COMPLEMENTS AND CLOSED SUBMODULES RELATIVE TO TORSION THEORIES

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ABSTRACT. A submodule of a module M is called to be closed if it has no proper essential extensions in M. A submodule X of M is called to be a complement if it is maximal with respect to $X \cap Y = 0$, for some submodule Y of M. It is well known that closed and complement submodule are the same. A module M is called to be extending (M has condition (C_1)) if any submodule of M is essential in a summand of M. It is known that quasi-injective module is extending. In this note we generalize this by using hereditary torsion theories and state related results.

1. INTRODUCTION

Throughout this paper R is a ring with a unit element, every right R-module is unital and Mod-R is the category of right R-modules. A subfunctor of the identity functor of Mod-R is called a preradical. For preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R | \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R | \sigma(M) = 0\}$ is the class of σ-torsion free right R-modules. A preradical t is called to be idempotent(a radical) if $t(t(M)) = t(M)(t(M/t(M)) = 0)$. Let $\mathcal{C}$ be a subclass of Mod-R. A torsion theory for $\mathcal{C}$ is a pair of $(\mathcal{T},\mathcal{F})$ of classes of objects of $\mathcal{C}$ such that (i) $\text{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$. (ii) If $\text{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$. (iii) If $\text{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$. It is well known that $(\mathcal{T}_t, \mathcal{F}_t)$ is a torsion theory for an idempotent radical t. A preradical t is called to be left exact if $t(N) = N \cap t(M)$ holds for any module M and its submodule N. For a preradical σ and a module M and its submodule N, N is called to be σ-dense submodule of M if $M/N \in \mathcal{T}_\sigma$. If N is an essential and σ-dense submodule of M, then N is called to be a σ-essential submodule of M(M is a σ-essential extension of N). If N is essential in M, we denote $N \subseteq^e M$. If N is σ-essential in M, we denote $N \subseteq^{\sigma e} M$. For an idempotent radical σ a module M is called to be σ-injective if the functor $\text{Hom}_R(\_, M)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$. We denote $E(M)$ the injective hull of a module M. For an idempotent radical σ, $E_\sigma(M)$ is called the σ-injective hull of a module M, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$. Then even if σ is not left exact, $E_\sigma(M)$ is σ-injective and a σ-essential extension of M, is a maximal σ-essential extension of M and is a minimal σ-injective extension of M. If N is σ-essential in M, then it holds that $E_\sigma(N) = E_\sigma(M)$. Let B be a submodule of a module M. We call B is σ-essentially closed in M if B has no proper σ-essential extension in M.

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2. COMPLEMENT AND CLOSED SUBMODULE

First we state σ-essentially closed submodules and complement submodules relative to torsion theories. Following proposition generalize Proposition 1.4 in [2].

Proposition 1. *Let σ be a left exact radical and B be a submodule of a module M. We denote $\overline{B}/B := \sigma(M/B)$. Then the following conditions from (1) to (9) are equivalent.*

(1) *B is essentially closed in $\overline{B}$.*
(2) *B is σ-essentially closed in M.*
(3) *B is a complement of a submodule in $\overline{B}$.*
(4) *If X is a complement of B in $\overline{B}$, then B is a complement of X in $\overline{B}$.*
(5) *It holds that $B = E_\sigma(B) \cap M$.*
(6) *If $B \subseteq X \subseteq^e \overline{B}$, then $X/B \subseteq^e \overline{B}/B$.*
(7) *It holds that $B = E(B) \cap \overline{B}$.*
(8) *There exists submodules M_1 and K of M such that $K \subseteq M_1$, $M/M_1 \in \mathcal{F}_\sigma$ and B is a complement of K in M_1.*
(9) *If $B \subseteq X \subseteq^{\sigma e} M$, then $X/B \subseteq^{\sigma e} M/B$.*

Proof. (2)→(1): Let B be σ-essentially closed in M. Let H a module such that $B \subseteq^e H \subseteq \overline{B}$. Since $H/B \subseteq \overline{B}/B = \sigma(M/B) \in \mathcal{T}_\sigma$, $H/B \in \mathcal{T}_\sigma$. Thus $B \subseteq^{\sigma e} H \subseteq M$, and so $H = B$ by (2).

