GROUP-GRADED AND GROUP-BIGRADED RINGS

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Abstract. Let I be a non-trivial finite multiplicative group with the unit element e and $A = \oplus_{x \in I} A_x$ an I-graded ring. We construct a Frobenius extension Λ of A and study when the ring extension A of A_e can be a Frobenius extension. Also, formulating the ring structure of Λ, we introduce the notion of I-bigraded rings and show that every I-bigraded ring is isomorphic to the I-bigraded ring Λ constructed above.

Introduction

Let I be a non-trivial finite multiplicative group with the unit element e and $A = \oplus_{x \in I} A_x$ an I-graded ring. In this note, assuming A_e is a local ring, we study when a ring extension A of A_e can be a Frobenius extension, the notion of which we recall below. Auslander-Gorenstein rings (see Definition 2) appear in various fields of current research in mathematics. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [5], [6] and [14], respectively). However, little is known about constructions of Auslander-Gorenstein rings. We have shown in [9, Section 3] that a left and right noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring. A Frobenius extension A of a left and right noetherian ring R is a typical example such that A admits an Auslander-Gorenstein resolution over R.

Now we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [11, 12] which we modify as follows (cf. [1, Section 1]). We use the notation A/R to denote that a ring A contains a ring R as a subring. We say that A/R is a Frobenius extension if the following conditions are satisfied: (F1) A is finitely generated as a left R-module; (F2) A is finitely generated projective as a right R-module; (F3) there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A. Note that ϕ induces a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of R. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let A/R be a Frobenius extension. Then A is an Auslander-Gorenstein ring if so is R, and the converse holds true if A is projective as a left R-module, and if A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. It should be noted that A is projective as a left R-module if A/R is of second kind.

The detailed version of this paper will be submitted for publication elsewhere.

To state our main theorem we have to construct a Frobenius extension Λ/A of first kind. Namely, we will define an appropriate multiplication on a free right A-module Λ with a basis $\{v_x\}_{x\in I}$ so that Λ/A is a Frobenius extension of first kind. Denote by $\{\gamma_x\}_{x\in I}$ the dual basis of $\{v_x\}_{x\in I}$ for the free left A-module $\mathrm{Hom}_A(\Lambda, A)$ and set $\gamma = \Sigma_{x\in I}\gamma_x$. Assume A_e is local, $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and A is reflexive as a right A_e-module. Our main theorem states that the following are equivalent: (1) $A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right A-modules; (2) There exist a unique $s \in I$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that $\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$ for all $x \in I$; (3) There exist a unique $s \in I$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in I$ (Theorem 18). Assume A/A_e is a Frobenius extension. We show that it is of second kind (Corollary 20), and that A is an Auslander-Gorenstein ring if and only if so is Λ (Theorem 21).

As we saw above, the ring Λ plays an essential role in our argument. Formulating the ring structure of Λ, we introduce the notion of group-bigraded rings as follows. A ring Λ together with a group homomorphism $\eta : I^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$ is said to be an I-bigraded ring, denoted by (Λ, η), if $1 = \sum_{x\in I} v_x$ with the v_x orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in I$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in I$. We conclude that every I-bigraded ring is isomorphic to the I-bigraded ring Λ constructed above (Proposition 24).

This note is organized as follows. In Section 1, we recall basic facts on Auslander-Gorenstein rings and Frobenius extensions. In Section 2, we construct a Frobenius extension Λ/A of first kind and study the ring structure of Λ. In Section 3, we prove the main theorem. In Section 4, we introduce the notion of group-bigraded rings and study the structure of such rings.

1. Preliminaries

For a ring R we denote by $\mathrm{rad}(R)$ the Jacobson radical of R, by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. Usually, the identity element of a ring is simply denoted by 1. Sometimes, we use the notation 1_R to stress that it is the identity element of the ring R. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}). Sometimes, we use the notation X_R (resp., ${}_RX$) to stress that the module X considered is a right (resp., left) R-module.

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander)**.** *Let R be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

(1) *In a minimal injective resolution $I^\bullet$ of R in* Mod-R*,* flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
(2) *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}*,* flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.

(3) *For any* $1 \leq i \leq n+1$, *any* $M \in \text{mod-}R$ *and any submodule* X *of* $\text{Ext}^i_R(M,R) \in \text{mod-}R^{\text{op}}$ *we have* $\text{Ext}^j_{R^{\text{op}}}(X,R) = 0$ *for all* $0 \leq j < i$.

