SERRE SUBCATEGORIES OF ARTINIAN MODULES

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Abstract. We study subcategories of the category of artinian modules. We prove that all wide subcategories of artinian modules are Serre subcategories. We also provide the bijection between the set of Serre subcategories of artinian modules and the set of specialization closed subsets of the set of closed prime ideals of some completed ring.

1. Introduction

Let R be a commutative noetherian ring and M be an R-module. We denote by $\mathrm{Mod}(R)$ the category of R-modules and R-homomorphisms and by $\mathrm{mod}(R)$ the full subcategory consisting of finitely generated R-modules. We also denote by $\mathrm{Spec}\ R$ the set of prime ideals of R and by $\mathrm{Ass}_R M$ the set of associated prime ideals of M. A subcategory of an abelian category is said to be a wide subcategory if it is closed under kernels, cokernels and extensions. We also say that a subcategory is a Serre subcategory if it is a wide subcategory which is closed under subobjects.

Classifying subcategories of a module category also has been studied by many authors. Classically, Gabriel [2] gives a bijection between the set of Serre subcategories of $\mathrm{mod}(R)$ and the set of specialization closed subsets of $\mathrm{Spec}\ R$. Recently, Takahashi [10] and Krause [4] proved the following.

Theorem 1. [5, Theorem 4.1][3, Corollary 2.6] *Let R be a noetherian ring. Then we have the following 1-1 correspondences;*

$$\begin{aligned}&\{\ \textit{subcategories of } \mathrm{mod}(R) \textit{ closed under submodules and extensions}\ \}\\ &\cong \{\ \textit{subsets of}\ \ \mathrm{Spec}\ R\ \}.\end{aligned}$$

Moreover this induces the bijection

$$\{\ \textit{Serre subcategories of } \mathrm{mod}(R)\ \} \cong \left\{\ \textit{specialization closed subsets of } \mathrm{Spec}\ R\ \right\}.$$

In addition, Takahashi [10] pointed out a property concerning wide subcategories of $\mathrm{mod}(R)$.

Theorem 2. [10, Theorem 3.1, Corollary 3.2] *Let R be a noetherian ring. Then every wide subcategory of $\mathrm{mod}(R)$ is a Serre subcategory of $\mathrm{mod}(R)$.*

In this note, we want to consider the artinian analogue of these results. We prove that all wide subcategories of artinian modules are Serre subcategories (Theorem 9). We also provide the bijection between the set of Serre subcategories and the set of specialization closed subsets of the set of closed prime ideals of some completed ring (Theorem 26). We

We refer to [3] for more details on the present article.

consider some completion of a ring (see Proposition 16), so that all of artinian modules can be regarded as modules over it.

Throughout the note, we always assume that R is a commutative ring with identity, and by a subcategory we mean a nonempty full subcategory which is closed under isomorphism.

2. WIDE SUBCATEGORIES OF ARTINIAN MODULES

In this section, we investigate wide subcategories of artinian modules.

Let M be an artinian R-module. We denote by $\mathrm{Soc}(M)$ the sum of simple submodules of M. Since $\mathrm{Soc}(M)$ is also artinian, there exist only finitely many maximal ideals $\mathfrak{m}$ of R for which $\mathrm{Soc}(M)$ has a submodule isomorphic to $R/\mathfrak{m}$. Let the distinct such maximal ideals be $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$. Set $J_M = \bigcap_{i=1}^{s} \mathfrak{m}$ and $\hat{R}^{(M)} = \varprojlim R/J_M^n$.

Lemma 3. [9, Lemma 2.2] *Each non-zero element $m \in M$ is annihilated by some power of J_M. Hence M has the natural structure of a module over $\hat{R}^{(M)}$ in such a way that a subset of M is an R-submodule if and only if it is an $\hat{R}^{(M)}$-submodule.*

Proof. We need in the present note how the $\hat{R}^{(M)}$-module structure is defined for an artinian module M. For this reason we briefly recall the proof of the lemma.

Since $\mathrm{Soc}(M) = \oplus_{i=1}^{s}(R/\mathfrak{m}_i)^{n_i}$, M can be embedded in $\oplus_{i=1}^{s}(E_R(R/\mathfrak{m}_i))^{n_i}$ where $E_R(R/\mathfrak{m})$ is an injective hull of $R/\mathfrak{m}$. Note that an element of $E_R(R/\mathfrak{m})$ is annihilated by some power of $\mathfrak{m}$. Hence one can show that each element of M is annihilated by some power of $\mathfrak{m}_1 \cdots \mathfrak{m}_s = J_M$.