(1)→(2): Let B be essentially closed in $\overline{B}$. Let N be a module such that $B \subseteq^{\sigma e} N \subseteq M$, and so $B \subseteq^e N$ and $N/B \in \mathcal{T}_\sigma$. Then $N/B \subseteq \sigma(M/B) = \overline{B}/B$. Thus it holds that $B \subseteq^e N \subseteq \overline{B}$. By (1), $B = N$.

(2)→(6): Suppose that B is σ-essentially closed in M and X an essential submodule of $\overline{B}$ containing B. Let Y/B be a submodule of $\overline{B}/B$ such that $X/B \cap Y/B = \overline{0}$. Then $X \cap Y = B$. Since X is essential in $\overline{B}$, $B = Y \cap X$ is essential in $Y \cap \overline{B} = Y$. Since $Y/B = Y/(Y \cap X) \cong (Y + X)/X \subseteq \overline{B}/X \twoheadleftarrow \overline{B}/B \in \mathcal{T}_\sigma$, B is σ-essential in Y. As B is σ-essentially closed in M, it follows that $Y = B$. Thus X/B is essential in $\overline{B}/B$.

(6)→(4): Let X be a complement of B in $\overline{B}$. Let B' be a complement of X in $\overline{B}$ containing B. Then $(X \oplus B) \cap B' = (X \cap B') \oplus B = B$. Thus $((X \oplus B)/B) \cap (B'/B) = \overline{0}$. Since $X \oplus B$ is essential in $\overline{B}$, it holds that $(X \oplus B)/B$ is essential in $\overline{B}/B$ by (6) . Since $((X \oplus B)/B) \cap (B'/B) = \overline{0}$, then $B' = B$, as desired.

(4)→(3): Since there exists a complement of B in $\overline{B}$, it is obvious.

(3)→(2): Let B be a complement of a submodule K of $\overline{B}$. Then B is essentially closed in $\overline{B}$. We show that B is σ-essentially closed in M. Let B' be a submodule of M such that B' is a σ-essential extension of B. Then $B \cap \overline{B} = B$ is essential in $B' \cap \overline{B}$. Since B is essentially closed in $\overline{B}$, $B = B' \cap \overline{B}$. Since $\mathcal{T}_\sigma \ni B'/B = B'/(B' \cap \overline{B}) \cong (B' + \overline{B})/\overline{B} \subseteq M/\overline{B} \cong (M/B)/\sigma(M/B) \in \mathcal{F}_\sigma$, it follows that $B' = B$, as desired.

(2)→(5): It is easily verified that $E_\sigma(B) \cap M$ is σ-essential extension of B in M. By (2), it follows that $E_\sigma(B) \cap M = B$.

(5)→(2): Let X be a module such that $B \subseteq X \subseteq M$ and B is σ-essential in X. Then $E_\sigma(B) = E_\sigma(X)$. By (5), $B = E_\sigma(B) \cap M$. Since $B \subseteq X \subseteq E_\sigma(X) \cap M = E_\sigma(B) \cap M = B$, it follows that $X = B$, as desired.

(1)→(7): Since $E(B) \cap \overline{B}$ is essential extension of B in $\overline{B}$, it holds that $B = E(B) \cap \overline{B}$.

(7)→(1): Let X be a module such that $B \subseteq X \subseteq \overline{B}$ such that X is σ-essential extension of B. Then it follows that $E(X) = E(B)$. Since $B \subseteq X \subseteq E(X) \cap \overline{B} = E(B) \cap \overline{B} = B$, it concludes that $B = X$.

(2)→(8): Let B be σ-essentially closed in M. Then $M/\overline{B} \in \mathcal{F}_\sigma$. We take a complement K of B in $\overline{B}$. Then $B \oplus K$ is essential in $\overline{B}$ and $(B \oplus K)/K$ is essential in $\overline{B}/K$. We take a complement L of K containing B in $\overline{B}$. Since $(B \oplus K)/K$ is σ-essential in $\overline{B}/K$, $(B \oplus K)/K$ is σ-essential in $(L \oplus K)/K$. Thus L is σ-essential extension of B. Thus by (2) $B = L$, and so B is a complement of K in $\overline{B}$.