(4) *For any* $1 \leq i \leq n+1$, *any* $X \in \text{mod-}R^{\text{op}}$ *and any submodule* M *of* $\text{Ext}^i_{R^{\text{op}}}(X,R) \in \text{mod-}R$ *we have* $\text{Ext}^j_R(M,R) = 0$ *for all* $0 \leq j < i$.

Proof. See e.g. [7, Theorem 3.7]. □

Definition 2 ([6])**.** A right and left noetherian ring R is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and inj dim $R =$ inj dim $R^{\text{op}} < \infty$.

It should be noted that for a right and left noetherian ring R we have inj dim $R =$ inj dim R^{op} whenever inj dim $R < \infty$ and inj dim $R^{\text{op}} < \infty$ (see [15, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [11, 12], which we modify as follows (cf. [1, Section 1]).

Definition 3. A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:

(F1) A is finitely generated as a left R-module;

(F2) A is finitely generated projective as a right R-module;

(F3) $A \cong \text{Hom}_R(A,R)$ as right A-modules.

In case R is a right and left noetherian ring, for any Frobenius extension A/R the isomorphism $A \xrightarrow{\sim} \text{Hom}_R(A,R)$ in Mod-A yields an Auslander-Gorenstein resolution of A over R in the sense of [9, Definition 3.5].

The next proposition is well-known and easily verified.

Proposition 4. *Let* A/R *be a ring extension and* $\phi : A \xrightarrow{\sim} \text{Hom}_R(A,R)$ *an isomorphism in* Mod-A*. Then the following hold.*

(1) *There exists a unique ring homomorphism* $\theta : R \to A$ *such that* $x\phi(1) = \phi(1)\theta(x)$ *for all* $x \in R$.

(2) *If* $\phi' : A \xrightarrow{\sim} \text{Hom}_R(A,R)$ *is another isomorphism in* Mod-A, *then there exists* $u \in A^\times$ *such that* $\phi'(1) = \phi(1)u$ *and* $\theta'(x) = u^{-1}\theta(x)u$ *for all* $x \in R$.

(3) ϕ *is an isomorphism of* R-A*-bimodules if and only if* $\theta(x) = x$ *for all* $x \in R$.

Definition 5 (cf. [11, 12])**.** A Frobenius extension A/R is said to be of first kind if $A \cong \text{Hom}_R(A,R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \text{Hom}_R(A,R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

Proposition 6. *If* A/R *is a Frobenius extension of second kind, then* A *is projective as a left* R*-module.*

Proof. Let $\phi : A \xrightarrow{\sim} \text{Hom}_R(A,R)$ be an isomorphism in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Then θ induces an equivalence $U_\theta : \text{Mod-}R^{\text{op}} \xrightarrow{\sim} \text{Mod-}R^{\text{op}}$ such that for any $M \in \text{Mod-}R^{\text{op}}$ we have $U_\theta M = M$ as an additive group and the left R-module structure of $U_\theta M$ is given by the law of composition $R \times M \to M, (x, m) \mapsto \theta(x)m$. Since ϕ yields an isomorphism of R-A-bimodules $U_\theta A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$, and since $\mathrm{Hom}_R(A, R)$ is projective as a left R-module, it follows that $U_\theta A$ and hence A are projective as left R-modules. □

Proposition 7. *For any Frobenius extensions Λ/A, A/R the following hold.*

(1) *Λ/R is a Frobenius extension.*

(2) *Assume Λ/A is of first kind. If A/R is of second (resp., first) kind, then so is Λ/R.*

Proof. (1) Obviously, (F1) and (F2) are satisfied. Also, we have

$$\begin{aligned}\Lambda &\cong \mathrm{Hom}_A(\Lambda, A)\\ &\cong \mathrm{Hom}_A(\Lambda, \mathrm{Hom}_R(A, R))\\ &\cong \mathrm{Hom}_R(\Lambda \otimes_A A, R)\\ &\cong \mathrm{Hom}_R(\Lambda, R)\end{aligned}$$

in Mod-Λ.

