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Edited by
Isao Kikumasa
Yamaguchi University

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第46回 環論および表現論シンポジウム報告集

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Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium and committee members are listed in the web page

http://fuji.cec.yamanashi.ac.jp/˜ring/h-of-ringsymp.html.

The present members of the committee are H. Asashiba (Shizuoka Univ.), S. Ikehata (Okayama Univ.), S. Kawata (Osaka City Univ.) and I. Kikumasa (Yamaguchi Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2014 will be held at Osaka City University for Sept. 13 (Sat.)–15 (Mon.) and the program will be arranged by A. Hida (Saitama Univ.).

Concerning several information on ring theory and representation theory of group and algebras containing schedules of meetings and symposiums as well as ring mailing list service for registered members, you should refer to the following ring homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/˜ring/ (in Japanese)
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http://fuji.cec.yamanashi.ac.jp/˜ring/japan/ (in English)

Shûichi Ikehata
Okayama Japan
March, 2014

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Preface

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Isao Kikumasa
Yamaguchi, Japan
March, 2014

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Source algebras and cohomology of block ideals of fintie group algebras

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Geigle-Lenzing spaces and canonical algebras in dimension d

12:00–12:30 松田 一徳（立教大学・JST CREST）

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The 46th Symposium on Ring Theory and Representation Theory (2013)

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Classifying τ-tilting modules over Nakayama algebras

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On the poset of pre-projective tilting modules over path algebras

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The dimension formula of the cyclic homology of truncated quiver algebras over a field of positive characteristic

CLASSIFYING τ-TILTING MODULES OVER NAKAYAMA ALGEBRAS

TAKAHIDE ADACHI

Abstract. In this article, we study τ-tilting modules over Nakayama algebras. We establish bijection between τ-tilting modules and triangulations of a regular polygon with a puncture.

Throughout this article, Λ is a basic connected finite dimensional algebra over an algebraically closed field K and by a module we mean a finite dimensional right module. For a Λ-module M with a minimal projective presentation $P^{-1} \overset{p}{\to} P^0 \to M \to 0$, we define a Λ-module τM by an exact sequence

$$0 \to \tau M \to \nu P^{-1} \overset{\nu p}{\to} \nu P^0,$$

where $\nu := \mathrm{Hom}_K(\mathrm{Hom}_\Lambda(-, \Lambda), K)$. In this article, the following quivers are useful:

$$\vec{A}_n : n \xrightarrow{\alpha_{n-1}} n{-}1 \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1$$

$$\vec{\Delta}_n : \begin{tikzcd} & 2 \arrow[dl, "\alpha_1"'] & 3 \arrow[l, "\alpha_2"'] & \\ 1 \arrow[dr, "\alpha_n"'] & & & \vdots \arrow[ul, "\alpha_3"'] \\ & n \arrow[r, "\alpha_{n-1}"'] & n{-}1 \arrow[ur, "\alpha_{n-2}"'] & \end{tikzcd}$$

In representation theory of finite dimensional algebra, tilting modules play an important role because they induce derived equivalences. Recently, the authors in [2] introduced the notion of τ-tilting modules, which is a generalization of (classical) tilting modules. They showed there are close relationships between τ-tilting modules and some important notions: *torsion classes, silting complexes, and cluster-tilting objects.* For background and results of τ-tilting modules, we refer to [2].

Our aim of this article is to give a generalization of the following well-known result. A Λ-module M is called *tilting* if $\mathrm{pd}M \leq 1$, $\mathrm{Ext}^1_\Lambda(M, M) = 0$ and $|M| = |\Lambda|$, where $\mathrm{pd}M$ is the projective dimension and $|M|$ the number of nonisomorphic indecomposable direct summands of M.