(8)→(2): Suppose that there exists submodules M_1 and K of M such that $K \subseteq M_1$, $M/M_1 \in \mathcal{F}_\sigma$ and B is a complement of K in M_1. Then B is essentially closed in M_1. We show that B is σ-essentially closed in M. Let B_1 be a submodule of M such that B is σ-essential in B_1. Then $B = B \cap M_1$ is essential in $B_1 \cap M_1 (\subseteq M_1)$. Since B is essentially closed in M_1, $B = B_1 \cap M_1$. Since $\mathcal{T}_\sigma \ni B_1/B = B_1/(B_1 \cap M_1) \cong (B_1 + M_1)/M_1 \subseteq M/M_1 \in \mathcal{F}_\sigma$, it follows that $B_1 = B$.

(2)→(9): Suppose that B is σ-essentially closed in M. Let X be a submodule of M such that $B \subseteq X \subseteq^{\sigma e} M$. Let Q be a submodule of M containing B such that $(X/B) \cap (Q/B) = 0$. Then $B = Q \cap X \subseteq^{e} Q \cap M = Q$. Since $Q/B = Q/(Q \cap X) \cong (Q + X)/X \subseteq M/X \in \mathcal{T}_\sigma$, it holds that $B \subseteq^{\sigma e} Q \subseteq M$. Since B is σ-essentially closed in M, $B = Q$, and so $(Q/B) = 0$. Thus X/B is σ-essential in M/B.

(9)→(2): Suppose that $B \subseteq^{\sigma e} X \subseteq M$. Let B' be a complement of B in M. Then $B \oplus B' \subseteq^{\sigma e} M$ and hence by (9) $(B \oplus B')/B \subseteq^{\sigma e} M/B$. Since $B \cap (B' \cap X) = 0$, $B' \cap X = 0$. Since $((B \oplus B')/B) \cap (X/B) = [(B \oplus B') \cap X]/B = [B \oplus (B' \cap X)]/B = 0$, $(X/B) = 0$, as desired. □

3. σ-QUASI-INJECTIVE MODULE

We call A σ-M-injective if $\mathrm{Hom}_R(-, A)$ preserves the exactness for any exact sequence $0 \to N \to M \to M/N \to 0$, where $M/N \in \mathcal{T}_\sigma$. The following proposition is a generalization of Theorem 15 in [1].

Proposition 2. *Let σ be a left exact radical. Then A is σ-M-injective if and only if $f(M) \subseteq A$ for any $f \in Hom_R(E_\sigma(M), E_\sigma(A))$.*

Proof. (←): Let σ be an idempotent radical and N be a submodule of M such that $M/N \in \mathcal{T}_\sigma$. Since $E_\sigma(M)/M \in \mathcal{T}_\sigma$ and $\mathcal{T}_\sigma$ is closed under taking extensions, it follows that $E_\sigma(M)/N \in \mathcal{T}_\sigma$. Consider the following diagram.

$$\begin{array}{ccccccc} 0 \to & N & \to & E_\sigma(M) & \to E_\sigma(M)/N \to 0 \\ & \downarrow_f & & \downarrow_g & \\ 0 \to & A & \longrightarrow & E_\sigma(A) & \end{array}$$

For any $f \in \mathrm{Hom}_R(N, A)$, f is extended to $g \in \mathrm{Hom}_R(E_\sigma(M), E_\sigma(A))$. By the assumption it follows that $g(M) \subseteq A$, and so f is extended to $g|_M \in \mathrm{Hom}_R(M, A)$, as desired.

(→): Let σ be a left exact radical and $f \in \mathrm{Hom}_R(E_\sigma(M), E_\sigma(A))$. Then $f|_{M \cap f^{-1}A} \in \mathrm{Hom}_R(M \cap f^{-1}(A), A)$. Since $M/(M \cap f^{-1}(A)) \simeq (M + f^{-1}(A))/f^{-1}(A) \simeq (f(M) + A)/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$, $M/(M \cap f^{-1}(A)) \in \mathcal{T}_\sigma$. Consider the following diagram.

$$\begin{array}{ccccc} 0 \to & M \cap f^{-1}(A) \to M \to M/(M \cap f^{-1}(A)) \to 0 \\ & {}_f\downarrow \quad \swarrow_g & \\ & A & \end{array}$$