(2) Let $\psi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A)$ be an isomorphism of A-Λ-bimodules and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Setting $\gamma = \psi(1)$ and $\alpha = \phi(1)$, as in (1), we have an isomorphism in Mod-Λ

$$\xi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, R), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

For any $x \in R$, we have

$$\begin{aligned}x\xi(1)(\mu) &= x\alpha(\gamma(\mu))\\ &= \alpha(\theta(x)\gamma(\mu))\\ &= \alpha(\gamma(\theta(x)\mu))\\ &= \xi(1)(\theta(x)\mu)\end{aligned}$$

for all $\mu \in \Lambda$ and $x\xi(1) = \xi(1)\theta(x)$. □

Definition 8 ([1])**.** A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 9 (cf. [1])**.** *For any Frobenius extension A/R the following hold.*

(1) *If R is an Auslander-Gorenstein ring, then so is A with* inj dim $A \le$ inj dim R.

(2) *Assume A is projective as a left R-module and A/R is split. If A is an Auslander-Gorenstein ring, then so is R with* inj dim $R =$ inj dim A.

Proof. (1) See [9, Theorem 3.6].

(2) It follows by [1, Proposition 1.7] that R is a right and left noetherian ring with inj dim $R =$ inj dim $R^{\mathrm{op}} =$ inj dim A. Let $A \to E^\bullet$ be a minimal injective resolution in Mod-A. For any $i \ge 0$, $\mathrm{Hom}_R(-, E^i) \cong \mathrm{Hom}_A(- \otimes_R A, E^i)$ as functors on Mod-R and E^i_R is injective, and $E^i \otimes_R - \cong E^i \otimes_A A \otimes_R -$ as functors on Mod-R^{op} and flat dim $E^i_R \le$ flat dim $E^i_A \le i$. Now, since R_R appears in A_R as a direct summand, it follows that R satisfies the Auslander condition. □

2. Graded rings

Throughout the rest of this note, I stands for a non-trivial finite multiplicative group with the unit element e.

Throughout this and the next sections, we fix a ring A together with a family $\{\delta_x\}_{x\in I}$ in $\mathrm{End}_{\mathbb{Z}}(A)$ satisfying the following conditions:

(D1) $\delta_x\delta_y = 0$ unless $x = y$ and $\sum_{x\in I}\delta_x = \mathrm{id}_A$;

(D2) $\delta_x(a)\delta_y(b) = \delta_{xy}(\delta_x(a)b)$ for all $a, b \in A$ and $x, y \in I$.

Namely, setting $A_x = \mathrm{Im}\,\delta_x$ for $x \in I$, $A = \oplus_{x\in I}A_x$ is an I-graded ring. In particular, A/A_e is a split ring extension.

To prove our main theorem (Theorem 18), we need an extension ring Λ of A such that Λ/A is a Frobenius extension of first kind. Let Λ be a free right A-module with a basis $\{v_x\}_{x\in I}$ and define a multiplication on Λ subject to the following axioms:

(M1) $v_xv_y = 0$ unless $x = y$ and $v_xv_x = v_x$ for all $x \in I$;

(M2) $av_x = \sum_{y\in I} v_y\delta_{yx^{-1}}(a)$ for all $a \in A$ and $x \in I$.

We denote by $\{\gamma_x\}_{x\in I}$ the dual basis of $\{v_x\}_{x\in I}$ for the free left A-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{x\in I} v_x\gamma_x(\lambda)$ for all $\lambda \in \Lambda$. It is not difficult to see that

$$\lambda\mu = \sum_{x,y\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)$$

for all $\lambda, \mu \in \Lambda$. Also, setting $\gamma = \sum_{x\in I}\gamma_x$, we define a mapping

$$\phi : \Lambda \to \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda.$$

Proposition 10. *The following hold.*

1. *Λ is an associative ring with $1 = \sum_{x\in I} v_x$ and contains A as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{x\in I} v_xa$.*
2. *ϕ is an isomorphism of A-Λ-bimodules, i.e., Λ/A is a Frobenius extension of first kind.*

Proof. (1) Let $\lambda \in \Lambda$. Obviously, $\sum_{x\in I} v_x \cdot \lambda = \lambda$. Also, by (D1) we have

$$\begin{aligned}\lambda \cdot \sum_{y\in I} v_y &= \sum_{x,y\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda)) \\ &= \sum_{x\in I} v_x\gamma_x(\lambda) \\ &= \lambda.\end{aligned}$$

Next, for any $\lambda, \mu, \nu \in \Lambda$ by (D2) we have

$$\begin{aligned}(\lambda\mu)\nu &= \sum_{x,y,z\in I} v_x\delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu))\gamma_z(\nu) \\ &= \sum_{x,y,z\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu))\gamma_z(\nu) \\ &= \lambda(\mu\nu).\end{aligned}$$

The remaining assertions are obvious.

