ON SOME FINITENESS QUESTIONS ABOUT HOCHSCHILD COHOMOLOGY OF FINITE-DIMENSIONAL ALGEBRAS

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Abstract. In this article, we give the dimensions of the Hochschild cohomology groups of certain finite-dimensional algebras $A = A_T(q_0, q_1, q_2, q_3)$ ($T \geq 0$; $q_0, q_1, q_2, q_3 \in K^\times$) and Λ_t ($t \geq 1$). Moreover, we study the Hochschild cohomology rings modulo nilpotence of A and Λ_t. We then see that A gives a negative answer to Happle's question whereas Λ_t is a counterexample to Snashall-Solberg's conjecture.

1. Introduction

Throughout this article, let K be an algebraically closed field. Let B be a finite-dimensional K-algebra. We denote by B^e the enveloping algebra $B^{\mathrm{op}} \otimes_K B$ of B. Here, note that there is a natural one-to-one correspondence between the family of right B^e-modules and that of B-B-bimodules. Recall that the ith *Hochschild cohomology group* $\mathrm{HH}^i(B)$ of B is defined to be the K-space $\mathrm{HH}^i(B) := \mathrm{Ext}^i_{B^e}(B, B)$ for $i \geq 0$. Also, the *Hochschild cohomology ring* $\mathrm{HH}^*(B)$ of B is defined to be the graded ring $\mathrm{HH}^*(B) := \bigoplus_{i\geq 0} \mathrm{HH}^i(B) = \bigoplus_{i\geq 0} \mathrm{Ext}^i_{B^e}(B, B)$, where the product is given by the Yoneda product. It is known that $\mathrm{HH}^*(B)$ is a graded commutative K-algebra. Let $\mathcal{N}_B$ be an ideal in $\mathrm{HH}^*(B)$ generated by all homogeneous nilpotent elements. Note that $\mathcal{N}_B$ is a homogeneous ideal. Then the graded K-algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$ is called the *Hochschild cohomology ring modulo nilpotence* of B. It is known that $\mathrm{HH}^*(B)/\mathcal{N}_B$ is a commutative K-algebra.

In this article, we consider the following question and conjecture:

(1) Happel's question ([9]). For a finite-dimensional algebra B, if $\mathrm{HH}^i(B) = 0$ for all $i \gg 0$, then is the global dimension of B finite?

(2) Snashall-Solberg's conjecture ([12]). For any finite-dimensional algebra B, the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}_B$ is finitely generated as an algebra.

In the papers [2, 3, 10], a negative answer to the question **(1)** was obtained, where the authors studied the Hochschild cohomology groups for several weakly symmetric algebras. On the other hand, a counterexample to the conjecture **(2)** was recently given by Xu ([14]) and Snashall ([11]).

In this article, we study the Hochschild cohomology groups and rings for two finite-dimensional algebras $A_T(q_0, q_1, q_2, q_3)$ ($T \geq 0$; $q_0, q_1, q_2, q_3 \in K^\times$) and Λ_t ($t \geq 1$) (see Section 2), and then see that the algebra $A_T(q_0, q_1, q_2, q_3)$ also gives a negative answer to **(1)** and that the algebra Λ_t is also a counterexample to **(2)**.

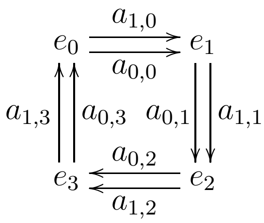
The detailed version of this paper will be submitted for publication elsewhere.

In Section 2, we define two finite-dimensional algebras $A_T(q_0, q_1, q_2, q_3)$ and Λ_t by using some finite quivers, where $T \geq 0$ and $t \geq 1$ are integers and q_i are elements in $K^\times$ for $i = 0, 1, 2, 3$. In Section 3, we give the dimensions of the Hochschild cohomology groups of $A_T(q_0, q_1, q_2, q_3)$, where the product $q_0q_1q_2q_3$ is not a roof of unity, and Λ_t for $t \geq 3$ (Theorems 1 and 3). In Section 4, we describe the structures of the Hochschild cohomology rings modulo nilpotence of $A_T(q_0, q_1, q_2, q_3)$ and Λ_t (Theorems 4 and 5).

