THE DIMENSION FORMULA OF THE CYCLIC HOMOLOGY OF TRUNCATED QUIVER ALGEBRAS OVER A FIELD OF POSITIVE CHARACTERISTIC

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Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Tokyo University of Science, Japan, 10-12 October 2013. In this paper, we give the dimension formula of the cyclic homology of truncated quiver algebras over a field of positive characteristic. This is done by using a mixed complex due to Cibils.

1. Introduction

Let Δ be a finite quiver and K a field. We fix a positive integer $m \geq 2$. The truncated quiver algebra is defined by $K\Delta/R_\Delta^m$ where R_Δ^m is the two-sided ideal of $K\Delta$ generated by the all paths of length m.

In [8], Sköldberg computes the Hochschild homology of a truncated quiver algebra A over a commutative ring using an explicit description of the minimal left A^e-projective resolution $\boldsymbol{P}$ of A. He also computes the Hochschild homology of quadratic monomial algebras. On the other hand, Cibils gives a useful projective resolution $\boldsymbol{Q}$ for more general algebras in [3].

If A is a K-algebra with a decomposition $A = E \oplus r$, where E is a separable subalgebra of A and r a two-sided ideal of A, then Cibils ([4]) gives the *E-normalized mixed complex*. Sköldberg [9] gives the chain maps between the left A^e-projective resolution given in [8] and $\boldsymbol{Q}$ above for a quadratic monomial algebra A, and he obtains the module structure of the cyclic homology by computing the E^2-term of a spectral sequence determined by the above mixed complex due to Cibils.

In [1], Ames, Cagliero and Tirao give chain maps between the left A^e-projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ of a truncated quiver algebra A over commutative ring. In this paper, by means of these chain maps, we obtain the dimension formula of the cyclic homology of truncated quiver algebras over a field.

On the other hand, by means of [7, Theorem 4.1.13], Taillefer [10] gives a dimension formula for the cyclic homology of truncated quiver algebras over a field of characteristic zero. Our result generalizes the formula into the case of the field of any characteristic.

2. Preliminaries

Let Δ be a finite quiver and $m(\geq 2)$ a positive integer. For $\alpha \in \Delta_1$, its source and target are denoted by $s(\alpha)$ and $t(\alpha)$, respectively. A path in Δ is a sequence of arrows

The detailed version of this paper has been submitted for publication elsewhere.

$\alpha_1\alpha_2\cdots\alpha_n$ such that $t(\alpha_i)=s(\alpha_{i+1})$ for $i=1,\ldots,n-1$. The set of all paths of length n is denoted by Δ_n.

By adjoining the element $\perp$, we will consider the following set (cf. [8], [9]):

$$\hat{\Delta}=\{\perp\}\cup\bigcup_{i=0}^{\infty}\Delta_i.$$

This set is a semigroup with the multiplication defined by

$$\delta\cdot\gamma=\begin{cases}\delta\gamma & \text{if}\quad t(\delta)=s(\gamma),\\ \perp & \text{otherwise},\end{cases}\qquad \delta,\gamma\in\bigcup_{i=0}^{\infty}\Delta_i,$$

and

$$\perp\cdot\gamma=\gamma\cdot\perp=\perp,\quad \gamma\in\hat{\Delta}.$$

Let K be a commutative ring. Then $K\hat{\Delta}$ is a semigroup algebra and the path algebra $K\Delta$ is isomorphic to $K\hat{\Delta}/(\perp)$. So, $K\Delta$ is a $\hat{\Delta}$-graded algebra with a basis consisting of the paths in Δ. Moreover, $K\Delta$ is $\mathbb{N}$-graded, that is, $K\Delta=\bigoplus_{i=0}^{\infty}K\Delta_i$. In particular, R_Δ^m is $\hat{\Delta}$-graded and $\mathbb{N}$-graded, thus the truncated quiver algebra $A=K\Delta/R_\Delta^m$ is a $\hat{\Delta}$-graded and $\mathbb{N}$-graded algebra.