Theorem 1. *Let $\Lambda = K\vec{A}_n$ be a path algebra. Then there is a one-to-one correspondence between*

(1) *the set* $\mathrm{tilt}\Lambda$ *of isomorphism classes of basic tilting Λ-modules,*
(2) *the set of triangulations of an $(n+2)$-regular polygon.*

The following theorem is our main result of this article. A Λ-module M is called *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$. A τ-rigid Λ-module M is called *τ-tilting* if $|M| = |\Lambda|$.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 2. [1] *Let Λ be a Nakayama algebra with $n := |\Lambda|$. Assume that the Loewy length of every indecomposable projective Λ-module is at least n. Then there is a one-to-one correspondence between*

(1) *the set $\tau\text{-tilt}\Lambda$ of isomorphism classes of basic τ-tilting Λ-modules,*
(2) *the set $\mathcal{T}(n)$ of triangulations of an n-regular polygon with a puncture.*

First, recall the definition and basic properties of Nakayama algebras. An algebra Λ is said to be *Nakayama* if every indecomposable projective Λ-module and every indecomposable injective Λ-module are *uniserial* (*i.e*, it has a unique composition series). We give a characterization of Nakayama algebras by using quivers.

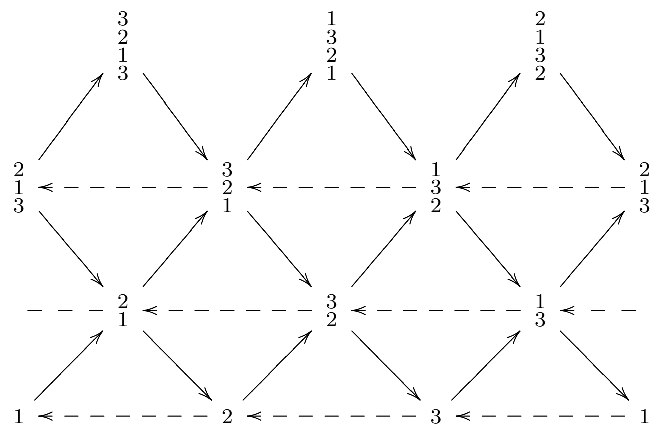
Proposition 3. [3, V.3.2] *A basic connected algebra Λ is Nakayama if and only if its quiver Q_Λ is one of $\vec{A}_n$ or $\vec{\Delta}_n$.*

In the following, we assume that Λ is a basic connected Nakayama algebra and $n := |\Lambda|$. We give a concrete description of indecomposable Λ-modules. We denote by P_i (respectively, S_i) the indecomposable projective (respectively, the simple) Λ-module corresponding to the vertex i in Q_Λ and $\ell(M)$ the Loewy length of a Λ-module M.

Proposition 4. [3, V.3.5, V.4.1 and V.4.2] *Let M be an indecomposable nonprojective Λ-module. Then there exists an indecomposable projective Λ-module P_i and an integer $1 \le t < \ell(P_i)$ such that $M \simeq P_i/\mathrm{rad}^t P_i$. In this case, we have $\tau M \simeq P_{i-1}/\mathrm{rad}^t P_{i-1}$ and $t = \ell(M) = \ell(\tau M)$.*

Every indecomposable Λ-module M is uniquely determined, up to isomorphism, by its simple top S_i and the Loewy length $t = \ell(M)$. In this case, M has a unique composition series with the associated composition factors $S_i = S_{i_1}, S_{i_2}, \cdots, S_{i_t}$, where $i_1, \cdots, i_t \in \{1, 2, \cdots, n\}$ with $i_{j+1} = i_j - 1 (\mathrm{mod}\ n)$ for any j.

Let Λ_n^r be a self-injective Nakayama algebra with $n = |\Lambda|$ and the Loewy length r. The Auslander-Reiten quiver of Λ_3^4 is given by the following:



Then the following modules are all τ-tilting Λ_3^4-modules:

$$\begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix},\quad \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 1,\quad \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 1,\quad \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 2,\quad \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 2,$$
$$\begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 2,\quad \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3,\quad \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus 3,\quad \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus 3,\quad \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus 1.$$

By the example above, note that every τ-tilting module does not have an indecomposable Λ-module M with the Loewy length $\ell(M) = 3$ as a direct summand. This is always the case as the following result shows. By Proposition 4, we can easily understand the existence of homomorphisms between indecomposable Λ-modules.