2. Algebras $A_T(q_0, q_1, q_2, q_3)$ and Λ_t

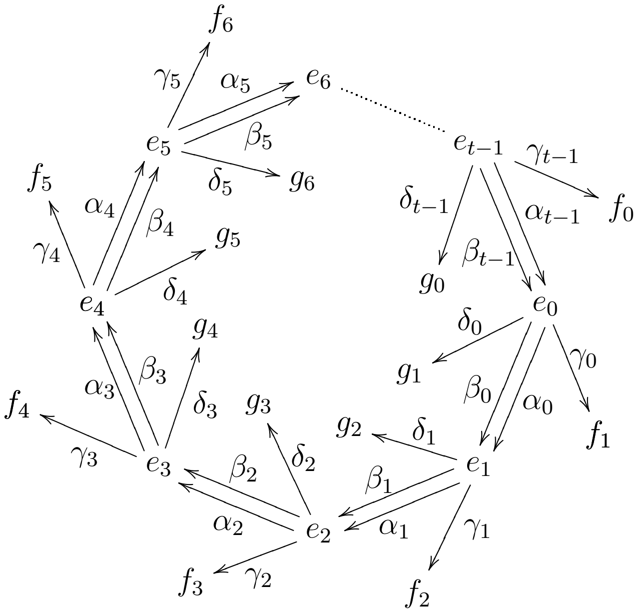
Let $T \geq 0$ and $t \geq 1$ be integers, and let q_0, q_1, q_2, q_3 be elements in $K^\times$. We definite the algebras $A = A_T(q_0, q_1, q_2, q_3)$ and Λ_t as follows:

(i) Let Γ be the following quiver with four vertices e_0, e_1, e_2, e_3 and eight arrows $a_{l,m}$ for $l = 0, 1$ and $m = 0, 1, 2, 3$:



Let $K\Gamma$ be the path algebra, and set $x_l := \sum_{m=0}^{3} a_{l,m} \in K\Gamma$ for $l = 0, 1$. Let I denote the ideal in $K\Gamma$ generated by the uniform elements $e_i x_0 x_1$, $e_i x_1 x_0$, $e_j(q_j x_0^{4T+2} + x_1^{4T+2})$, and $e_k(q_k x_1^{4T+2} + x_0^{4T+2})$ for $0 \leq i \leq 3$, $j = 0, 2$ and $k = 1, 3$. We then define the algebra $A = A_T(q_0, q_1, q_2, q_3)$ by $A = A_T(q_0, q_1, q_2, q_3) := K\Gamma/I$.

(ii) Let Δ be the following quiver with $3t$ vertices e_i, f_i, g_i for $i = 0, \ldots, t-1$ and $4t$ arrows $\alpha_i, \beta_i, \gamma_i, \delta_i$ for $i = 0, \ldots, t-1$:



Let I be the ideal in the path algebra $K\Delta$ generated by the elements $\alpha_i\alpha_{i+1}$, $\beta_i\beta_{i+1}$, $\alpha_i\delta_{i+1}$, $\beta_i\gamma_{i+1}$, $\alpha_i\beta_{i+1}+\beta_i\alpha_{i+1}$ for $i=0,\dots,t-1$ (where $\alpha_t:=\alpha_0$, $\beta_t:=\beta_0$, $\gamma_t:=\gamma_0$ and $\delta_t:=\delta_0$). Define the algebra Λ_t by $\Lambda_t:=K\Delta/I$.

We see that $A=A_T(q_0,q_1,q_2,q_3)$ is a self-injective special biserial algebra for all $T\geq 0$ and $q_i \in K^\times$ $(i=0,1,2,3)$. In particular, if $T=0$, then $A_0(q_0,q_1,q_2,q_3)$ is a Koszul algebra for all $q_i\in K^\times$ $(i=0,1,2,3)$. Also, Λ_t is a Koszul algebra for $t\geq 1$.

3. The Hochschild cohomology groups for A and Λ_t

In this section, we give dimensions of the Hochschild cohomology groups $\mathrm{HH}^n(A)$ $(n\geq 0)$, where the product $q_0q_1q_2q_3\in K^\times$ is not a roof of unity, and $\mathrm{HH}^n(\Lambda_t)$ $(n\geq 0)$ for $t\geq 3$.

Theorem 1 ([5])**.** *Suppose that the product $q_0q_1q_2q_3\in K^\times$ is not a root of unity. Then*

(a) *For $m\geq 0$ and $0\leq r\leq 3$,*

$$\dim_K \mathrm{HH}^{4m+r}(A)=\begin{cases} 2T+1 & \text{if } m=r=0\\ 2T+3 & \text{if } m=0,\ r=1 \text{ and } \operatorname{char}K \mid 2T+1\\ 2T+2 & \text{if } m=0,\ r=1 \text{ and } \operatorname{char}K \nmid 2T+1\\ 2T+2 & \text{if } m=0,\ r=2 \text{ and } \operatorname{char}K \mid 2T+1\\ 2T+1 & \text{if } m=0,\ r=2 \text{ and } \operatorname{char}K \nmid 2T+1\\ 2T+2 & \text{if } m\geq 1,\ r=0 \text{ and } \operatorname{char}K \mid 2T+1,\\ & \text{or if } m\geq 1,\ r=1 \text{ and } \operatorname{char}K \mid 2T+1\\ 2T & \text{if } m\geq 1,\ r=0 \text{ and } \operatorname{char}K \nmid 2T+1,\\ & \text{or if } m\geq 1,\ r=1 \text{ and } \operatorname{char}K \nmid 2T+1\\ 2T & \text{if } m\geq 0 \text{ and } r=2,\\ & \text{or if } m\geq 0 \text{ and } r=3. \end{cases}$$

(b) $\mathrm{HH}^n(A)=0$ *for all $n\geq 3$ if and only if $T=0$.*

Remark 2. If $T=0$, then since the global dimension of $A_0(q_0,q_1,q_2,q_3)$ is infinite for all $q_i\in K^\times$ $(i=0,1,2,3)$, by Theorem 1 (b) we have got a negative answer to Happel's question **(1)**.