Thus by the assumption $f|_{M\cap f^{-1}(A)}$ is extended to $g \in \mathrm{Hom}_R(M, A)$, and so $(g-f)(M\cap f^{-1}(A)) = 0$. Hence we obtain $ker(g - f) \supseteq M \cap f^{-1}(A)$. If $x \in (g - f)^{-1}(A)$, then there exists an $a \in A$ such that $g(x) - f(x) = a$, and then $f(x) = g(x) - a \in A$ and so $x \in f^{-1}(A)$. It follows that $(g - f)^{-1}(A) \subseteq f^{-1}(A)$, and so $M \cap (g - f)^{-1}(A) \subseteq M \cap f^{-1}(A) \subseteq \ker(g - f)$. If $a = (g - f)(m) \in (g - f)M \cap A$ for $a \in A$ and $m \in M$, then $m \in (g - f)^{-1}a \subseteq M \cap (g - f)^{-1}A \subseteq \ker(g - f)$, and so $0 = (g - f)(m) = a$. Thus it follows that $(g - f)M \cap A = 0$. Since A is essential in $E_\sigma(A)$, $(g - f)M = 0$, and so we obtain that $f(M) = g(M) \subseteq A$, as desired. □

We obtain the following corollary as a torsion theoretic generalization of the Johnson Wong theorem [4] by putting $M = A$ in Proposition 2. We call a module A σ-quasi-injective if A is σ-A-injective.

Corollary 3. *Let σ be a left exact radical. Then A is σ-quasi-injective if and only if $f(A) \subseteq A$ for any $f \in Hom_R(E_\sigma(A), E_\sigma(A))$.*

The following lemma generalizes Proposition 2.3 in [3].

Lemma 4. *If A is σ-quasi-injective and $E_\sigma(A) = M \oplus N$, then $A = (M \cap A) \oplus (N \cap A)$.*

Proof. Let $p_M(p_N)$ be a canonical projection from $E_\sigma(A)$ to $M(N)$ respectively. Then by Corollay 3, it follows that $p_M(A) \subseteq A$ and $p_N(A) \subseteq A$. If $A \ni a = m + n \in M + N$ for $m \in M$ and $n \in N$, then $A \ni p_M(a) = p_M(m + n) = m \in M$, and so $m \in A \cap M$, and it is similarly proved that $n \in A \cap N$. Thus $A \subseteq (M \cap A) \oplus (N \cap A)$, as desired. □

4. (σ-C_i) CONDITIONS

Next we consider (C_i) conditions relative to torsion theories. For (C_i) conditions, see [5]. We call a module M σ-quasi-injective if for any σ-dense submodule N of M, $\mathrm{Hom}_R(\_, M)$ preserves the exactness of a short exact sequence $0 \to N \to M \to M/N \to 0$. The following proposition generalize Proposition 2.1 in [5]. We call a module M has (σ-C_1) if every σ-dense submodule of M is essential in a summand of M. We call a module M has (σ-C_2) if a σ-dense submodule A of M is isomorphic to a summand A_1 of M, then A is a summand of M.

From now on we assume that σ is a left exact radical.

Proposition 5. *Any σ-quasi-injective module M has (σ-C_1) and (σ-C_2).*

Proof. (σ-C_1): Let N be a σ-dense submodule of a σ-quasi-injective module M. Consider the exact sequence $0 \to M/N \to E_\sigma(M)/N \to E_\sigma(M)/M \to 0$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, it follows that $E_\sigma(M)/N \in \mathcal{T}_\sigma$. Since $\mathcal{T}_\sigma$ is closed under taking factor modules, it holds that $E_\sigma(M)/E_\sigma(N) \in \mathcal{T}_\sigma$. As $E_\sigma(N)$ is σ-injective, there exists a submodule E of $E_\sigma(M)$ such that $E_\sigma(M) = E_\sigma(N) \oplus E$. Since M is σ-quasi-injective, it follows that $M = (M \cap E_\sigma(N)) \oplus (E \cap M)$ by Lemma 4. Thus N is σ-essential in $M \cap E_\sigma(N)$ which is a summand of M, as desired. (σ-C_2): Since M is σ-quasi-injective,

M is σ-M-injective. As A_1 is a direct summand of M, A_1 is σ-M-injective. Consider the following exact sequence.

$$\begin{array}{l} 0 \to A \overset{g}{\to} M \to M/A \to 0 \text{ (with } M/A \in \mathcal{T}_\sigma) \\ \quad\ \downarrow_h \quad \downarrow_f \\ \quad\ A_1 \subseteq_\oplus M \end{array}$$

, where h is isomorphism from A to A_1 and f is a homomorphism from M to A_1 such that $fg = h$. It is easily verified that A is a summand of M. □

We call a module M has (σ-C_3) if M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$ and $M/(M_1 \oplus M_2) \in \mathcal{T}_\sigma$, then $M_1 \oplus M_2$ is a summand of M. We call a module M has (σ-C_3') if M_1 and M_2 are summands of M such that $M_1, M/M_2 \in \mathcal{T}_\sigma$ and $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M. It is easily verified that (σ-C_3)$\Rightarrow$(σ-C_3'). The following proposition generalize Proposition 2.2 in [5].