(2) Let $\lambda \in \operatorname{Ker} \phi$. For any $y \in I$ we have $0 = \gamma(\lambda v_y) = \sum_{x \in I} \delta_{xy^{-1}}(\gamma_x(\lambda))$ and $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $x \in I$. Thus for any $x \in I$ we have $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $y \in I$ and by (D1) $\gamma_x(\lambda) = 0$, so that $\lambda = 0$. Next, for any $f = \sum_{x \in I} a_x \gamma_x \in \operatorname{Hom}_A(\Lambda, A)$, setting $\lambda = \sum_{x,z \in I} v_x \delta_{xz^{-1}}(a_z)$, by (D1) we have

$$\begin{aligned}(\gamma\lambda)(v_y) &= \gamma(\lambda v_y) \\ &= \sum_{x \in I} \delta_{xy^{-1}}(\gamma_x(\lambda)) \\ &= \sum_{x,z \in I} \delta_{xy^{-1}}(\delta_{xz^{-1}}(a_z)) \\ &= a_y \\ &= f(v_y)\end{aligned}$$

for all $y \in I$ and $f = \gamma\lambda$. Finally, for any $a \in A$ by (D1) we have

$$\begin{aligned}(\gamma a)(\lambda) &= \gamma(a\lambda) \\ &= \sum_{x,y \in I} \delta_{yx^{-1}}(a)\gamma_x(\lambda) \\ &= a\gamma(\lambda)\end{aligned}$$

for all $\lambda \in \Lambda$ and $\gamma a = a\gamma$. □

Remark 11. Denote by $|I|$ the order of I. If $|I| \cdot 1_A \in A^\times$, then Λ/A is a split ring extension.

Lemma 12. *The following hold.*

(1) $v_x \lambda v_y = v_x \delta_{xy^{-1}}(\gamma_x(\lambda))$ *for all* $\lambda \in \Lambda$ *and* $x, y \in I$.
(2) $v_x \Lambda v_y = v_x A_{xy^{-1}}$ *for all* $x, y \in I$.
(3) $v_x a \cdot v_y b = v_x ab$ *for all* $x, y, z \in I$ *and* $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.

Proof. Immediate by the definition. □

Setting $\Lambda_{x,y} = v_x \Lambda v_y$ for $x, y \in I$, we have $\Lambda = \oplus_{x,y \in I} \Lambda_{x,y}$ with $\Lambda_{x,y}\Lambda_{z,w} = 0$ unless $y = z$ and $\Lambda_{x,y}\Lambda_{y,z} \subseteq \Lambda_{x,z}$ for all $x, y, z \in I$. Also, setting $\lambda_{x,y} = \delta_{xy^{-1}}(\gamma_x(\lambda)) \in A_{xy^{-1}}$ for $\lambda \in \Lambda$ and $x, y \in I$, we have a group homomorphism

$$\eta : I^{\mathrm{op}} \to \operatorname{Aut}(\Lambda), x \mapsto \eta_x$$

such that $\eta_x(\lambda)_{y,z} = \lambda_{yx^{-1},zx^{-1}}$ for all $\lambda \in \Lambda$ and $x, y, z \in I$. We denote by Λ^I the subring of Λ consisting of all λ such that $\eta_x(\lambda) = \lambda$ for all $x \in I$.

Proposition 13. *The following hold.*

(1) $\eta_y(v_x) = v_{xy}$ *for all* $x, y \in I$.
(2) $\Lambda^I = A$.
(3) $(\lambda\mu)_{x,z} = \sum_{y \in I} \lambda_{x,y}\mu_{y,z}$ *for all* $\lambda, \mu \in \Lambda$ *and* $x, z \in I$.