Theorem 3 ([6])**.** *Let $t\geq 3$. Then,*

(1) $\dim_K \mathrm{HH}^0(\Lambda_t)=1$ *and* $\dim_K \mathrm{HH}^1(\Lambda_t)=2$.

(2) *For an integer $n\geq 2$, write $n=mt+r$ for integers $m\geq 0$ and r with $0\leq r\leq t-1$.*

 (a) *If t is even, m is even, or* $\operatorname{char}K=2$, *then*

$$\dim_K \mathrm{HH}^n(\Lambda_t)=\begin{cases} mt-1 & \text{if } r=0\\ 2mt+2t & \text{if } r=1\\ 2mt+2t+1 & \text{if } r=2\\ 0 & \text{if } 3\leq r\leq t-1 \end{cases}$$

(b) *If t is odd, m is odd and* $\operatorname{char} K \neq 2$*, then*

$$\dim_K \mathrm{HH}^n(\Lambda_t) = \begin{cases} 0 & \text{if } r = 0, \text{ or } 3 \leq r \leq t-1 \\ 2t & \text{if } r = 1, \text{ or } r = 2. \end{cases}$$

4. The Hochschild cohomology rings modulo nilpotence of A and Λ_t

In this section, we describe the structures of the Hochschild cohomology rings modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}_A$, where $T = 0$ and the product $q_0q_1q_2q_3 \in K^\times$ is not a roof of unity, and $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ for $t \geq 3$.

Theorem 4 ([5])**.** *Let $T = 0$ and $q_i \in K^\times$ for $0 \leq i \leq 3$. Suppose that the product $q_0q_1q_2q_3$ is not a root of unity. Then $\mathrm{HH}^*(A)$ is a 4-dimensional local algebra, and $\mathrm{HH}^*(A)/\mathcal{N}_A$ is isomorphic to K.*

Let m be an integer, and denote by C_m the quotient ring

$$K[z_0, \ldots, z_m]/\left\langle z_iz_j - z_kz_l \mid 0 \leq i,j,k,l \leq m;\ i+j = k+l \right\rangle$$

of the polynomial ring $K[z_0, \ldots, z_m]$ in $m+1$ variables. Then we have the following:

Theorem 5 ([6])**.** *Let $t \geq 3$.*

(a) *If t is even or* $\operatorname{char} K = 2$*, then $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ is isomorphic to the graded subalgebra of C_t with a K-basis*

$$\{1_K\} \cup \{z_i^j \mid j \geq 1;\ 1 \leq i \leq t-1\} \cup \{z_i^k z_{i+1}^{l-k} \mid l \geq 2;\ 1 \leq k \leq l-1;\ 0 \leq i \leq t-1\},$$

where $\deg z_i = t$ $(i = 0, \ldots, t)$.

(b) *If t is odd and* $\operatorname{char} K \neq 2$*, then $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ is isomorphic to the graded subalgebra of C_{2t} with a K-basis*

$$\{1_K\} \cup \{z_i^j \mid j \geq 1;\ 1 \leq i \leq 2t-1\} \cup \{z_i^k z_{i+1}^{l-k} \mid l \geq 2;\ 1 \leq k \leq l-1;\ 0 \leq i \leq 2t-1\},$$

where $\deg z_i = 2t$ $(i = 0, \ldots, 2t)$.

By Theorem 5, we easily have the following corollary, which tells us that Λ_t $(t \geq 3)$ is a counterexample to the conjecture **(2)**:

Corollary 6 ([6])**.** *For $t \geq 3$, $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ is not finitely generated as an algebra.*

Remark 7. Since Λ_t is a Koszul algebra for $t \geq 1$, the graded centre $Z_{\mathrm{gr}}(E(\Lambda_t))$ of the Ext algebra $E(\Lambda_t) := \bigoplus_{i \geq 0} \mathrm{Ext}^i_{\Lambda_t}(\Lambda_t/\mathrm{rad}\,\Lambda_t, \Lambda_t/\mathrm{rad}\,\Lambda_t)$ of Λ_t is isomorphic to $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ as graded algebras (see [1]). Hence, for $t \geq 3$, $Z_{\mathrm{gr}}(E(\Lambda_t))$ is also isomorphic to the algebra described in Theorem 5 and is not finitely generated as an algebra.

In [11], Snashall gave the following new question:

Snashall's question ([11]). Can we give necessary and sufficient conditions on a finite-dimensional algebra for its Hochschild cohomology ring modulo nilpotence to be finitely generated as an algebra?

It is known that the Hochschild cohomology rings modulo nilpotence for following finite-dimensional algebras are finitely generated:

• group algebras ([4, 13]) • self-injective algebras of finite representation type ([7])
• monomial algebras ([8]) • algebras with finite global dimension ([9])

However, a definitive answer to the question above has not been obtained yet.

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