Let $x \in M$ and $\hat{r} = (r_n + J_M^n)_{n\in\mathbb{N}} \in \hat{R}^{(M)}$. Suppose that $J_M^k x = 0$. It is straightforward to check that M has the structure of an $\hat{R}^{(M)}$-module such that $\hat{r}x = r_k x$.

□

By virtue of Lemma 3, each artinian R-module can be regarded as a module over some complete semi-local ring. We note that the Matlis duality theorem holds over a noetherian complete semi-local ring (cf. [6, Theorem 1.6]). It is the strategy of the note that we replace the categorical property on a subcategory of finitely generated (namely, noetherian) modules with that of artinian modules by using Matlis duality. We denote by $\mathrm{Art}(R)$ the subcategory consisting of artinian R-modules. The following lemma holds from the Matlis duality theorem.

Lemma 4. *Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_s)$ be a noetherian complete semi-local ring and set $E = \oplus_{i=1}^{s} E_R(R/\mathfrak{m}_i)$. For each subcategory $\mathcal{X}$ of $\mathrm{Mod}(R)$, we denote by $\mathcal{X}^\vee = \{M^\vee \mid M \in \mathcal{X}\}$ where $(-)^\vee = \mathrm{Hom}_R(-, E)$. Then the following assertions hold.*

1. *If $\mathcal{X}$ is a subcategory of $\mathrm{Art}(R)$ (resp. $\mathrm{mod}(R)$) which is closed under quotient modules (resp. submodules) and extensions, then $\mathcal{X}^\vee$ is a subcategory of $\mathrm{mod}(R)$ (resp. $\mathrm{Art}(R)$) which is closed under submodules (resp. quotient modules) and extensions.*
2. *If $\mathcal{X}$ is a wide subcategory of $\mathrm{Art}(R)$ (resp. $\mathrm{mod}(R)$), then $\mathcal{X}^\vee$ is also a wide subcategory of $\mathrm{mod}(R)$ (resp. $\mathrm{Art}(R)$).*
3. *If $\mathcal{X}$ is a Serre subcategory of $\mathrm{Art}(R)$ (resp. $\mathrm{mod}(R)$), then $\mathcal{X}^\vee$ is also a Serre subcategory of $\mathrm{mod}(R)$ (resp. $\mathrm{Art}(R)$).*

Definition 5. Let M be an R-module. For a nonnegative integer n, we inductively define a subcategory $\mathrm{Wid}_R^n(M)$ of $\mathrm{Mod}(R)$ as follows:

(1) Set $\mathrm{Wid}_R^0(M) = \{M\}$.
(2) For $n \geq 1$, let $\mathrm{Wid}_R^n(M)$ be a subcategory of $\mathrm{Mod}(R)$ consisting of all R-modules X having an exact sequence of either of the following three forms:

$$\begin{aligned} &A \to B \to X \to 0, \\ &0 \to X \to A \to B, \\ &0 \to A \to X \to B \to 0 \end{aligned}$$

where $A, B \in \mathrm{Wid}_R^{n-1}(M)$.

Remark 6. Let M be an R-module. Then the following hold.

(1) There is an ascending chain $\{M\} = \mathrm{Wid}_R^0(M) \subseteq \mathrm{Wid}_R^1(M) \subseteq \cdots \subseteq \mathrm{Wid}_R^n(M) \subseteq \cdots \subseteq \mathrm{Wid}_R(M)$ of subcategories $\mathrm{Mod}(R)$. Here we denote by $\mathrm{Wid}_R(M)$ the smallest wide subcategory of $\mathrm{Mod}(R)$ which contains M.
(2) $\bigcup_{n\geq 0} \mathrm{Wid}_R^n(M)$ is wide and the equality $\mathrm{Wid}_R(M) = \bigcup_{n\geq 0} \mathrm{Wid}_R^n(M)$ holds.
(3) If M is artinian, then $\bigcup_{n\geq 0} \mathrm{Wid}_R^n(M)$, hence $\mathrm{Wid}_R(M)$, is a subcategory of $\mathrm{Art}(R)$.