Proof. (1) Since $\eta_y(v_x)_{z,w} = \delta_{zw^{-1}}(\gamma_{zy^{-1}}(v_x))$ for all $z, w \in I$, we have

$$\eta_y(v_x)_{z,w} = \begin{cases} 1 & \text{if } z = w \text{ and } x = zy^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $a \in A$, since $\eta_x(a)_{y,z} = a_{yx^{-1},zx^{-1}} = \delta_{(yx^{-1})(zx^{-1})^{-1}}(a) = \delta_{yz^{-1}}(a) = a_{y,z}$ for all $x, y, z \in I$, we have $a \in \Lambda^I$. Conversely, for any $\lambda \in \Lambda^I$ we have $\delta_{y^{-1}}(\gamma_x(\lambda)) = \lambda_{x,yx} = \eta_{x^{-1}}(\lambda)_{e,y} = \lambda_{e,y} = \delta_{y^{-1}}(\gamma_e(\lambda))$ for all $x, y \in I$, so that $\gamma_x(\lambda) = \gamma_e(\lambda)$ for all $x \in I$.

(3) For any $\lambda, \mu \in \Lambda$ and $x, z \in I$ by (D2) we have

$$\begin{aligned}(\lambda\mu)_{x,z} &= \sum_{y\in I} \delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)) \\ &= \sum_{y\in I} \delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu)) \\ &= \sum_{y\in I} \lambda_{x,y}\mu_{y,z}.\end{aligned}$$

□

Remark 14. We have $\eta_y(v_x a_x)v_y b_y = v_{xy} a_x b_y$ for all $a_x \in A_x$ and $b_y \in A_y$.

Proposition 15. *The following hold.*

(1) $\mathrm{End}_\Lambda(v_x\Lambda) \cong A_e$ *as rings for all* $x \in I$.
(2) $v_x\Lambda \not\cong v_y\Lambda$ *in* Mod-Λ *for all* $x, y \in I$ *with* $A_{xy^{-1}}A_{yx^{-1}} \subsetneq \mathrm{rad}(A_e)$.

Proof. (1) We have $\mathrm{End}_\Lambda(v_x\Lambda) \cong v_x\Lambda v_x \cong A_e$ as rings.

(2) For any $f : v_x\Lambda \to v_y\Lambda$ and $g : v_y\Lambda \to v_x\Lambda$ in Mod-Λ, since $f(v_x) = v_y a$ with $a \in A_{yx^{-1}}$ and $g(v_y) = v_x b$ with $b \in A_{xy^{-1}}$, we have $g(f(v_x)) = v_x ba$ with $ba \in \mathrm{rad}(A_e)$. □

The proposition above asserts that if A_e is local and $A_x A_{x^{-1}} \subsetneq \mathrm{rad}(A_e)$ for all $x \neq e$ then Λ is semiperfect and basic. We refer to [3] for semiperfect rings.

3. Auslander-Gorenstein rings

In this section, we will ask when A/A_e is a Frobenius extension.

Lemma 16. *For any* $x \in I$ *the following hold.*

(1) $av_x = v_x a$ *for all* $a \in A_e$ *and* Λv_x *is a* Λ-A_e-*bimodule.*
(2) $\Lambda v_x = \sum_{y\in I} v_y A_{yx^{-1}}$.
(3) $A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y\in I} v_y \delta_{yx^{-1}}(a)$ *as* A-A_e-*bimodules.*
(4) *If* Λv_x *is reflexive as a right* A_e-*module, then* $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong A_e$ *as rings.*

Proof. (1) and (2) Immediate by the definition.

(3) By (2) we have a bijection $f_x : A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y\in I} v_y\delta_{yx^{-1}}(a)$. Since every $\delta_{yx^{-1}}$ is a homomorphism in Mod-A_e, so is f_x. Finally, for any $a, b \in A$ we have

$$\begin{aligned} a\cdot(\sum_{y\in I} v_y\delta_{yx^{-1}}(b)) &= \sum_{y,z\in I} v_z\delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b) \\ &= \sum_{z\in I} v_z(\sum_{y\in I}\delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b)) \\ &= \sum_{z\in I} v_z\delta_{zx^{-1}}(\sum_{y\in I}\delta_{zy^{-1}}(a)b) \\ &= \sum_{z\in I} v_z\delta_{zx^{-1}}(ab) \end{aligned}$$

and f_x is a homomorphism in Mod-A^{op}.