Lemma 5. *Assume that $M = P_i/\mathrm{rad}^j P_i$ and $N = P_k/\mathrm{rad}^l P_k$. Then the following are equivalent.*

(1) $\mathrm{Hom}_\Lambda(M, N) \neq 0$.
(2) $i \in \{k, k-1, \cdots, k-l+1 \pmod n\}$ *and* $k-l+1 \in \{i, i-1, \cdots, i-j+1 \pmod n\}$.

Thus we have the following result.

Proposition 6. *Let M be an indecomposable nonprojective Λ-module. Then M is τ-rigid if and only if $\ell(M) < n$.*

Proof. By Proposition 4, we can assume that $M = P_i/\mathrm{rad}^j P_i$ and $\tau M = P_{i-1}/\mathrm{rad}^j P_{i-1}$. Then we have

$$\begin{aligned}\mathrm{Hom}_\Lambda(M, \tau M) \neq 0 &\overset{5}{\Leftrightarrow} \begin{cases} i \in \{i-1, i-2, \cdots, i+j \pmod n\} \\ i+j \in \{i, i-1, \cdots, i-j+1 \pmod n\}\end{cases} \\ &\Leftrightarrow \ell(M) \geq n.\end{aligned}$$

□

To study a connection between τ-tilting modules and triangulations of an n-regular polygon with a puncture, recall the definition and basic properties of triangulations. Let $\mathcal{G}_n$ be an n-regular polygon with a puncture. We label the points of $\mathcal{G}_n$ counterclockwise around the boundary by $1, 2, \cdots, n \pmod n$. Let $i, j \in \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. An *inner* arc $\langle i, j\rangle$ in $\mathcal{G}_n$ is a path from the point i to the point j homotopic to the boundary path $i, i+1, \cdots, i+l = j \pmod n$ such that $1 < l \leq n$. Then we call i an *initial* point, j a *terminal* point and $\ell(\langle i, j\rangle) := l$ the *length* of the inner arc. A *projective* arc $\langle \bullet, j\rangle$ in $\mathcal{G}_n$ is a path from the puncture to the point j. Then we call j a *terminal* point. An *admissible* arc is an inner arc or a projective arc.

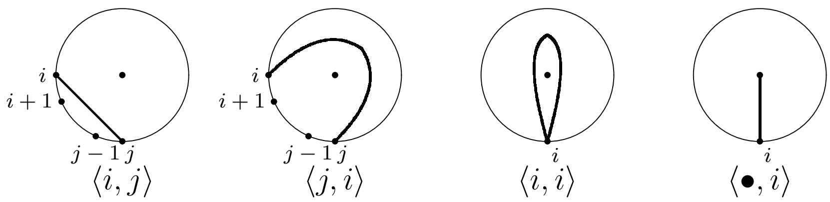


FIGURE 1. Admissible arcs in a polygon with a puncture

Two admissible arcs in $\mathcal{G}_n$ are called *compatible* if they do not intersect in $\mathcal{G}_n$ (except their initial and terminal points). A *triangulation* of $\mathcal{G}_n$ is a maximal set of distinct pairwise compatible admissible arcs. Note that the set of all projective arcs gives a triangulation of $\mathcal{G}_n$. Triangulations have the following property.

Proposition 7. *Each triangulation of $\mathcal{G}_n$ consists of exactly n admissible arcs and contains at least one projective arc.*

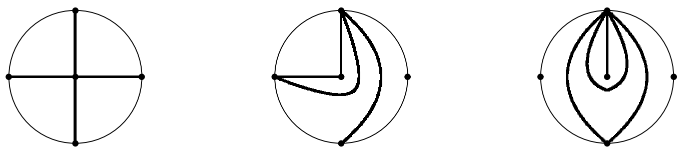


FIGURE 2. Triangulations of $\mathcal{G}_4$

We give a bijection between indecomposable τ-rigid Λ-modules and admissible arcs in $\mathcal{G}_n$. By Proposition 6, every indecomposable nonprojective τ-rigid Λ-module M is uniquely determined by its simple top S_j and its simple socle S_k. Thus we denote by $(k-2, j)$ the τ-rigid module M above. Moreover, we put $(\bullet, j) := P_j$ for any j. We denote by τ-rigidΛ the set of isomorphism classes of indecomposable τ-rigid Λ-modules.