Proposition 6. *If a module M has (σ-C_2), then M has (σ-C_3').*

Proof. Let M_1 and M_2 be summands of M such that $M_1, M/M_2 \in \mathcal{T}_\sigma$ and $M_1 \cap M_2 = 0$. Since M_1 is a summand of M, there exists a submodule M_1^* such that $M = M_1 \oplus M_1^*$. Let π be a projection $M = M_1 \oplus M_1^* \to M_1^*$. By modular law, $M_1 \oplus M_2 = M \cap (M_1 \oplus M_2) = (M_1 \oplus M_1^*) \cap (M_1 \oplus M_2) = M_1 \oplus (M_1^* \cap (M_1 \oplus M_2))$. Thus $\pi(M_2) = \pi(M_1 \oplus M_2) = \pi(M_1 \oplus (M_1^* \cap (M_1 \oplus M_2))) = M_1^* \cap (M_1 \oplus M_2)$. Thus $M_1 \oplus M_2 = M_1 \oplus \pi(M_2)$ and $\pi(M_2) \subseteq M_1^*$. Then $\ker \pi|_{M_2} = \ker \pi \cap M_2 = M_1 \cap M_2 = 0$, $\pi|_{M_2} : M_2 \twoheadrightarrow \pi(M_2)(\subseteq M)$ is an isomorphism. Since $M_1^*/\pi(M_2) \simeq M/M_2 \in \mathcal{T}_\sigma$ and $M/M_1^* \simeq M_1 \in \mathcal{T}_\sigma$, the middle term of $0 \to M_1^*/\pi(M_2) \to M/\pi(M_2) \to M/M_1^* \to 0$ is in $\mathcal{T}_\sigma$. Thus $\pi(M_2)$ is σ-dense submodule of M. Thus we get $\pi(M_2) \subseteq^\oplus M$ by (σ-C_2). Thus there exists a module X such that $M = X \oplus \pi(M_2)$. By modular law, $M_1^* = (X \cap M_1^*) \oplus \pi(M_2)$. Thus $M = M_1 \oplus M_1^* = M_1 \oplus (X \cap M_1^*) \oplus \pi(M_2) = (M_1 \oplus \pi(M_2)) \oplus (X \cap M_1^*) = M_1 \oplus M_2 \oplus (X \cap M_1^*)$, and so $M_1 \oplus M_2 \subseteq^\oplus M$. □

We call a module of M σ-continuous if it has (σ-C_1) and (σ-C_2). We call a module M σ-quasi-continuous if it has (σ-C_1) and (σ-C_3'). We have just seen that the following implications hold: σ-injective $\Rightarrow$ σ-quasi-injective $\Rightarrow$ σ-continuous $\Rightarrow$ σ-quasi-continuous $\Rightarrow$ σ-C_1

Proposition 7. *A module M has (σ-C_1) if and only if every essentially closed σ-dense-submodule of M is a summand of M.*

Proof. $\Rightarrow$): Let N be an essentially closed σ-dense submodule of M. Since $M/N \in \mathcal{T}_\sigma$, there exists a decomposition $M = X \oplus Y$ such that $N \subseteq^{\sigma e} X \subseteq M$. As N is essentially closed in M and so $N = X$. Thus $M = N \oplus Y$.

$\Leftarrow$): Let N be a σ-dense submodule of M. Let X be a complement of N in M and Y be a complement of X in M containing N . Then Y is essentially closed σ-dense in M. By the assumption Y is a summand of M. We show that N is essential in Y. If N is not essential in Y, there exists a nonzero submodule H of Y such that $N \cap H = 0$. If $N \cap (X \oplus H) \ni n = x + h$, where $n \in N, x \in X$ and $h \in H$. Then $x = n - h \in X \cap Y = 0$. Thus $x = 0$, and so $n = h \in N \cap H = 0$. Therefore $N \cap (X \oplus H) = 0$. By construction of X, $X = X \oplus H$, and so $H = 0$. Thus N is essential in Y. Thus if $M/N \in \mathcal{T}_\sigma$, then there exists a submodule Y of M such that $N \subseteq^e Y$ and Y is a summand of M. □

