QUANTUM PLANES AND ITERATED ORE EXTENSIONS

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Abstract. Quantum projective planes are well studied in noncommutative algebraic geometry. However, there has never been a precise definition of a quantum affine plane. In this paper, we define a quantum affine plane, and classify quantum affine planes by using 3-iterated quadratic Ore extensions of $k$.

1. Preliminaries

Throughout this paper, we fix an algebraically closed field $k$ of characteristic 0, and we assume that all vector spaces and algebras are over $k$. In this paper, a graded algebra means a connected graded algebra finitely generated over $k$. A connected graded algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ such that $A_0 = k$. We denote by $\text{GrMod} A$ the category of graded right $A$-modules. An AS-regular algebra defined below is one of the main objects of study in noncommutative algebraic geometry.

Definition 1 ([1]). A noetherian connected graded algebra $A$ is called a $d$-dimensional AS-regular algebra if

- $\text{gl.dim} A = d < \infty$, and
- $\text{Ext}^i_A(k, A) \cong \begin{cases} k & i = d \\ 0 & i \neq d. \end{cases}$

One of the first achievements of noncommutative algebraic geometry was classifying all 3-dimensional AS-regular algebras by Artin, Tate and Van den Bergh using geometric techniques [2]. In this paper, we will use their classification only in the quadratic case.

Let $T(V)$ be the tensor algebra on $V$ over $k$ where $V$ is a finite dimensional vector space. We say that $A$ is a quadratic algebra if $A$ is a graded algebra of the form $T(V)/(I)$ where $I \subseteq V \otimes_k V$ is a subspace and $(I)$ is the two-sided ideal of $T(V)$ generated by $I$. For a quadratic algebra $A = T(V)/(I)$, we define

$\mathcal{V}(I) = \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) | f(p, q) = 0 \text{ for all } f \in I\}$.

Definition 2 ([5]). A quadratic algebra $A = T(V)/(I)$ is called geometric if there exists a geometric pair $(E, \tau)$ where $E \subseteq \mathbb{P}(V^*)$ is a closed $k$-subscheme and $\tau$ is a $k$-automorphism of $E$ such that

- $\mathcal{V}(I) = \{(p, \tau(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) | p \in E\}$, and
- $I = \{f \in V \otimes_k V | f(p, \tau(p)) = 0 \text{ for all } p \in E\}$.

Let $A = T(V)/(I)$ be a quadratic algebra. If $A$ satisfies the condition $(\text{G1})$, then $A$ determines a geometric pair $(E, \tau)$. If $A$ satisfies the condition $(\text{G2})$, then $A$ is determined by a geometric pair $(E, \tau)$, so we will write $A = \mathcal{A}(E, \tau)$. All 3-dimensional quadratic

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AS-regular algebras are geometric by [2]. Moreover, it follows that they can be classified in terms of geometric pairs \((E, \tau)\), where \(E\) is either \(\mathbb{P}^2\) or a cubic curve in \(\mathbb{P}^2\) by [2].

2. Ore extensions.

Ore extensions are defined as follows:

**Definition 3 ([4]).** Let \(R\) be an algebra, \(\sigma\) an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation (i.e., \(\delta : R \to R\) is a linear map such that \(\delta(ab) = \delta(a)b + \sigma(a)\delta(b)\) for all \(a, b \in R\)). Then \(\sigma, \delta\) uniquely determine an algebra \(S\) satisfying the following two properties:

- \(S = R[z]\) as a left \(R\)-module.
- For any \(a \in R\), \(za = \sigma(a)z + \delta(a)\).

The algebra \(S\) is denoted by \(R[z; \sigma, \delta]\) and is called the Ore extension of \(R\) associated to \(\sigma\) and \(\delta\). Then we define an \(n\)-iterated Ore extension of \(k\) by

\[
k[z_1; \sigma_1, \delta_1][z_2; \sigma_2, \delta_2] \cdots [z_n; \sigma_n, \delta_n].
\]

Iterated graded Ore extensions of \(k\) are defined below.