Proposition 8. *The following hold.*

(1) *There is a bijection*

$$\tau\text{-rigid}\Lambda \longrightarrow \{\langle i,j\rangle \mid i,j \in \mathbb{Z}_n,\ \ell(\langle i,j\rangle) \le \ell(P_j)\} \coprod \{\langle \bullet, i\rangle \mid i \in \mathbb{Z}_n\}$$

given by $(i,j) \mapsto \langle i,j\rangle$ *for* $i \in \{1,2,\cdots,n\} \sqcup \{\bullet\}$ *and* $j \in \{1,2,\cdots,n\}$.

(2) *For any* $i,k \in \{1,2,\cdots,n\} \coprod \{\bullet\}$ *and* $j,l \in \{1,2,\cdots,n\}$, $(i,j) \oplus (k,l)$ *is* τ*-rigid if and only if* $\langle i,j\rangle$ *and* $\langle k,l\rangle$ *are compatible.*

Proof. (1) By Proposition 6, every indecomposable τ-rigid Λ-module M is either a projective Λ-module or a Λ-module satisfying $\ell(M) < n$. Thus we can easily check the map is a well-defined bijection.

(2) It follows from Lemma 5. □

Instead of proving Theorem 2, we give a proof of the following theorem which is a generalization of Theorem 2. Let $\ell_i := \ell(P_i)$ for any $i \in \{1,2,\cdots,n\}$. We denote by $\mathcal{T}(n;\ell_1,\ell_2,\cdots,\ell_n)$ the subset of $\mathcal{T}(n)$ consisting of triangulations such that the length of every inner arc with the terminal point j is at most ℓ_j for any $j \in \{1,2,\cdots,n\}$.

Theorem 9. *Let* Λ *be a Nakayama algebra. Then the map in Proposition 8 induces a bijection*

$$\tau\text{-tilt}\Lambda \longrightarrow \mathcal{T}(n;\ell_1,\ell_2,\cdots,\ell_n).$$

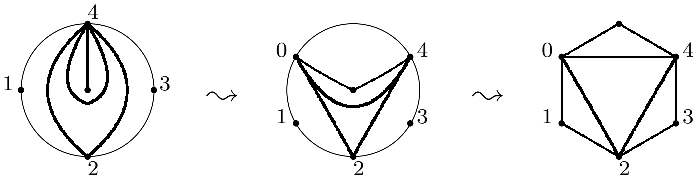
Proof. By Proposition 7 and 8, we can easily check that the map is a bijection. □

As an application of Theorem 9, we have the following corollary.

Corollary 10. *Let* $\Lambda = K\vec{A}_n$ *be a path algebra. Then the map in Proposition 8 induces a bijection*

$$\text{tilt}\Lambda \to \mathcal{U}(n) := \{X \in \mathcal{T}(n) \mid \langle \bullet, n\rangle \in X\}.$$

Note that $\mathcal{U}(n)$ can identify the set of triangulations of an $(n+2)$-regular polygon. As a result, we can recover Theorem 1.



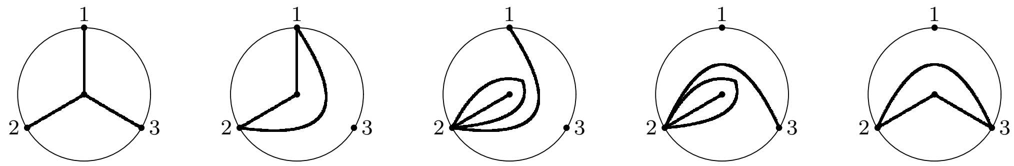
Proof. By Theorem 9, there is a bijection between τ-tiltΛ and $\mathcal{T}(n; 1, 2, \cdots, n)$. Since Λ is hereditary, we have tilt$\Lambda = \tau$-tiltΛ. We have only to show that