Proposition 8. *For a submodule A of a module M, if A is σ-essentially closed in a summand of M, then A is σ-essentially closed in M.*

Proof. Let $M = M_1 \oplus M_2$ with A σ-essentially closed in M_1. Let π denote the projection $M_1 \oplus M_2 \twoheadrightarrow M_1$. Assume that $A \subseteq^{\sigma e} B \subseteq M$. It is easy to see that $A = \pi(A) \subseteq^{\sigma e} \pi(B) \subseteq M_1$. Since A is σ-essentially closed in M_1, $\pi(B) = A \subseteq B$, and so $(1-\pi)(B) \subseteq B$. Since $(1-\pi)(B) \cap A = 0$ and $A \subseteq^e B$, $(1-\pi)(B) = 0$. Thus $A \subseteq^{\sigma e} B = \pi(B) \subseteq M_1$. Since A is σ-essentially closed in M_1, it holds that $A = B$. □

Lemma 9. *If $M = A \oplus B$ and $A \subseteq^e K \subseteq M$, then $K = A$.*

Proof. By modular law it follows that $K = A \oplus (K \cap B)$, and so $A \cap (K \cap B) = 0$. Since A is essential in K, $K \cap B = 0$, and so $K = A \oplus (K \cap B) = A$. □

The following proposition generalize Theorem 2.8 in [5].

Proposition 10. *Consider the following conditions.*

It holds that (3) $\Leftrightarrow$ (4) $\to$ (1) $\to$ (2). *If* $\ker f \in \mathcal{T}_\sigma$ *for any idempotent* $f \in End_R(E_\sigma(M))$, *then* (2) $\to$ (3) *holds.*

(1) *M has (σ-C_1) and (σ-C_3).*
(2) *$M = X \oplus Y$ for σ-dense submodules X, Y of M such that X is a complement of Y in M and Y is a complement of X in M.*
(3) *$f(M) \subseteq M$ for any idempotent f in $End_R(E_\sigma(M))$.*
(4) *If $E_\sigma(M) = \oplus E_i$, then $M = \oplus\ (M \cap E_i)$.*

Proof. (1)$\to$(2): Let X and Y be σ-dense submodules of M such that X is a complement of Y in M and Y is a complement of X in M. Since X and Y are essentially closed in M, X and Y are direct summands of M by (σ-C_1). Then $X \oplus Y$ is σ-essential in M. By (σ-C_3), $X \oplus Y$ is a direct summand of M, and so $M = X \oplus Y \oplus Z \supseteq^e (X \oplus Y)$. Therefore it follows that $Z = 0$, and so $M = X \oplus Y$.

(2)$\to$(3): We assume that $\ker f \in \mathcal{T}_\sigma$ for any idempotent $f \in \mathrm{End}_R(E_\sigma(M))$. Let $A_1 = M \cap f(E_\sigma(M))$ and $A_2 = M \cap (1-f)(E_\sigma(M))$. Then $A_1 \cap A_2 = 0$. Since $E_\sigma(M) = f(E_\sigma(M)) \oplus \ker f$ for any idempotent f in $\mathrm{End}_R(E_\sigma(M))$ and $M/A_i \simeq (M + f(E_\sigma(M)))/f(E_\sigma(M)) \subseteq E_\sigma(M)/f(E_\sigma(M)) \simeq \ker f \in \mathcal{T}_\sigma$, $M/A_i \in \mathcal{T}_\sigma$ for $i = 1, 2$. Let B_1 be a complement of A_1 containing A_2 in M and B_2 be a complement of B_1 containing A_2 in M. Then by (2) $M = B_1 \oplus B_2$. Let π be a projection $B_1 \oplus B_2 \twoheadrightarrow B_1$. We claim that $M \cap (f-\pi)(M) = 0$. Let $x, y \in M$ such that $(f-\pi)(x) = y$. Then $f(x) = y + \pi(x) \in M$, and so $f(x) \in A_1$. Moreover $(1-f)(x) \in M$, and so $(1-f)(x) \in A_2$. Therefore $x = f(x) \oplus (1-f)(x) \in A_1 \oplus A_2 \subseteq B_1 \oplus B_2 = M$. $\pi(x) = \pi(f(x)) + \pi(1-f)(x) = f(x) + 0$, and so $y = 0$. Thus $M \cap (f-\pi)(M) = 0$. Since M is essential in $E_\sigma(M)$, $(f-\pi)(M) = 0$, and so $f(M) = \pi(M) \subseteq M$.