**Definition 4.** Let \(A\) be a graded algebra, \(\sigma\) a graded automorphism of \(A\) and \(\delta\) a graded \(\sigma\)-derivation (i.e., \(\delta : A \to A\) is a linear map of degree \(\ell\) for some \(\ell \in \mathbb{N}\) such that \(\delta(ab) = \delta(a)b + \sigma(a)\delta(b)\) for all \(a, b \in A\)). Then \(\sigma, \delta\) uniquely determine a graded algebra \(B\) satisfying the following two properties

- \(B = A[z]\) with \(\deg(z) = \ell\) as a graded left \(A\)-module.
- For any \(a \in A\), \(za = \sigma(a)z + \delta(a)\).

The graded algebra \(B\) is denoted by \(A[z; \sigma, \delta]\) and is called the graded Ore extension of \(A\) associated to \(\sigma\) and \(\delta\). Then we define an \(n\)-iterated graded Ore extension of \(k\) by

\[
k[z_1; \sigma_1, \delta_1][z_2; \sigma_2, \delta_2] \cdots [z_n; \sigma_n, \delta_n].
\]

If \(\deg(x_i) = 1\) for any \(i \in \{1, 2, \cdots, n\}\), the above algebra is a quadratic algebra. Then we call it an \(n\)-iterated quadratic Ore extension of \(k\).

It is known that \(n\)-iterated quadratic Ore extensions of \(k\) are \(n\)-dimensional quadratic AS-regular algebras. Moreover, if \(n \leq 2\), then \(n\)-dimensional AS-regular algebras are \(n\)-iterated graded Ore extensions of \(k\) by [6]. In this paper, we answer the question which \(3\)-dimensional quadratic AS-regular algebras are \(3\)-iterated quadratic Ore extensions of \(k\).

**Theorem 5.** Let \(A = A(E, \tau)\) be a \(3\)-dimensional quadratic AS-regular algebra. Then there exists a \(3\)-iterated quadratic Ore extension \(B\) such that \(\text{GrMod}A \cong \text{GrMod}B\) if and only if \(E\) is not a elliptic curve.

3. Quantum planes

**Definition 6 ([3]).** Let \(R\) be an algebra. We denote by \(\text{Mod}R\) the category of right \(R\)-modules. We define the noncommutative affine scheme \(\text{Spec}_{\text{nc}}R\) associated to \(R\) by the pair \((\text{Mod}R, R)\).

Let \(\text{Tails}A\) be the the quotient category \(\text{GrMod}A/\text{Tors}A\) where \(\text{Tors}A\) is the full subcategory of \(\text{GrMod}A\) consisting of direct limits of modules finite dimensional over \(k\), and let \(\pi\) be the canonical functor \(\text{GrMod}A \to \text{Tails}A\).
**Definition 7** ([3]). Let $A$ be a graded algebra. We define the noncommutative projective scheme $\text{Proj}_{\text{nc}} A$ associated to $A$ by the pair $(\text{Tails} A, \pi A)$.

The simplest surface in algebraic geometry is the affine plane, which is $\text{Spec} \ k[x, y]$, so the simplest noncommutative surface must be a quantum affine plane, which should be $\text{Spec}_{\text{nc}} R$, where $R$ is a noncommutative analogue of $k[x, y]$. Since a skew polynomial algebra $R = k(x, y)/(xy - \lambda yx)$ is the simplest example of a noncommutative analogue of $k[x, y]$ in noncommutative algebraic geometry, it can be regarded as a coordinate ring of a quantum affine plane. However, there has never been a precise definition of quantum affine plane. In the projective case, if $A$ is a $(d + 1)$-dimensional quadratic AS-regular algebra, then we call $\text{Proj}_{\text{nc}} A$ a $d$-dimensional quantum projective space ($q\mathbb{P}^d$). In particular, if $A$ is a 3-dimensional quadratic AS-regular algebra, then we call $\text{Proj}_{\text{nc}} A$ a quantum projective plane ($q\mathbb{P}^2$).

