{1,n}$ *in* $\mathrm{HH}^{m+n}(B)$ *for* $l > 0$, $m, n > 0$.

It follows from Theorem 4, 5 and 6 that if q is a root of unity then $\mathrm{HH}^*(B)/\mathcal{N}$ is not finitely generated as an algebra. So B negates Snashall-Solberg's conjecture.

References

[1] Green, E. L., Marcos, E. N., Snashall, N., *The Hochschild cohomology ring of a one point extension*, Comm. Algebra **31** (2003), 2019–2037.

[2] Obara, D., *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation*, Comm. in Algebra **40** (2012), 1724–1761.

[3] Obara, D., *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation* II, Preprint to appear in Comm. in Algebra.

[4] Snashall, N., *Support varieties and the Hochschild cohomology ring modulo nilpotence*, In Proceedings of the 41st Symposium on Ring Theory and Representation Theory, Shizuoka, Japan, Sept. 5–7, 2008; Fujita, H., Ed.; Symp. Ring Theory Represent. Theory Organ. Comm., Tsukuba, 2009, pp. 68–82.

[5] Snashall, N., Solberg, Ø, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. **88** (2004), 705–732.

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Sinjuku-ku, Tokyo 162-8601 JAPAN
E-mail address: d_obara@rs.tus.ac.jp