QUANTUM PLANES AND ITERATED ORE EXTENSIONS

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Abstract. Quantum projective planes are well studied in noncommutative algebraic geometry. However, there has never been a precise definition of a quantum affine plane. In this paper, we define a quantum affine plane, and classify quantum affine planes by using 3-iterated quadratic Ore extensions of k.

1. Preliminaries

Throughout this paper, we fix an algebraically closed field k of characteristic 0, and we assume that all vector spaces and algebras are over k. In this paper, a graded algebra means a connected graded algebra finitely generated over k. A connected graded algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$ such that $A_0 = k$. We denote by $\mathrm{GrMod}A$ the category of graded right A-modules. An AS-regular algebra defined below is one of the main objects of study in noncommutative algebraic geometry.

Definition 1 ([1]). A noetherian connected graded algebra A is called a d-dimensional AS-regular algebra if

- $\mathrm{gl.dim}A = d < \infty$, and
- $\mathrm{Ext}^i_A(k, A) \cong \begin{cases} k & i = d \\ 0 & i \neq d. \end{cases}$

One of the first achievements of noncommutative algebraic geometry was classifying all 3-dimensional AS-regular algebras by Artin, Tate and Van den Bergh using geometric techniques [2]. In this paper, we will use their classification only in the quadratic case.

Let $T(V)$ be the tensor algebra on V over k where V is a finite dimensional vector space. We say that A is a quadratic algebra if A is a graded algebra of the form $T(V)/(I)$ where $I \subseteq V \otimes_k V$ is a subspace and (I) is the two-sided ideal of $T(V)$ generated by I. For a quadratic algebra $A = T(V)/(I)$, we define

$$\mathcal{V}(I) = \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) |\ f(p, q) = 0 \text{ for all } f \in I\}.$$

Definition 2 ([5]). A quadratic algebra $A = T(V)/(I)$ is called geometric if there exists a geometric pair (E, τ) where $E \subseteq \mathbb{P}(V^*)$ is a closed k-subscheme and τ is a k-automorphism of E such that

(G1) $\mathcal{V}(I) = \{(p, \tau(p) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) |\ p \in E\}$, and
(G2) $I = \{f \in V \otimes_k V |\ f(p, \tau(p)) = 0 \text{ for all } p \in E\}$.

Let $A = T(V)/(I)$ be a quadratic algebra. If A satisfies the condition **(G1)**, then A determines a geometric pair (E, τ). If A satisfies the condition **(G2)**, then A is determined by a geometric pair (E, τ), so we will write $A = \mathcal{A}(E, \tau)$. All 3-dimensional quadratic

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AS-regular algebras are geometric by [2]. Moreover, it follows that they can be classified in terms of geometric pairs (E, τ), where E is either $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$ by [2].

2. Ore extensions.

Ore extensions are defined as follows:

Definition 3 ([4])**.** Let R be an algebra, σ an automorphism of R and δ a σ-derivation (i.e., $\delta : R \to R$ is a linear map such that $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in R$). Then σ, δ uniquely determine an algebra S satisfying the following two properties;

- $S = R[z]$ as a left R-module .
- For any $a \in R$, $za = \sigma(a)z + \delta(a)$.

The algebra S is denoted by $R[z; \sigma, \delta]$ and is called the Ore extension of R associated to σ and δ. Then we define an n-iterated Ore extension of k by

$$k[z_1; \sigma_1, \delta_1][z_2; \sigma_2, \delta_2] \cdots [z_n; \sigma_n, \delta_n].$$

Iterated graded Ore extensions of k are defined bellow.

Definition 4. Let A be a graded algebra, σ a graded automorphism of A and δ a graded σ-derivation (i.e., $\delta : A \to A$ is a linear map of degree ℓ for some $\ell \in \mathbb{N}$ such that $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in A$). Then σ, δ uniquely determine a graded algebra B satisfying the following two properties

- $B = A[z]$ with $\deg(z) = \ell$ as a graded left A-module.
- For any $a \in A$, $za = \sigma(a)z + \delta(a)$.