(4) Since the canonical homomorphism

$$\Lambda v_x \to \mathrm{Hom}_{A_e^{\mathrm{op}}}(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e), A_e), \lambda \mapsto (f \mapsto f(\lambda))$$

is an isomorphism, $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(\Lambda v_x)^{\mathrm{op}} \cong v_x\Lambda v_x \cong A_e$ as rings. □

It follows by Lemma 16(1) that $\delta_e\gamma_e : \Lambda \to A_e$ is a homomorphism of A_e-A_e-bimodules and Λ/A_e is a split ring extension.

Lemma 17. *For any $x, y \in I$ and $a, b \in A$ we have*

$$v_x a\cdot(\sum_{z\in I} v_z\delta_{zy^{-1}}(b)) = v_x(\sum_{z\in I}\delta_{xz^{-1}}(a)\delta_{zy^{-1}}(b))$$

Proof. Immediate by the definition. □

Theorem 18. *Assume A_e is local, $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and A is reflexive as a right A_e-module. Then the following are equivalent.*

(1) *$A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right A-modules.*

(2) *There exist a unique $s \in I$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that*

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

for all $x \in I$.

(3) *There exist a unique $s \in I$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that*

$$\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$$

for all $x \in I$.

Proof. (1) ⇒ (2). Let $A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), 1 \mapsto \alpha$ in Mod-A. Then, since by Proposition 10(2) $\Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda$ in Mod-Λ, by adjointness we have an isomorphism in Mod-Λ

$$\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

By Proposition 15(1) $\Lambda = \oplus_{x\in I} v_x\Lambda$ with the $\mathrm{End}_\Lambda(v_x\Lambda)$ local. Also, by (1) and (4) of Lemma 16

$$\mathrm{Hom}_{A_e}(\Lambda, A_e) \cong \oplus_{x\in I}\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)$$

with the $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e))$ local. Now, according to Proposition 15(2), it follows by the Krull-Schmidt theorem that there exists a unique $s \in I$ such that

$$\phi_{s,e} : v_s\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_e, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

Thus, setting $\alpha_s = \alpha|_{A_s}$, by Lemmas 16(2) and 17 we have

$$\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab))).$$

It then follows again by Lemmas 16(2) and 17 that

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

for all $x \in I$.

(2) $\Rightarrow$ (3). Since $A = \oplus_{x\in I} A_{sx} = \oplus_{x\in I} A_{x^{-1}}$, and since $A_{sx}A_{x^{-1}} \subseteq A_s$ for all $x \in I$, ψ induces $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in I$.

(3) $\Rightarrow$ (1). Setting $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for each $x \in I$, the ψ_x yields $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$. □

Remark 19. In the theorem above, α_s is an isomorphism and $A_e \xrightarrow{\sim} \mathrm{End}_{A_e}(A_s)$ canonically.

Proof. For any $b \in A_e$, setting $f : A_e \to A_e, 1 \mapsto b$, we have $f = \psi_e(a)$ and hence $b = \alpha_s(a)$ for some $a \in A_s$. Also, Ker α_s = Ker $\psi_s = 0$. Then, since the composite $A_e \to \mathrm{End}_{A_e}(A_s) \to \mathrm{Hom}_{A_e}(A_s, A_e)$ is an isomorphism, the last assertion follows. □

Corollary 20. *Assume A_e is local and $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$. If A/A_e is a Frobenius extension, then it is of second kind.*

Proof. Set $t = \alpha_s^{-1}(1) \in A_s$. Then for any $u \in A_s$ there exists $f \in \mathrm{End}_{A_e}(A_s)$ such that $u = f(t)$ and hence $u = at$ for some $a \in A_e$. Thus $A_et = A_s$ and there exists $\theta \in \mathrm{Aut}(A_e)$ such that $\theta(a)t = ta$ for all $a \in A_e$. Then $(\alpha_s\theta(a))(t) = \alpha_s(\theta(a)t) = \alpha_s(ta) = \alpha_s(t)a = a = (a\alpha_s)(t)$ and $\alpha_s\theta(a) = a\alpha_s$ for all $a \in A_e$. Now, setting $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$, we have $(a\psi(1))(b) = a\alpha_s(\delta_s(b)) = (a\alpha_s)(\delta_s(b)) = (\alpha_s\theta(a))(\delta_s(b)) = \alpha_s(\theta(a)\delta_s(b)) = \alpha_s(\delta_s(\theta(a)b)) = (\psi(1)\theta(a))(b)$ for all $a, b \in A$, so that $a\psi(1) = \psi(1)\theta(a)$ for all $a \in A$. □