For an $\mathbb{N}$-graded vector space V, V_+ is defined by $V_+=\bigoplus_{i\geq 1}V_i$.

Let Δ be a finite quiver. For a path γ, $|\gamma|$ denotes the length of γ. A path γ is said to be a cycle if $|\gamma|\geq 1$ and its source and target coincide. The period of a cycle γ is defined by the smallest integer i such that $\gamma=\delta^j$ $(j\geq 1)$ for a cycle δ of length i, which is denoted by $\operatorname{per}\gamma$. A cycle is said to be a basic cycle if the length of the cycle coincides with its period. It is also called a proper cycle [5]. Denote by Δ_n^{c} (respectively Δ_n^{b}) the set of cycles (respectively basic cycles) of length n. Let $G_n=\langle t_n\rangle$ be the cyclic group of order n and the path $\alpha_1\cdots\alpha_{n-1}\alpha_n$ a cycle where α_i is an arrow in Δ. Then we define the action of G_n on Δ_n^{c} by $t_n\cdot(\alpha_1\cdots\alpha_{n-1}\alpha_n):=\alpha_n\alpha_1\cdots\alpha_{n-1}$, and $\Delta_n^{\mathrm{c}}/G_n$ denotes the set of all G_n-orbits on Δ_n^{c}. Similarly, G_n acts on Δ_n^{b}, and $\Delta_n^{\mathrm{b}}/G_n$ denotes the set of all G_n-orbits on Δ_n^{b}. For $\bar{\gamma}\in\Delta_n^{\mathrm{c}}/G_n$, we define the period $\operatorname{per}\bar{\gamma}$ of $\bar{\gamma}$ by $\operatorname{per}\gamma$. For convenience we use the notation $\Delta_0^{\mathrm{c}}/G_0$ for the set of vertices Δ_0. Throughout this paper, $\alpha_i (i\geq 0)$ denotes an arrow in Δ.

3. The Hochschild homology of truncated quiver algebras

In this section, we intoroduce the Hochschild homology of truncated quiver algebra in [8].

Theorem 1 ([8, Theorem 1]). *The following is a projective $\hat{\Delta}$-graded resolution of A as a left A^e-module*:

$$\boldsymbol{P}:\cdots\xrightarrow{d_{i+1}}P_i\xrightarrow{d_i}\cdots\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\xrightarrow{\epsilon}A\longrightarrow 0.$$

Here the modules are defined by

$$P_i=A\otimes_{K\Delta_0}K\Gamma^{(i)}\otimes_{K\Delta_0}A,$$

where $\Gamma^{(i)}$ *is given by*

$$\Gamma^{(i)} = \begin{cases} \Delta_{cm} & \text{if } i = 2c \ (c \geq 0), \\ \Delta_{cm+1} & \text{if } i = 2c+1 \ (c \geq 0), \end{cases}$$

and the differentials are defined by

$$d_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta) \\ = \sum_{j=0}^{m-1} \alpha\alpha_1 \cdots \alpha_j \otimes \alpha_{1+j} \cdots \alpha_{(c-1)m+1+j} \otimes \alpha_{(c-1)m+2+j} \cdots \alpha_{cm}\beta,$$

and

$$d_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta) = \alpha\alpha_1 \otimes \alpha_2 \cdots \alpha_{cm+1} \otimes \beta - \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1}\beta.$$