The graded algebra B is denoted by $A[z; \sigma, \delta]$ and is called the graded Ore extension of A associated to σ and δ. Then we define an n-iterated graded Ore extension of k by

$$k[z_1; \sigma_1, \delta_1][z_2; \sigma_2, \delta_2] \cdots [z_n; \sigma_n, \delta_n].$$

If $\deg(x_i) = 1$ for any $i \in \{1, 2, \cdots, n\}$, the above algebra is a quadratic algebra. Then we call it an n-iterated quadratic Ore extension of k.

It is known that n-iterated quadratic Ore extensions of k are n-dimensional quadratic AS-regular algebras. Moreover, if $n \leqslant 2$, then n-dimensional AS-regular algebras are n-iterated graded Ore extensions of k by [6]. In this paper, we answer the question which 3-dimensional quadratic AS-regular algebras are 3-iterated quadratic Ore extensions of k.

Theorem 5. *Let $A = \mathcal{A}(E, \tau)$ be a 3-dimensional quadratic AS-regular algebra. Then there exists a 3-iterated quadratic Ore extension B such that* $\mathrm{GrMod}A \cong \mathrm{GrMod}B$ *if and only if E is not a elliptic curve.*

3. Quantum planes

Definition 6 ([3])**.** Let R be an algebra. We denote by $\mathrm{Mod}R$ the category of right R-modules. We define the noncommutative affine scheme $\mathrm{Spec}_{\mathrm{nc}}R$ associated to R by the pair $(\mathrm{Mod}R, R)$.

Let $\mathrm{Tails}A$ be the the quotient category $\mathrm{GrMod}A/\mathrm{Tors}A$ where $\mathrm{Tors}A$ is the full subcategory of $\mathrm{GrMod}A$ consisting of direct limits of modules finite dimensional over k, and let π be the canonical functor $\mathrm{GrMod}A \to \mathrm{Tails}A$.

Definition 7 ([3])**.** Let A be a graded algebra. We define the noncommutative projective scheme $\mathrm{Proj}_{\mathrm{nc}}A$ associated to A by the pair $(\mathrm{Tails}A, \pi A)$.

The simplest surface in algebraic geometry is the affine plane, which is Spec $k[x, y]$, so the simplest noncommutative surface must be a quantum affine plane, which should be $\mathrm{Spec}_{\mathrm{nc}}R$, where R is a noncommutative analogue of $k[x, y]$. Since a skew polynomial algebra $R = k\langle x, y\rangle/(xy - \lambda yx)$ is the simplest example of a noncommutative analogue of $k[x, y]$ in noncommutative algebraic geometry, it can be regarded as a coordinate ring of a quantum affine plane. However, there has never been a precise definition of quantum affine plane. In the projective case, if A is a $(d + 1)$-dimensional quadratic AS-regular algebra, then we call $\mathrm{Proj}_{\mathrm{nc}}A$ a d-dimensional quantum projective space (q-$\mathbb{P}^d$). In particular, if A is a 3-dimensional quadratic AS-regular algebra, then we call $\mathrm{Proj}_{\mathrm{nc}}A$ a quantum projective plane (q-$\mathbb{P}^2$).

In algebraic geometry, the following result is well known. If A is a polynomial algebra $k[x, y, z]$ and $u \in A_1$, then

$$\begin{array}{ccccc} \mathrm{Proj}\, A & = & \mathrm{Proj}\, A/(u) & \cup & \mathrm{Spec}\, A[u^{-1}]_0 \\ \wr\| & & \wr\| & & \wr\| \\ \mathbb{P}^2 & & \mathbb{P}^1 & & \mathbb{A}^2 \end{array}.$$

Meanwhile, if A be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element (i.e., $uA = Au$), then $\mathrm{Proj}_{\mathrm{nc}}A$ is a q-$\mathbb{P}^2$ and $\mathrm{Proj}_{\mathrm{nc}}A/(u)$ is a q-$\mathbb{P}^1$. Following the above facts, we define a quantum affine plane as follows.

Definition 8. Let A be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element (i.e., $uA = Au$), then we define a quantum affine plane by

$$\mathrm{Spec}_{\mathrm{nc}}A[u^{-1}]_0$$

where $A[u^{-1}]_0$ is the degree zero part of the noncommutative graded localization of A.