Definition 7. Let J be an ideal of R. For each R-module M, we denote by $\Gamma_J(M)$ the set of elements of M which are annihilated by some power of J, namely $\Gamma_J(M) = \bigcup_{n\in\mathbb{N}}(0 :_M J^n)$. An R-module M is said to be J-torsion if $M = \Gamma_J(M)$. We denote by $\mathrm{Art}_J(R)$ the subcategory consisting of artinian J-torsion R-modules.

Proposition 8. *Let M be an artinian R-module. Then* $\mathrm{Wid}_R(M)$ *and* $\mathrm{Wid}_{\hat{R}^{(M)}}(M)$ *are equivalent as subcategories of* $\mathrm{Art}(\hat{R}^{(M)})$.

Proof. Since a J-torsioness is closed under taking kernels, cockerels and extension, we can naturally identify $\mathrm{Wid}_R(M)$ with a subcategory of $\mathrm{Art}_{J_M}(R)$. □

Theorem 9. *Let R be a noetherian ring. Then every wide subcategory of* $\mathrm{Art}(R)$ *is a Serre subcategory of* $\mathrm{Art}(R)$.

Proof. Let $\mathcal{X}$ be a wide subcategory of $\mathrm{Art}(R)$. It is sufficiently to show that $\mathcal{X}$ is closed under submodules. Assume that $\mathcal{X}$ is not closed under submodules. Then there exists an R-module X in $\mathcal{X}$ and R-submodule M of X such that M does not belong to $\mathcal{X}$. Applying Lemma 3 to X, X is a module over the complete semi-local ring $\hat{R} := \hat{R}^{(J_X)}$ and M is an $\hat{R}$-submodule of X. Now we consider the wide subcategory $\mathrm{Wid}_R(X)$. By virtue of Proposition 8, $\mathrm{Wid}_R(X) = \mathrm{Wid}_{\hat{R}}(X)$ as a subcategory of $\mathrm{Art}(\hat{R})$. Since $\hat{R}$ is a complete semi-local ring, by Matlis duality, we have the equivalence of the categories $\mathrm{Wid}_{\hat{R}}(X) \cong \{\mathrm{Wid}_{\hat{R}}(X)^\vee\}^{op} \cong \mathrm{Wid}_{\hat{R}}(X^\vee)^{op}$ where $(-)^\vee = \mathrm{Hom}_{\hat{R}}(-, E_{\hat{R}}(\hat{R}/J_X\hat{R}))$. Since $\mathrm{Wid}_{\hat{R}}(X^\vee)$ is a wide subcategory of finitely generated $\hat{R}$-modules, it follows from Theorem 2 that $\mathrm{Wid}_{\hat{R}}(X^\vee)$ is a Serre subcategory. Thus $M^\vee$ is contained in $\mathrm{Wid}_{\hat{R}}(X^\vee)$. Using Matlis duality again, we conclude that M must be contained in $\mathrm{Wid}_{\hat{R}}(X) = \mathrm{Wid}_R(X)$, hence also in $\mathcal{X}$. This is a contradiction, so that $\mathcal{X}$ is closed under submodules.

□

3. Classifying subcategories of artinian modules

In this section, we shall give the artinian analogue of the classification theorem of subcategories of finitely generated modules (Theorem 26). First, we state the notion and the basic properties of attached prime ideals which play a key role of our theorem. For the detail, we recommend the reader to look at [8, 9] and [5, §6 Appendix.] .

Definition 10. Let M be an R-module. We say that M is secondary if for each $a \in R$ the endomorphism of M defined by the multiplication map by a is either surjective or nilpotent.

Remark 11. If M is secondary then $\mathfrak{p} = \sqrt{\mathrm{ann}_R(M)}$ is a prime ideal and M is said to be $\mathfrak{p}$-secondary.

Definition 12. Let M be an R-module.

(1) $M = S_1 + \cdots + S_r$ is said to be a secondary representation if S_i is a secondary submodule of M for all i. And we also say that the representation is minimal if the prime ideals $\mathfrak{p}_i = \sqrt{\mathrm{ann}_R(S_i)}$ are all distinct, and none of the S_i is redundant

(2) A prime ideal $\mathfrak{p}$ is said to be an attached prime ideal of M if M has a $\mathfrak{p}$-secondary quotient. We denote by $\mathrm{Att}_R M$ the set of the attached prime ideals of M.

Remark 13. Let M be an R-module.