The augmentation $\varepsilon\colon A \otimes_{K\Delta_0} K\Delta_0 \otimes_{K\Delta_0} A \cong A \otimes_{K\Delta_0} A \longrightarrow A$ *is defined by*

$$\varepsilon(\alpha \otimes \beta) = \alpha\beta.$$

Theorem 2 ([8, Theorem 2])**.** *Let* K *be a commutative ring and* A *a truncated quiver algebra* $K\Delta/R_\Delta^m$ *and* $q = cm + e$ *for* $0 \leq e \leq m-1$*. Then the degree* q *part of the pth Hochschild homology* $HH_p(A)$ *is given by*

$$HH_{p,q}(A) = \begin{cases} K^{a_q} & \text{if } 1 \leq e \leq m-1 \text{ and } 2c \leq p \leq 2c+1, \\ \bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \operatorname{Ker}\left(\cdot\frac{m}{\gcd(m,r)} : K \longrightarrow K\right)\right)^{b_r} & \\ \qquad \text{if } e = 0 \text{ and } 0 < 2c-1 = p, & \\ \bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \operatorname{Coker}\left(\cdot\frac{m}{\gcd(m,r)} : K \longrightarrow K\right)\right)^{b_r} & \\ \qquad \text{if } e = 0 \text{ and } 0 < 2c = p, & \\ K^{\#\Delta_0} & \text{if } p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we set $a_q := \#(\Delta_q^{\mathrm{c}}/G_q)$ *and* $b_r := \#(\Delta_r^{\mathrm{b}}/G_r)$*.*

4. Main result

In this section, by means of chain maps which are given by Ames, Cagliero, Tirao, we determine the dimension formula of the cyclic homology of truncated quiver algebra.

Lemma 3 ([3, Lemma 1.1])**.** *Let* Δ *be a finite quiver,* I *an admissible ideal,* $K\Delta_0$ *the subalgebra of* $A = K\Delta/I$ *generated by* Δ_0 *and* r *the Jacobson radical of* A*. The following* *is a projective resolution of A as a left A^e-module*:

$$\boldsymbol{Q} : \cdots \longrightarrow A \otimes_{K\Delta_0} r^{\otimes^i_{K\Delta_0}} \otimes_{K\Delta_0} A \xrightarrow{d_i} A \otimes_{K\Delta_0} r^{\otimes^{i-1}_{K\Delta_0}} \otimes_{K\Delta_0} A \longrightarrow \cdots$$
$$\longrightarrow A \otimes_{K\Delta_0} r \otimes_{K\Delta_0} A \xrightarrow{d_1} A \otimes_{K\Delta_0} A \xrightarrow{d_0} A \longrightarrow 0,$$

where

$$d_0(\lambda[\quad]\mu) = \lambda\mu,$$
$$d_i(\lambda[x_1|\cdots|x_i]\mu) = \lambda x_1[x_2|\cdots|x_i]\mu + \sum_{j=1}^{i-1}(-1)^i\lambda[x_1|\cdots|x_jx_{j+1}|\cdots|x_i]\mu$$
$$+ (-1)^i\lambda[x_1|\cdots|x_{i-1}]x_i\mu \quad \text{for } i \geq 1,$$

and we use the bar notation $\lambda[x_1|\cdots|x_i]\mu$ for $\lambda \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes \mu$.

Cibils constructs the following mixed complex.

Theorem 4 ([4], [9])**.** *Let Δ be a finite quiver, K a field, and $A = K\Delta/I$ for I a homogeneous ideal. Define the mixed complex $(C_{K\Delta_0}(A), b, B)$ by*

$$C_{K\Delta_0}(A)_n = A \otimes_{K\Delta_0^e} A_+^{\otimes^n_{K\Delta_0}},$$

and

$$b(x_0[x_1|\cdots|x_n]) = x_0x_1[x_2|\cdots|x_n]$$
$$+ \sum_{i=1}^{n-1}(-1)^i x_0[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$$
$$+ (-1)^n x_nx_0[x_1|\cdots|x_{n-1}],$$
$$B(x_0[x_1|\cdots|x_n]) = \sum_{i=0}^{n}(-1)^{in}[x_i|\cdots|x_n|x_0|\cdots|x_{i-1}].$$

Then $HH_n(C_{K\Delta_0}(A)) = HH_n(A)$ and $HC_n(C_{K\Delta_0}(A)) = HC_n(A)$.