(3)$\to$(4): Let $E_\sigma(M) = \underset{i\in I}{\oplus} E_i$, then it is clear that $M \supseteq \underset{i\in I}{\oplus}(M \cap E_i)$. Let m be an element of $M \subseteq E_\sigma(M) = \underset{i\in I}{\oplus} E_i$. Then there exists a finite index subset F of I such that $m \in \underset{i\in F}{\oplus} E_i$. Write $E_\sigma(M) = (\underset{i\in F}{\oplus} E_i) \oplus (\underset{i\in I-F}{\oplus} E_i)$. Then there exist orthogonal idempotents $f_i \in \mathrm{End}_R(E_\sigma(M)) (i \in F)$ such that $E_i = f_i(E_\sigma(M))$. Since $f_i(M) \subseteq M$ by (3), $m = \sum_{i\in F} f_i(m) \in \underset{i\in F}{\oplus}(M \cap E_i)$. Thus $M \subseteq \underset{i\in I}{\oplus}(M \cap E_i)$, and $M = \underset{i\in I}{\oplus}(M \cap E_i)$.

(4)→(1): Let A be a σ-dense submodule of M. Consider the following exact sequence. $0 \to M/A \to E_\sigma(M)/A \to E_\sigma(M)/M \to 0$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, $E_\sigma(M)/A \in \mathcal{T}_\sigma$. As $E_\sigma(M)/A \twoheadrightarrow E_\sigma(M)/E_\sigma(A)$, $E_\sigma(M)/E_\sigma(A) \in \mathcal{T}_\sigma$. Thus $0 \to E_\sigma(A) \to E_\sigma(M) \to E_\sigma(M)/E_\sigma(A) \to 0$ splits. Then $E_\sigma(M) = E_\sigma(A) \oplus E$. By (4) $M = (M \cap E_\sigma(A)) \oplus (M \cap E)$. Since $(M \cap E_\sigma(A))/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$, A is σ-essential in $M \cap E_\sigma(A)$ which is a direct summand of M. Thus M has (σ-C_1).

Let M_1 and M_2 be direct summands of M such that $M_1 \cap M_2 = 0$ and $M/M_1, M_2 \in \mathcal{T}_\sigma$. Then $M/(M_1 \oplus M_2) \in \mathcal{T}_\sigma$. Consider the following exact sequence. $0 \to M/(M_1 \oplus M_2) \to E_\sigma(M)/(M_1 \oplus M_2) \to E_\sigma(M)/M \to 0$. Thus $E_\sigma(M)/(M_1 \oplus M_2) \in \mathcal{T}_\sigma$. Thus $E_\sigma(M)/(E_\sigma(M_1) \oplus E_\sigma(M_2)) \in \mathcal{T}_\sigma$. Thus $0 \to E_\sigma(M_1) \oplus E_\sigma(M_2) \to E_\sigma(M) \to E_\sigma(M)/(E_\sigma(M_1) \oplus E_\sigma(M_2)) \to 0$ splits. Thus there exists a submodule E of $E_\sigma(M)$ such that $E_\sigma(M) = E_\sigma(M_1) \oplus E_\sigma(M_2) \oplus E$. Then by (4) $M = (M \cap E_\sigma(M_1)) \oplus (M \cap E_\sigma(M_2)) \oplus (M \cap E)$. Since M_i is a summand of M and M_i is essential in $M \cap E_\sigma(M_i)$, $M_i = M \cap E_\sigma(M_i)$ by Lemma 9. Thus $M = M_1 \oplus M_2 \oplus (M \cap E)$, as desired. Thus M has (σ-C_3).

(4)→(3): $\mathrm{End}_R(E_\sigma(M)) \ni f = f^2$, then $E_\sigma(M) = f(E_\sigma(M)) \oplus f^{-1}(0)$. By (4) $M = (M \cap f(E_\sigma(M))) \oplus (M \cap f^{-1}(0))$. For any $m \in M$, there exists $x \in M \cap f(E_\sigma(M))$ and $y \in M \cap f^{-1}(0)$ such that $m = x + y$. Then $f(m) = f(x) + f(y) = x + 0 \in M$, and so $f(M) \subseteq M$. □

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