$$\mathcal{U}(n) = \mathcal{T}(n; 1, 2, \cdots, n)$$

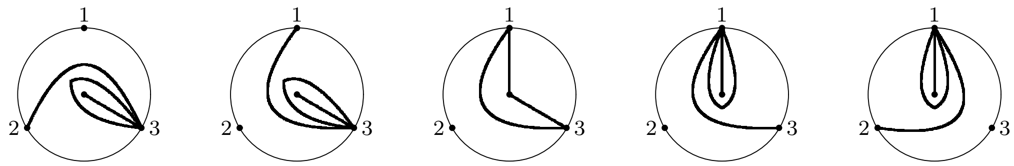
Indeed, assume that $X \in \mathcal{U}(n)$. Since X contains the projective arc $\langle \bullet, n\rangle$, we have $\ell(\langle i, j\rangle) \leq j$ for each inner arc $\langle i, j\rangle \in X$. Thus, we have $X \in \mathcal{T}(n; 1, 2, \cdots, n)$. Conversely, assume that $X \in \mathcal{T}(n; 1, 2, \cdots, n)$. Clearly, the projective arc $\langle \bullet, n\rangle$ is compatible with all admissible arc in X. Thus, we have $\langle \bullet, n\rangle \in X$, and hence $X \in \mathcal{U}(n)$. $\square$

Finally, we give an example of Theorem 9.

Example 11. Let $\Lambda := \Lambda_3^r$ be a self-injective Nakayama algebra with $r \geq 3$. Then we have the followng description of τ-tilting Λ-modules.



$P_1 \oplus P_2 \oplus P_3$ $\quad$ $P_1 \oplus P_2 \oplus 1$ $\quad$ $\begin{smallmatrix}2\\1\end{smallmatrix} \oplus P_2 \oplus 1$ $\quad$ $\begin{smallmatrix}2\\1\end{smallmatrix} \oplus P_2 \oplus 2$ $\quad$ $P_3 \oplus P_2 \oplus 2$



$P_3 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 2$ $\quad$ $P_3 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3$ $\quad$ $P_3 \oplus P_1 \oplus 3$ $\quad$ $\begin{smallmatrix}1\\3\end{smallmatrix} \oplus P_1 \oplus 3$ $\quad$ $\begin{smallmatrix}1\\3\end{smallmatrix} \oplus P_1 \oplus 1$

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MUTATION AND MUTATION QUIVERS OF SYMMETRIC SPECIAL BISERIAL ALGEBRAS

TAKUMA AIHARA

Abstract. The notion of mutation plays crucial roles in representation theory of algebras. Two kinds of mutation are well-known: tilting/silting mutation and quiver-mutation. In this paper, we focus on tilting mutation for symmetric algebras. Introducing mutation of SB quivers, we explicitly give a combinatorial description of tilting mutation of symmetric special biserial algebras. As an application, we generalize Rickard's star theorem. We also introduce flip of Brauer graphs and apply our results to Brauer graph algebras. Moreover we study tilting quivers of symmetric algebras and show that a Brauer graph algebra is tilting-discrete if and only if its Brauer graph is of type odd.

Key Words: special biserial algebra, mutation of SB quivers, tilting mutation, Brauer graph algebra, flip of Brauer graphs.

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1. Introduction

In representation theory of algebras, the notion of mutation plays an important role. We refer to two kinds of mutation: quiver-mutation and tilting/silting mutation. Quiver-mutation was introduced by Fomin-Zelevinsky [FZ] to develop a combinatorial approach to canonical bases of quantum groups, and yields the notion of Fomin-Zelevinsky cluster algebras which has spectacular growth thanks to the many links with a wide range of subjects of mathematics.

Tilting mutation, which is a special case of silting mutation [AI], was introduced by Riedtmann-Schofield [RS] and Happel-Unger [HU] to investigate the structure of the derived category. For example, Bernstein-Gelfand-Ponomarev reflection functors [BGP], Auslander-Platzeck-Reiten tilting modules [APR] and Okuyama-Rickard tilting complexes [O, R2] are special cases of tilting mutation. In the case that a given algebra is symmetric, tilting mutation yields infinitely many tilting complexes, which are extremely important complexes from Morita theoretic viewpoint of derived categories [R1]. It is because they give rise to derived equivalences which preserve many homological properties.

The aim of this paper is to find some similarities between the effects of tilting mutation and Fomin-Zelevinsky quiver-mutations.