In particular, if A is a truncated quiver algebra $K\Delta/R^m_\Delta(m \geq 2)$, then the map B in $(C_{K\Delta_0}(A), b, B)$ respects the Δ^c_*/G_*-grading (cf. [9]). Furthermore if we consider the double complex $\mathcal{B}C$ associate to this mixed complex and filter the total complex $\operatorname{Tot}\mathcal{B}C$ by the column filtration, then the resulting spectral sequence is Δ^c_*/G_*-graded. Thus $HC_n(A)$ is Δ^c_*/G_*-graded. Moreover, for $\bar{\gamma} \in \Delta^c_*/G_*$ the degree $\bar{\gamma}$ part of the E^1-term of this spectral sequence is $E^1_{p,q,\bar{\gamma}} = HH_{q-p,\bar{\gamma}}(A)$.

On the other hand, Ames, Cagliero and Tirao find the chain maps between the left A^e-projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ of a truncated quiver algebra A over an arbitrary field as follows:

Proposition 5 ([1]). *Define the map $\iota : \boldsymbol{P} \longrightarrow \boldsymbol{Q}$ as follows*:

$$\iota_0(\alpha \otimes \beta) = \alpha[\quad]\beta,\ \iota_1(\alpha \otimes \alpha_1 \otimes \beta) = \alpha[\alpha_1]\beta,$$

$$\begin{aligned}&\iota_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta)\\ &\quad = \sum_{0 \le j_1, \dots, j_c \le m-2} \alpha[\alpha_1 \cdots \alpha_{1+j_1}|\alpha_{2+j_1}|\alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2}|\alpha_{4+j_1+j_2}| \cdots \\ &\qquad |\alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c}|\alpha_{2c+j_1+\cdots+j_c}]\alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{cm}\beta,\end{aligned}$$

$$\begin{aligned}&\iota_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta)\\ &\quad = \sum_{0 \le j_1, \dots, j_c \le m-2} \alpha[\alpha_1|\alpha_2 \cdots \alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1} \cdots \alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}| \cdots \\ &\qquad |\alpha_{2c+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c} \cdots \alpha_{cm+1}\beta.\end{aligned}$$

Then, ι is a chain map.

Proposition 6 ([1]). *Let m_i be a positive integer for any $i \ge 1$. Suppose that x_i is the path $\alpha_{m_1+\cdots+m_{i-1}+1} \cdots \alpha_{m_1+\cdots+m_i}$ of length m_i. Define the map $\pi : \boldsymbol{Q} \longrightarrow \boldsymbol{P}$ as follows*:

$$\pi_0(\alpha[\quad]\beta) = \alpha \otimes \beta,$$

$$\pi_1(\alpha[x_1]\beta) = \sum_{j=1}^{m_1} \alpha\alpha_1 \cdots \alpha_{j-1} \otimes \alpha_j \otimes \alpha_{j+1} \cdots \alpha_{m_1}\beta,$$

$$\pi_{2c}(\alpha[x_1|x_2|\cdots|x_{2c}]\beta) = \begin{cases} \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1} \cdots \alpha_{m_1+\cdots+m_{2c}}\beta \\ \qquad\qquad \text{if} \quad m_{2i-1} + m_{2i} \ge m\ (1 \le i \le c), \\ 0 \qquad \text{otherwise}, \end{cases}$$

$$\pi_{2c+1}(\alpha[x_1|x_2|\cdots|x_{2c+1}]\beta) = \begin{cases} \sum_{j=1}^{m_1} \alpha\alpha_1 \cdots \alpha_{j-1} \otimes \alpha_j \cdots \alpha_{j+cm} \otimes \\ \qquad \alpha_{j+cm+1} \cdots \alpha_{m_1+\cdots+m_{2c+1}}\beta \\ \qquad\qquad \text{if} \quad m_{2i} + m_{2i+1} \ge m\ (1 \le i \le c), \\ 0 \qquad \text{otherwise}. \end{cases}$$

Then, π is a chain map and $\pi\iota = \mathrm{id}_{\boldsymbol{P}}$.