(1) If $M = S_1 + \cdots + S_r$ is a minimal representation and $p_i = \sqrt{\mathrm{ann}_R(S_i)}$. then $\mathrm{Att}_R M = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}$. See [5, Theorem 6.9] .

(2) Given a submodule $N \subseteq M$, we have

$$\mathrm{Att}_R M/N \subseteq \mathrm{Att}_R M \subseteq \mathrm{Att}_R(N) \cup \mathrm{Att}_R M/N.$$

See [5, Theorem 6.10] .

(3) It is known that if M is artinian then M has a secondary representation. Thus it has a minimal one. See [5, Theorem 6.11] .

In the rest of this section, we always assume that R is a noetherian ring. The following observation tells us that we should consider a larger set than Spec R to classify subcategories of artinian modules.

Example 14. Let $(R, \mathfrak{m})$ be a noetherian local ring and $\mathcal{X}$ a Serre subcategory of $\mathrm{Art}(R)$. By virtue of Lemma 3, $\mathrm{Art}(R)$ is equivalent to $\mathrm{Art}(\hat{R})$ where $\hat{R}$ is an $\mathfrak{m}$-adic completion of R. Now we consider $\mathcal{X}$ as a subcategory of $\mathrm{Art}(\hat{R})$. Since $\mathcal{X}^\vee$ is a Serre subcategory of $\mathrm{mod}(\hat{R})$ (Lemma 4), $\mathcal{X}^\vee$, hence $\mathcal{X}$, corresponds to the specialization closed subset of Spec $\hat{R}$ by Theorem 1. That is, there is the bijection between the set of Serre subcategories of $\mathrm{Art}(R)$ and the set of specialization closed subsets of Spec $\hat{R}$.

As mentioned in Lemma 3, we can determine some complete semi-local rings for each artinian module respectively, so that the artinian module has the module structure over such a completed ring. Now we attempt to treat all the artinian R-modules as modules over the same completed ring. For this, we consider the following set of ideals of R:

$$\mathcal{T} = \{ \ I \mid \text{the length of } R/I \text{ is finite } \}.$$

The set $\mathcal{T}$ forms a directed set ordered by inclusion. Then we can consider the inverse system $\{R/I, f_{I,I'}\}$ where $f_{I,I'}$ are natural surjections. We denote $\varprojlim_{I\in\mathcal{T}} R/I$ by $\hat{R}_{\mathcal{T}}$.

The proof of the following lemma will go through similarly to the proof of Lemma 3.

Lemma 15. *Every artinian R-module has the structure of an $\hat{R}_{\mathcal{T}}$-module in such a way that a subset of an artinian R-module M is an R-submodule if and only if it is an $\hat{R}_{\mathcal{T}}$-submodule. Consequently, we have an equivalence of categories* $\mathrm{Art}(R) \cong \mathrm{Art}(\hat{R}_{\mathcal{T}})$.

We set another family of ideals of R as

$$\mathcal{J} = \{\ \mathfrak{m}_1^{k_1}\cdots\mathfrak{m}_s^{k_s} \mid \mathfrak{m}_i \text{ is a maximal ideal of } R,\ k_i \in \mathbb{N}\}.$$

It is also a directed set ordered by inclusion and we denote by $\hat{R}_{\mathcal{J}}$ its inverse limit on the system via natural surjections.

Next we consider a direct product of rings

$$\prod_{\mathfrak{n}\in\max(R)} \hat{R}_{\mathfrak{n}}$$

where $\max(R)$ is the set of maximal ideals of R and $\hat{R}_{\mathfrak{m}}$ is an $\mathfrak{m}$-adic completion of R. We regard the ring as a topological ring by a product topology, namely the linear topology defined by ideals which are of the form $\mathfrak{m}_1^{k_1}\hat{R}_{\mathfrak{m}_1}\times\cdots\times\mathfrak{m}_s^{k_s}\hat{R}_{\mathfrak{m}_s}\times\prod_{\mathfrak{n}\neq\mathfrak{m}_1,\cdots,\mathfrak{m}_s}\hat{R}_{\mathfrak{n}}$ for some $\mathfrak{m}_i\in\max(R)$ and $k_i\in\mathbb{N}$. For the rings $\hat{R}_{\mathcal{T}}$, $\hat{R}_{\mathcal{J}}$ and $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$, we have the following.