In algebraic geometry, the following result is well known. If $A$ is a polynomial algebra $k[x, y, z]$ and $u \in A_1$, then

$$\text{Proj} A = \text{Proj} A/(u) \cup \text{Spec} A[u^{-1}]_0.$$ 

Meanwhile, if $A$ be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element (i.e., $uA = Au$), then $\text{Proj}_{\text{nc}} A$ is a $q\mathbb{P}^2$ and $\text{Proj}_{\text{nc}} A/(u)$ is a $q\mathbb{P}^1$. Following the above facts, we define a quantum affine plane as follows.

**Definition 8.** Let $A$ be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element (i.e., $uA = Au$), then we define a quantum affine plane by

$$\text{Spec}_{\text{nc}} A[u^{-1}]_0$$

where $A[u^{-1}]_0$ is the degree zero part of the noncommutative graded localization of $A$.

**Example 9.** The algebra $A = k(x, y, z)/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$ where $0 \neq \alpha, \beta, \gamma \in k$ is a 3-dimensional quadratic AS-regular algebra. Then $A$ has a normal element $x \in A_1$, and one can show that $A[x^{-1}]_0 \cong k(s, t)/(st - \alpha \beta \gamma ts)$.

4. **Classification of Quantum Affine Planes**

In this section, we will classify quantum affine planes. We define $\text{Spec}_{\text{nc}} R$ and $\text{Spec}_{\text{nc}} R'$ are isomorphic if there exists an equivalence functor $F : \text{Mod} R \to \text{Mod} R'$ such that $F(R) \cong R'$. Since $\text{Spec}_{\text{nc}} R$ and $\text{Spec}_{\text{nc}} R'$ are isomorphic if and only if $R \cong R'$, we call the coordinate ring $A[u^{-1}]_0$ a quantum affine plane.

Although it is difficult to find normal elements of a given algebra in general, we can find normal elements of 3-dimensional quadratic AS-regular algebras by using geometric pairs.

**Lemma 10.** Let $A = \mathcal{A}(E, \tau)$ be a 3-dimensional quadratic AS-regular algebra, and let $u \in A_1$. Then $u \in A_1$ is a normal element if and only if

1. $\mathcal{V}(u) \subset E$,
2. $\tau(\mathcal{V}(u)) = \mathcal{V}(u)$. 

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In addition, the following lemma is very useful to classify quantum affine planes.

Lemma 11. Let $A$ be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element. Then there exist a 3-iterated quadratic Ore extension $B$ and a normal element $v \in B_1$ which satisfy

$$\text{GrMod} A \cong \text{GrMod} B \text{ and } A[u^{-1}]_0 \cong B[v^{-1}]_0.$$

By using the above lemmas,

Theorem 12. Every quantum affine plane is isomorphic to exactly one of the following:

$$k\langle s, t \rangle/(st - \lambda ts) =: S_\lambda \ (0 \neq \lambda \in k)$$
$$k\langle s, t \rangle/(st - \lambda ts + 1) =: T_\lambda \ (0 \neq \lambda \in k)$$
$$k\langle s, t \rangle/(ts - st + t)$$
$$k\langle s, t \rangle/(ts - st + t^2)$$
$$k\langle s, t \rangle/(ts - st + t^2 + 1)$$

where

$$S_\lambda \cong S_{\lambda'} \Leftrightarrow \lambda' = \lambda \pm 1, \ T_\lambda \cong T_{\lambda'} \Leftrightarrow \lambda' = \lambda \pm 1.$$

All of the above algebras are 2-iterated (ungraded) Ore extensions of $k$. Hence, we see that quantum affine planes have nice properties like a polynomial algebras. For example, they are noetherian domains and have finite global dimension.

References