By investigating the basis of the Hochschild homology and finding the chain maps between the projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$, we are able to compute $B : HH_{p,\bar{\gamma}}(A) \longrightarrow HH_{p+1,\bar{\gamma}}(A)$ induced by the differential of the Cibils' mixed complex. Moreover, for $\bar{\gamma} \in \Delta_t^c/G_t$ we are able to determine the degree $\bar{\gamma}$ part of the E^2-term of the spectral sequence associated with the Cibils' mixed complex. Therefore we have the following result.

Theorem 7 ([6, Theorem 5.1]). *Suppose that $m \ge 2$ and $A = K\Delta/R_\Delta^m$. Then the dimension formula of the cyclic homology of A is given by, for $c \ge 0$,*

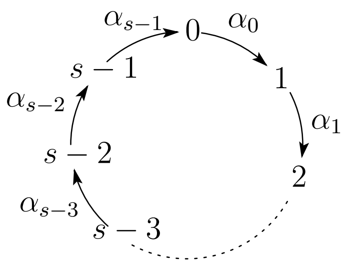
$$\dim_K HC_{2c}(A) = \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e} + \sum_{c'=0}^{c-1}\sum_{e=1}^{m-1} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta | c'm+e}} b_r$$

$$+\sum_{c'=1}^{c}\sum_{\substack{r>0\\ \text{s.t. } r|c'm,\\ \gcd(m,r)\zeta|m}} b_r+\sum_{c'=1}^{c}\sum_{\substack{r>0\\ \text{s.t.} r\zeta|\gcd(m,r)c'}}(\gcd(m,r)-1)b_r,$$

$$\dim_K HC_{2c+1}(A)=\sum_{\substack{r>0\\ \text{s.t. } r|(c+1)m}}(\gcd(m,r)-1)b_r+\sum_{c'=0}^{c}\sum_{e=1}^{m-1}\sum_{\substack{r>0\\ \text{s.t. } r\zeta|c'm+e}} b_r$$
$$+\sum_{c'=1}^{c+1}\sum_{\substack{r>0\\ \text{s.t. } r|c'm,\\ \gcd(m,r)\zeta|m}} b_r+\sum_{c'=1}^{c}\sum_{\substack{r>0\\ \text{s.t. } r\zeta|\gcd(m,r)c'}}(\gcd(m,r)-1)b_r.$$

Remark 8. If $\zeta=0$, then the above result coincides with the result of Taillefer in [10].

Example 9 ([6, Example 5.3]). Let K be a field of characteristic ζ and Δ the following quiver:



Suppose $m\geq 2$ and $A=K\Delta/R_\Delta^m$, which is called a truncated cycle algebra in [2].
Since

$$a_r=\begin{cases}1 & \text{if } s|r,\\ 0 & \text{otherwise,}\end{cases}\qquad b_r=\begin{cases}1 & \text{if } s=r,\\ 0 & \text{otherwise,}\end{cases}$$

we have, for $c\geq 0$,

$$\dim_K HC_{2c}(A)=s+\left[\frac{(c+1)m-1}{s}\right]-\left[\frac{cm}{s}\right]+\sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right]-\left[\frac{c'm}{s\zeta}\right]\right)$$
$$+\left(\left[\frac{m}{\gcd(m,s)\zeta}\right]-\left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c}\left(\left[\frac{c'm}{s}\right]-\left[\frac{c'm-1}{s}\right]\right)$$
$$+(\gcd(m,s)-1)\left[\frac{\gcd(m,s)c}{s\zeta}\right],$$

and

$$\dim_K HC_{2c+1}(A) = (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right] - \left[\frac{(c+1)m-1}{s}\right] + \left[\frac{\gcd(m,s)c}{s\zeta}\right]\right)$$
$$+\left(\left[\frac{m}{\gcd(m,s)\zeta}\right] - \left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c+1}\left(\left[\frac{c'm}{s}\right] - \left[\frac{c'm-1}{s}\right]\right)$$
$$+\sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right).$$

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