Proposition 16. [1, §2.13. Proposition 17] *There is an isomorphism of topological rings*

$$\hat{R}_{\mathcal{T}} \cong \hat{R}_{\mathcal{J}} \cong \prod_{\mathfrak{n}\in\max(R)} \hat{R}_{\mathfrak{n}}.$$

Remark 17. Let M be an artinian R-module. It follows from Lemma 15 that M is also an artinian $\hat{R}$-module. Then the radical of $\mathrm{ann}_{\hat{R}}(M)$ is a closed ideal. To show this, it suffices to prove that the inclusion $\sqrt{\mathrm{ann}_{\hat{R}}(M)} \supseteq \cap_{I\in\mathcal{T}}(\sqrt{\mathrm{ann}_{\hat{R}}(M)}+I)$ holds. Take an arbitrary element $\hat{a}\in\cap_{I\in\mathcal{T}}(\sqrt{\mathrm{ann}_{\hat{R}}(M)}+I)$. Then there exist some elements $\hat{b}_I\in\mathrm{ann}_{\hat{R}}(M)$ and $\hat{c}_I\in I$ such that $\hat{a}=\hat{b}_I+\hat{c}_I$ for all I. Let $x\in M$ and suppose that $Ix=0$ for some $I\in\mathcal{T}$. Since $\hat{b}_I\in\sqrt{\mathrm{ann}_{\hat{R}}(M)}$, $\hat{b}_I^k\in\mathrm{ann}_{\hat{R}}(M)$. Hence we see that $\hat{a}^kx=(\hat{b}_I+\hat{c}_I)^kx=0$ holds, so that $\hat{a}\in\sqrt{\mathrm{ann}_{\hat{R}}(M)}$. Consequently, $\mathrm{Att}_{\hat{R}}M$ is a subset of the set of closed prime ideals of $\hat{R}$.

For closed prime ideals of $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$, we have the following result.

Proposition 18. *Every proper closed prime ideal of $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$ is of the form $\mathfrak{p}\times\prod_{\mathfrak{n}\in\max(R),\mathfrak{m}\neq\mathfrak{n}}\hat{R}_{\mathfrak{n}}$ for some prime ideal $\mathfrak{p}\in\mathrm{Spec}\,\hat{R}_{\mathfrak{m}}$. Hence we can identify the set of closed prime ideals of $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$ with the disjoint union of* $\mathrm{Spec}\,\hat{R}_{\mathfrak{m}}$, *i.e.* $\coprod_{\mathfrak{n}\in\max(R)}\mathrm{Spec}\,\hat{R}_{\mathfrak{n}}$.

We can equate the rings $\hat{R}_{\mathcal{T}}$, $\hat{R}_{\mathcal{J}}$ and $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$ by virtue of Proposition 16. In the rest of this note we always denote them by $\hat{R}$ and identify the set of closed prime ideals of $\hat{R}$ with $\coprod_{\mathfrak{n}\in\max(R)}\mathrm{Spec}\,\hat{R}_{\mathfrak{n}}$.

Lemma 19. [7, Exercise 8.49] *Let M be an artinian R-module. Assume $\mathrm{Ass}_R M = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$. Then M is the direct sums of the submodules $\Gamma_{\mathfrak{m}_i}(M)$, that is $M = \oplus_{i=1}^s \Gamma_{\mathfrak{m}_i}(M)$. Here we denote by $\Gamma_{\mathfrak{m}_i}(M)$ the $\mathfrak{m}_i$-torsion submodule of M.*

Remark 20. Let M be an $\mathfrak{m}$-torsion R-module. Then M has the structure of an $\hat{R}$-module and an $\hat{R_{\mathfrak{m}}}$-module. Note that the $\hat{R_{\mathfrak{m}}}$-module action on M is identical with the action by means of the natural inclusion $\hat{R_{\mathfrak{m}}} \to \prod_{\mathfrak{n}\in\max(R)} \hat{R_{\mathfrak{n}}} \cong \hat{R}$. We also note from Lemma 15 or Lemma 3 that N is an $\hat{R}$-submodule (resp. a quotient $\hat{R}$-module) of M if and only if it is an $\hat{R_{\mathfrak{m}}}$-submodule (resp. a quotient $\hat{R_{\mathfrak{m}}}$-module) of M.