The following problem is naturally asked:

Problem 1. Give an explicit description of the endomorphism algebra of a tilting complex given by tilting mutation.

The detailed version of this paper will be submitted for publication elsewhere.

In this paper we give a complete answer to this problem for symmetric special biserial algebras, which is one of the important classes of algebras in representation theory. Some of special biserial algebras were first studied by Gelfand-Ponomarev [GP], and also naturally appear in modular representation theory of finite groups [Al, E]. Moreover such an algebra is always representation-tame and the classification of all indecomposable modules of such an algebra was provided in [WW, BR]. The derived equivalence classes of special biserial algebras were also discussed in [BHS, K, KR].

To realize our goal, we start with describing symmetric special biserial algebras in terms of combinatorial data, which we call *SB quivers*. Moreover we will study symmetric special biserial algebras from graph theoretic viewpoint, which is described by *Brauer graphs*. Indeed, we have the result below (see Lemma 8 and Proposition 26):

Proposition 2. *There exist one-to-one correspondences among the following three classes:*

(1) *Symmetric special biserial algebras;*
(2) *Special quivers with cycle-decomposition (*SB quivers*);*
(3) *Brauer graphs.*

We introduce *mutation of SB quivers* (see Definition 15, 18 and 21), which is similar to Fomin-Zelevinsky quiver-mutation. Moreover we will show that mutation of SB quivers corresponds to a certain operation on Brauer graphs, which we call *flip* and is a generalization of mutation/flip of Brauer trees introduced in [A].

The main theorem of this paper is the following:

Theorem 3 (Theorem 14 and Theorem 29)**.** *The following three operations are compatible each other:*

(1) *Tilting mutation of symmetric special biserial algebras;*
(2) *Mutation of SB quivers;*
(3) *Flip of Brauer graphs.*

We note that certain special cases of the compatibility of (1) and (3) in Theorem 3 were given by [K, An] (see Remark 30).

As an application of Theorem 3, we generalize "Rickard's star theorem" for Brauer tree algebras, which gives nice representatives of Brauer tree algebras up to derived equivalence [R2, M]. We introduce *Brauer double-star algebras*, as the corresponding class for Brauer tree algebras, and prove the following (see Section 4.3 for the details):

Theorem 4 (Theorem 31)**.** *Any Brauer graph algebra is derived equivalent to a Brauer double-star algebra whose Brauer graph has the same number of the edges and the same multiplicities of the vertices.*

As an application of Theorem 4, we deduce Rickard's star theorem (Corollary 33).

Finally we study tilting quivers which were introduced in [AI] to observe the behavior of tilting mutation. We are interested in the connectedness of tilting quivers. A symmetric algebra is said to be *tilting-connected* if its tilting quiver is connected. It was proved in [AI, A1] that a symmetric algebra is tilting-connected if it is either local or representation-finite. On the other hand, an example of symmetric algebras which are not tilting-connected was found by Grant, Iyama and the author. In this paper, we discuss the tilting-connectedness of Brauer graph algebras and aim to understand when a Brauer graph algebra is tilting-connected. We introduce Brauer graphs of *type odd*, and have the main theorem (see Section 5 for the details).

Theorem 5 (Theorem 39)**.** *Any Brauer graph algebra with a Brauer graph of type odd is tilting-connected.*

2. Symmetric special biserial algebras

This section is devoted to introducing the notion of SB quivers. We will give a relationship between symmetric special biserial algebras and SB quivers. Moreover we study tilting mutation, which is a special case of silting mutation introduced by [AI].

Throughout this paper, we use the following notation.

Notation. Let A be a finite dimensional algebra over an algebraically closed field k.

(1) We always assume that A is basic and indecomposable.
(2) We often write $A = kQ/I$ where Q is a finite quiver with relations I. The sets of vertices and arrows of Q are denoted by Q_0 and Q_1, respectively.
(3) We denote by $\mathrm{mod}\, A$ the category of finitely generated right A-modules. A simple (respectively, indecomposable projective) A-module corresponding to a vertex i of Q is denoted by S_i (respectively, by P_i). We always mean that a module is finitely generated.