Theorem 21. *Assume A_e is local, $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$, and A/A_e is a Frobenius extension. Then A is an Auslander-Gorenstein ring if and only if so is Λ.*

Proof. The "only if" part follows by Propositions 9(1) and 10(2). Assume Λ is an Auslander-Gorenstein ring. By Proposition 10(2) Λ/A is a Frobenius extension of first kind, and by Corollary 20 A/A_e is a Frobenius extension of second kind. Thus by Proposition 7 Λ/A_e is a Frobenius extension of second kind. Also, by Lemma 16(1) Λ/A_e is split. Hence by Propositions 6 and 9(2) A_e is an Auslander-Gorenstein ring and by Proposition 9(1) so is A. □

4. Bigraded rings

Formulating the ring structure of Λ constructed in Section 2, we make the following.

Definition 22. A ring Λ together with a group homomorphism

$$\eta : I^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$$

is said to be an I-bigraded ring, denoted by (Λ, η), if $1 = \sum_{x\in I} v_x$ with the v_x orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in I$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in I$.

Throughout this section, we fix an I-bigraded ring (Λ, η). Set $A_x = v_x\Lambda v_e$ for $x \in I$ and $A = \oplus_{x\in I} A_x$. Note that $\eta_y(A_x) = v_{xy}\Lambda v_y$ for all $x, y \in I$. For any $a_x \in A_x$ and $b_y \in A_y$ we define the multiplication $a_x \cdot b_y$ in A as the multiplication $\eta_y(a_x)b_y$ in Λ (cf. Remark 14).

Proposition 23. *The following hold.*

(1) *A is an associative ring with $1 = v_e$.*
(2) *A is an I-graded ring.*

Proof. (1) For any $a_x \in A_x$, $b_y \in A_y$ and $c_z \in A_z$ we have

$$\begin{aligned}(a_x \cdot b_y) \cdot c_z &= \eta_y(a_x)b_y \cdot c_z \\ &= \eta_z(\eta_y(a_x)b_y)c_z \\ &= \eta_{yz}(a_x)\eta_z(b_y)c_z \\ &= a_x \cdot (b_y \cdot c_z).\end{aligned}$$

Also, for any $a_x \in A_x$ we have $v_e \cdot a_x = \eta_x(v_e)a_x = v_xa_x = a_x$ and $a_x \cdot v_e = \eta_e(a_x)v_e = a_xv_e = a_x$.

(2) Obviously, $A_xA_y \subseteq A_{xy}$ for all $x, y \in I$. □

In the following, for each $x \in I$ we denote by $\delta_x : A \to A_x$ the projection. Then, setting $\lambda_{x,y} = v_x\lambda v_y$ for $\lambda \in \Lambda$ and $x, y \in I$, we have a mapping $\varphi : A \to \Lambda$ such that $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a))$ for all $a \in A$ and $x, y \in I$.

Proposition 24. *The following hold.*

(1) *$\varphi : A \to \Lambda$ is an injective ring homomorphism with* $\mathrm{Im}\ \varphi = \Lambda^I$.
(2) *$v_x\Lambda v_y = v_x\varphi(A_{xy^{-1}})$ for all $x, y \in I$.*
(3) *$\{v_x\}_{x\in I}$ is a basis for the right A-module Λ.*
(4) *$\varphi(a)v_x = \sum_{y\in I} v_y\varphi(\delta_{yx^{-1}}(a))$ for all $a \in A$ and $x \in I$.*
(5) *$v_x\varphi(a)v_y\varphi(b) = v_x\varphi(ab)$ for all $x, y, z \in I$ and $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.*

Proof. (1) Obviously, φ is a monomorphism of additive groups. Also, we have

$$\varphi(v_e)_{x,y} = \begin{cases} v_x & \text{if } x = y, \\ 0 & \text{otherwise}\end{cases}$$

and $\varphi(1_A) = 1_\Lambda$. Let $a_x \in A_x, b_y \in A_y$ and $z, w \in I$. Since $\varphi(a_x \cdot b_y)_{z,w} = \varphi(\eta_y(a_x)b_y)_{z,w} = \eta_w(\delta_{zw^{-1}}(\eta_y(a_x)b_y))$, $\varphi(a_x \cdot b_y)_{z,w} = 0$ unless $xy = zw^{-1}$. If $xy = zw^{-1}$, then $\eta_w(\delta_{zw^{-1}}(\eta_y(a_x)b_y)) =$