Example 9. The algebra $A = k\langle x, y, z\rangle/(yz-\alpha zy, zx-\beta xz, xy-\gamma yx)$ where $0 \neq \alpha, \beta, \gamma \in k$ is a 3-dimensional quadratic AS-regular algebra. Then A has a normal element $x \in A_1$, and one can show that $A[x^{-1}]_0 \cong k\langle s, t\rangle/(st - \alpha\beta\gamma ts)$.

4. Classification of quantum affine planes

In this section, we will classify quantum affine planes. We define $\mathrm{Spec}_{\mathrm{nc}}R$ and $\mathrm{Spec}_{\mathrm{nc}}R'$ are isomorphic if there exists an equivalence functor $F : \mathrm{Mod}R \to \mathrm{Mod}R'$ such that $F(R) \cong R'$. Since $\mathrm{Spec}_{\mathrm{nc}}R$ and $\mathrm{Spec}_{\mathrm{nc}}R'$ are isomorphic if and only if $R \cong R'$, we call the coordinate ring $A[u^{-1}]_0$ a quantum affine plane.

Although it is difficult to find normal elements of a given algebra in general, we can find normal elements of 3-dimensional quadratic AS-regular algebras by using geometric pairs.

Lemma 10. *Let $A = \mathcal{A}(E, \tau)$ be a 3-dimensional quadratic AS-regular algebra, and let $u \in A_1$. Then $u \in A_1$ is a normal element if and only if*

(1) $\mathcal{V}(u) \subset E$,
(2) $\tau(\mathcal{V}(u)) = \mathcal{V}(u)$.

In addition, the following lemma is very useful to classify quantum affine planes.

Lemma 11. *Let A be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element. Then there exist a 3-iterated quadratic Ore extension B and a normal element $v \in B_1$ which satisfy*

$$\mathrm{GrMod}A \cong \mathrm{GrMod}B \text{ and } A[u^{-1}]_0 \cong B[v^{-1}]_0.$$

By using the above lemmas,

Theorem 12. *Every quantum affine plane is isomorphic to exactly one of the following:*

$$\begin{aligned}
&k\langle s,t\rangle/(st-\lambda ts) =: S_\lambda \ (0 \neq \lambda \in k)\\
&k\langle s,t\rangle/(st-\lambda ts+1) =: T_\lambda \ (0 \neq \lambda \in k)\\
&k\langle s,t\rangle/(ts-st+t)\\
&k\langle s,t\rangle/(ts-st+t^2)\\
&k\langle s,t\rangle/(ts-st+t^2+1)
\end{aligned}$$

where

$$S_\lambda \cong S_{\lambda'} \Leftrightarrow \lambda' = \lambda^{\pm 1}, \quad T_\lambda \cong T_{\lambda'} \Leftrightarrow \lambda' = \lambda^{\pm 1}.$$

All of the above algebras are 2-iterated (ungraded) Ore extensions of k. Hence, we see that quantum affine planes have nice properties like a polynomial algebras. For example, they are noetherian domains and have finite global dimension.

References

[1] M. Artin and W. Schelter, Graded Algebras of Global Dimension 3, *Adv. Math.* **66** (1987), 171–216.

[2] M. Artin, J. Tate and M. Van den Bergh, "*Some Algebras Associated to Automorphisms of Elliptic Curves*" in The Grothendieck Festschrift, Vol. **1**, Progr. Math. **86**, Birkhauser, Boston 1990, 33–85.

[3] M. Artin and J. J. Zhang, Noncommutative Projective Schemes, *Adv. Math.* **109** (1994), 228–287.

[4] K. R. Goodearl and R. B. Warfield Jr. , *An Introduction to Noncommutative Noetherian Rings*, Second Edition, London Mathematical Society Student Texts, Vol. **61**, Cambridge University Press, Cambridge, 2004.

[5] I. Mori,"*Noncommutative Projective Schemes and Point Schemes*" in Algebras, Rings and Their Representations, World Sci., Hackensack, N.J., 2006, 215–239.

[6] D. R. Stephenson and J. J. Zhang, Growth of Graded Noetherian Rings, *Proc. Amer. Math. Soc.*, **125** (1997), 1593–1605.

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