A quiver of the form $\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$ with n arrows is called an *n-cycle* (for simplicity, *cycle*). We mean 1-cycle by *loop*.

Let us start with introducing SB quivers.

Definition 6. We say that a finite connected quiver Q is *special* if any vertex i of Q is the starting point of at most two arrows and also the end point of at most two arrows. For a special quiver Q with at least one arrow, a set $\mathsf{C} = \{C_1, C_2, \cdots, C_v\}$ of cycles in Q with a function $\mathsf{mult} : \mathsf{C} \to \mathbb{N}$ is said to be a *cycle-decomposition* if it satisfies the following conditions:

(1) Each C_ℓ is a subquiver of Q with at least one arrow such that $Q_0 = (C_1)_0 \cup \cdots \cup (C_v)_0$ and $Q_1 = (C_1)_1 \amalg \cdots \amalg (C_v)_1$: For any $\alpha \in Q_1$, we denote by C_α a unique cycle in C which contains α.
(2) Any vertex of Q belongs to at most two cycles.
(3) $\mathsf{mult}(C_\ell) > 1$ if C_ℓ is a loop.

A *SB quiver* is a pair (Q, C) of a special quiver Q and its cycle-decomposition C.

Let (Q, C) be a SB quiver. For each cycle C in C, we call $\mathsf{mult}(C)$ the *multiplicity* of C.

For any arrow α of Q, we denote by $\mathsf{na}(\alpha)$ a unique arrow β such that $\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ appears in C_α.

We construct a finite dimensional algebra from a SB quiver.

Definition 7. Let (Q, C) be a SB quiver. An ideal $I_{(Q,\mathsf{C})}$ of kQ is generated by the following three kinds of elements:

(1) $(\alpha_t\alpha_{t+1}\cdots\alpha_{t+s-1})^m\alpha_t$ for each cycle C in C of the form

$$\begin{tikzcd} i_1 \arrow[r, "\alpha_1"] & i_2 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{s-1}"] & i_s \arrow[lll, bend left, "\alpha_s"] \end{tikzcd}$$

and $t = 1, 2, \cdots, s$, where $m = \mathsf{mult}(C)$ and the indices are considered in modulo s.

(2) $\alpha\beta$ if $\beta \neq \mathsf{na}(\alpha)$.

(3) $(\alpha_1\alpha_2\cdots\alpha_s)^m - (\beta_1\beta_2\cdots\beta_t)^{m'}$ whenever we have a diagram

$$\begin{tikzcd} \cdots \arrow[r] & i_s \arrow[dr, "\alpha_s"] & & i'_t \arrow[dl, "\beta_t"'] & \cdots \arrow[l] \\ & & i \arrow[dl, "\alpha_1"] \arrow[dr, "\beta_1"'] & & \\ \cdots & i_2 \arrow[l, "\alpha_2"'] & & i'_2 \arrow[r, "\beta_2"] & \cdots \end{tikzcd}$$

where $C_{\alpha_\ell} = C_{\alpha_1}, C_{\beta_{\ell'}} = C_{\beta_1}$ for any $1 \leq \ell \leq s, 1 \leq \ell' \leq t$ and $m = \mathsf{mult}(C_{\alpha_1}), m' = \mathsf{mult}(C_{\beta_1})$.

We define a k-algebra $A := A_{(Q,\mathsf{C})}$ associated with (Q, C) by $A = kQ/I_{(Q,\mathsf{C})}$. Then the algebra $A_{(Q,\mathsf{C})}$ is finite dimensional and symmetric. The cycle-decomposition C is also said to be the *cycle-decomposition* of $A_{(Q,\mathsf{C})}$.

An algebra $A := kQ/I$ is said to be *special biserial* if Q is special and for any arrow β of Q, there is at most one arrow α with $\alpha\beta \notin I$ and at most one arrow γ with $\beta\gamma \notin I$.

Thanks to [Ro] (see Proposition 26), we have the following result.