$\eta_{yw}(a_x)\eta_w(b_y)$. On the other hand,

$$(\varphi(a_x)\varphi(b_y))_{z,w} = \sum_{u\in I} \varphi(a_x)_{z,u}\varphi(b_y)_{u,w} = \sum_{u\in I} \eta_u(\delta_{zu^{-1}}(a_x))\eta_w(\delta_{uw^{-1}}(b_y)).$$

Thus $(\varphi(a_x)\varphi(b_y))_{z,w} = 0$ unless $zu^{-1} = x$ and $uw^{-1} = y$, i.e., $zw^{-1} = xy$. If $zw^{-1} = xy$, then $\sum_{u\in I} \eta_u(\delta_{zu^{-1}}(a_x))\eta_w(\delta_{uw^{-1}}(b_y)) = \eta_{yw}(a_x)\eta_w(b_y)$. As a consequence, $\varphi(a_x \cdot b_y)_{z,w} = (\varphi(a_x)\varphi(b_y))_{z,w}$. The first assertion follows.

Next, for any $a \in A$ and $x, y, z \in I$ we have

$$\begin{aligned}\eta_x(\varphi(a))_{y,z} &= v_y\eta_x(\varphi(a))v_z \\ &= \eta_x(v_{yx^{-1}}\varphi(a)v_{zx^{-1}}) \\ &= \eta_x(\varphi(a)_{yx^{-1},zx^{-1}}) \\ &= \eta_x(\eta_{zx^{-1}}(\delta_{yz^{-1}}(a))) \\ &= \eta_z(\delta_{yz^{-1}}(a)) \\ &= \varphi(a)_{y,z},\end{aligned}$$

so that $\mathrm{Im}\,\varphi \subseteq \Lambda^I$. Conversely, let $\lambda \in \Lambda^I$. Then $\lambda_{x,y} = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e}$ for all $x, y \in I$. Thus, setting $a = \sum_{x\in I} \lambda_{x,e}$, we have $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a)) = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e} = \lambda_{x,y}$ for all $x, y \in I$ and $\varphi(a) = \lambda$.

(2) Let $x, y \in I$ and $a \in A_{xy^{-1}}$. For any $z \neq y$ we have $\delta_{xz^{-1}}(a) = 0$ and hence $v_x\varphi(a)v_z = \varphi(a)_{x,z} = \eta_z(\delta_{xz^{-1}}(a)) = 0$. Thus $v_x\varphi(a) = \varphi(a)_{x,y} = \eta_y(a)$. It follows that $v_x\Lambda v_y = \eta_y(v_{xy^{-1}}\Lambda v_e) = \eta_y(A_{xy^{-1}}) = v_x\varphi(A_{xy^{-1}})$.

(3) This follows by (2).

(4) Note that $\eta_x(\delta_{yx^{-1}}(a)) = v_y\eta_x(\delta_{yx^{-1}}(a))$ for all $y \in I$. Thus $\varphi(a)v_x = \sum_{y\in I} v_y\varphi(a)v_x = \sum_{y\in I} \eta_x(\delta_{yx^{-1}}(a)) = \sum_{y\in I} v_y\eta_x(\delta_{yx^{-1}}(a))$. Also,

$$\begin{aligned}v_y\varphi(\delta_{yx^{-1}}(a)) &= \sum_{z\in I} v_y\varphi(\delta_{yx^{-1}}(a))v_z \\ &= \sum_{z\in I} v_y\eta_z(\delta_{yz^{-1}}(\delta_{yx^{-1}}(a))) \\ &= v_y\eta_x(\delta_{yx^{-1}}(a))\end{aligned}$$

for all $y \in I$.

(5) This follows by (2) and (4). □

Let us call the I-bigraded ring constructed in Section 2 standard. Then the proposition above asserts that every I-bigraded ring is isomorphic to a standard one. Namely, according to Lemma 12, $\varphi : A \to \Lambda$ can be extended to an isomorphism of I-bigraded rings.

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