Proposition 21. *Let M be an $\mathfrak{m}$-torsion R-module. Then*

$$\mathrm{Att}_{\hat{R}} M = \mathrm{Att}_{\hat{R_{\mathfrak{m}}}} M$$

as a subset of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\, \hat{R_{\mathfrak{n}}}$.

Combing Proposition 21 with Lemma 19, we have the following corollary.

Corollary 22. *Let M be an artinian R-module. Then*

$$\mathrm{Att}_{\hat{R}} M = \coprod_{\mathfrak{m}\in\mathrm{Ass}_R M} \mathrm{Att}_{\hat{R_{\mathfrak{m}}}} \Gamma_{\mathfrak{m}}(M)$$

as a subset of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\, \hat{R_{\mathfrak{n}}}$.

Let us state the result which is a key to classify the subcategory of the category of noetherian modules.

Theorem 23. [10, Corollary 4.4][4, Corollary 2.6] *Let M and N be finitely generated R-modules. Then M can be generated from N via taking submodules and extension if and only if $\mathrm{Ass}_R M \subseteq \mathrm{Ass}_R N$.*

The following lemma is due to Sharp [8].

Lemma 24. [8, 3.5.] *Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_s)$ be a commutative noetherian complete semi-local ring and set $E = \oplus_{i=1}^s E_R(R/\mathfrak{m}_i)$. For an artinian R-module M, we have*

$$\mathrm{Att}_R M = \mathrm{Ass}_R \mathrm{Hom}_R(M, E).$$

The next claim is reasonable as the artinian analogue of Theorem 23.

Theorem 25. *Let M and N be artinian R-modules. Then M can be generated from N via taking quotient modules and extensions as R-modules if and only if $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N$.*

Proof. Suppose that M is contained in quot-ext$_R(N)$. It is clear from the property of attached prime ideals (Remark 13) that $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N$ holds.

Conversely, suppose that $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N$. First, we shall show that we may assume that M and N are $\mathfrak{m}$-torsion R-modules for some maximal ideal $\mathfrak{m}$. In fact, M (resp. N) can be decomposed as $M = \oplus_{\mathfrak{m}\in\mathrm{Ass}_R M} \Gamma_{\mathfrak{m}}(M)$ (resp. $N = \oplus_{\mathfrak{n}\in\mathrm{Ass}_R N} \Gamma_{\mathfrak{n}}(N)$) and the assumption implies that $\mathrm{Att}_{\hat{R_{\mathfrak{m}}}} \Gamma_{\mathfrak{m}}(M) \subseteq \mathrm{Att}_{\hat{R_{\mathfrak{m}}}} \Gamma_{\mathfrak{m}}(N)$ for all $\mathfrak{m} \in \mathrm{Ass}_R M$ by Corollary 22. If we show that $\Gamma_{\mathfrak{m}}(M)$ is contained in quot-ext$_R(\Gamma_{\mathfrak{m}}(N))$, we can get the assertion since quot-ext$_R(N)$ is closed under direct sums and direct summands.

Let M and N be $\mathfrak{m}$-torsion R-modules and E be an injective hull of $\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}}$ as an $\hat{R_\mathfrak{m}}$-module. Since M and N are also artinian $\hat{R_\mathfrak{m}}$-modules, $M^\vee$ and $N^\vee$ are finitely generated $\hat{R_\mathfrak{m}}$-modules by Matlis duality, where $(-)^\vee = \mathrm{Hom}_{\hat{R_\mathfrak{m}}}(-, E)$. Since $\mathrm{Att}_{\hat{R_\mathfrak{m}}} M$ (resp. $\mathrm{Att}_{\hat{R_\mathfrak{m}}} N$) is equal to $\mathrm{Ass}_{\hat{R_\mathfrak{m}}} M^\vee$ (resp. $\mathrm{Ass}_{\hat{R_\mathfrak{m}}} N^\vee$) (Lemma 24), the inclusion

$$\mathrm{Ass}_{\hat{R_\mathfrak{m}}} M^\vee \subseteq \mathrm{Ass}_{\hat{R_\mathfrak{m}}} N^\vee$$

holds. By virtue of Theorem 23, we conclude that $M^\vee$ can be generated from $N^\vee$ via taking submodules and extensions, i.e. $M^\vee \in \text{sub-ext}_{\hat{R_\mathfrak{m}}}(N^\vee)$. Hence it follows from Matlis duality and Lemma 4 that

$$M^{\vee\vee} \cong M \in \text{sub-ext}_{\hat{R_\mathfrak{m}}}(N^\vee)^\vee = \text{quot-ext}_{\hat{R_\mathfrak{m}}}(N).$$

Since artinian $\hat{R_\mathfrak{m}}$-modules are also artinian R-modules (cf. Lemma 3), we conclude that $M \in \text{quot-ext}_R(N)$.

□

We define by Ψ the map sending a subcategory $\mathcal{X}$ of $\mathrm{Art}(R)$ to

$$\mathrm{Att}\mathcal{X} = \cup_{M\in\mathcal{X}} \mathrm{Att}_{\hat{R}} M$$

and by Φ the map sending a subset S of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$ to

$$\{M \in \mathrm{Art}(R)\ |\mathrm{Att}_{\hat{R}} M \subseteq S\}.$$

Note from Corollary 22 that $\Psi(\mathcal{X})$ is a subset of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$. On the other hand, it follows from Remark 13 (2) that $\Phi(S)$ is closed under quotient modules and extensions.

Now we state the main theorem of this note.

Theorem 26. *Let R be a commutative noetherian ring. Then Ψ and Φ induce an inclusion preserving bijection between the set of subcategories of $\mathrm{Art}(R)$ which are closed under quotient modules and extensions and the set of subsets of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$.*

Moreover, they also induce an inclusion preserving bijection between the set of Serre subcategories of $\mathrm{Art}(R)$ and the set of specialization closed subsets of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$.

Proof. We show the first assertion of the theorem.

Let $\mathcal{X}$ be a subcategory of $\mathrm{Art}(R)$ which is closed under quotient modules and extensions. The subcategory $\Phi\Psi(\mathcal{X})$ consists of all artinian R-modules M with $\mathrm{Att}_{\hat{R}} M \subseteq \cup_{X\in\mathcal{X}} \mathrm{Att}_{\hat{R}} X$. It is clear that $\mathcal{X}$ is a subcategory of $\Phi\Psi(\mathcal{X})$. Let M be an artinian R-module with $\mathrm{Att}_{\hat{R}} M \subseteq \cup_{X\in\mathcal{X}} \mathrm{Att}_{\hat{R}} X$. For each ideal $\mathfrak{P} \in \mathrm{Att}_{\hat{R}} M$, there exists $X^{(\mathfrak{P})} \in \mathcal{X}$ such that $\mathfrak{P} \in \mathrm{Att}_{\hat{R}} X^{(\mathfrak{P})}$. Take the direct sums of such objects, that is $X = \oplus_{\mathfrak{P}\in\mathrm{Att}_{\hat{R}} M} X^{(\mathfrak{P})}$. X is also an object of $\mathcal{X}$, since $\mathrm{Att}_{\hat{R}} M$ is a finite set and $\mathcal{X}$ is closed under finite direct sums. It follows from the definition of X that $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} X$. By virtue of Theorem 25, M is contained in $\text{quot-ext}_R(X)$, so that M in $\mathcal{X}$. Hence we have the equality $\mathcal{X} = \Phi\Psi(\mathcal{X})$.

Let S be a subset of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$. It is trivial that the set $\Psi\Phi(S)$ is contained in S. Let $\mathfrak{p}$ be a prime ideal in S. Take a maximal ideal $\mathfrak{m}$ so that $\mathfrak{p}$ is a prime ideal of $\hat{R_\mathfrak{m}}$. We consider an $\hat{R_\mathfrak{m}}$-module $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$. Then we have the equality:

$$\mathrm{Att}_{\hat{R_\mathfrak{m}}} E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}}) = \mathrm{Ass}_{\hat{R_\mathfrak{m}}} \hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}} = \{\mathfrak{p}\}.$$

Note that $E_{\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ is artinian as an R-module. Indeed, we have the equality $E_{\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}}) = E_R(R/\mathfrak{m}R)$ as R-modules ([5, Theorem 18.6 (iii)]), so that it is an artinian R-module since $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ is an $\hat{R_\mathfrak{m}}$-submodule (thus an R-submodule) of $E_{\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$. Hence $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ is an artinian R-module which is a $\mathfrak{p}$-secondary $\hat{R_\mathfrak{m}}$-module. Consequently, $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ belongs to $\Phi(S)$, so that $\mathfrak{p} \in \Psi\Phi(S)$. $\square$

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