Lemma 8. *The assignment $(Q, \mathsf{C}) \mapsto A_{(Q,\mathsf{C})}$ gives rise to a bijection between the isoclasses of SB quivers and those of symmetric special biserial algebras.*

Example 9. (1) Let Q be the quiver

$$\begin{tikzcd} & 1 \arrow[dl, shift left, "\alpha"] \arrow[dr, shift left, "\gamma'"] & \\ 2 \arrow[ur, shift left, "\alpha'"] \arrow[rr, shift left, "\beta"] & & 3 \arrow[ll, shift left, "\beta'"] \arrow[ul, shift left, "\gamma"] \end{tikzcd}$$

with the relations $I := \langle \alpha\beta, \beta\gamma, \gamma\alpha, \alpha'\gamma', \gamma'\beta', \beta'\alpha', \alpha'\alpha - \beta\beta', \beta'\beta - \gamma\gamma', \gamma'\gamma - \alpha\alpha' \rangle$. Then the algebra $A := kQ/I$ is symmetric special biserial associated with the SB quiver (Q, C) where the cycle-decomposition is

$$\mathsf{C} = \left\{ \left(1 \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} 2 \right), \left(2 \underset{\beta'}{\overset{\beta}{\rightleftarrows}} 3 \right), \left(3 \underset{\gamma'}{\overset{\gamma}{\rightleftarrows}} 1 \right) \right\}$$

such that the multiplicity of every cycle is 1.

(2) Let Q be the quiver

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[loop above, "\delta"] & \\ 2 \arrow[rr, "\beta"'] & & 3 \arrow[ul, "\gamma"'] \end{tikzcd}$$

with the relations $I := \langle \gamma\alpha, (abcd)^2 a \mid \{a,b,c,d\} = \{\alpha,\beta,\gamma,\delta\}\rangle$. Then $A := kQ/I$ is a symmetric special biserial algebra which is isomorphic to $A_{(Q,\mathsf{C})}$, where C is the cycle-decomposition

$$\mathsf{C} = \left\{ \left(\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\ 1 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"] \end{tikzcd} \right) \right\}$$

with the multiplicity 2.

(3) Let Q be the quiver

$$\begin{tikzcd} & 1 \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\alpha'"] & \\ 3 \arrow[ur, "\gamma"] & 2 \arrow[l, "\beta"] \arrow[r, "\beta'"'] & 4 \arrow[ul, "\gamma'"'] \end{tikzcd}$$

with cycle-decomposition $\mathsf{C} = \{C_1, C_2\}$ where

$$C_1 = \left(\begin{tikzcd} & 1 \arrow[d, "\alpha"] \\ 3 \arrow[ur, "\gamma"] & 2 \arrow[l, "\beta"] \end{tikzcd} \right), C_2 = \left(\begin{tikzcd} 1 \arrow[d, "\alpha'"'] & \\ 2 \arrow[r, "\beta'"'] & 4 \arrow[ul, "\gamma'"'] \end{tikzcd} \right)$$

and $\mathsf{mult}(C_1) = \mathsf{mult}(C_2) = 1$. Then we have an isomorphism $A_{(Q,\mathsf{C})} \simeq kQ/I$ where $I = \langle \alpha\beta', \alpha'\beta, \gamma\alpha', \gamma'\alpha, (abc)a \mid \{a,b,c\} = \{\alpha,\beta,\gamma\}, \{\alpha',\beta',\gamma'\}\rangle$.

We know that the property of being symmetric special biserial is derived invariant.

Proposition 10. *Let A and B be finite dimensional algebras. Suppose that A and B are derived equivalent. If A is a symmetric special biserial algebra, then so is B.*

Proof. Combine [R1] and [P]. □

Next, we recall the notion of tilting mutation. We refer to [AI] for details.
The bounded derived category of $\operatorname{mod} A$ is denoted by $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$.
We give the definition of tilting complexes.

Definition 11. Let A be a finite dimensional algebra. We say that a bounded complex T of finitely generated projective A-modules is *tilting* if it satisfies $\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)}(T, T[n]) = 0$ for any integer $n \neq 0$ and produces the complex A concerned in degree 0 by taking direct summands, mapping cones and shifts.

The following result shows the importance of tilting complexes.

Theorem 12. [R1] *Let A and B be finite dimensional algebras. Then A and B are derived equivalent if and only if there exists a tilting complex T of A such that B is Morita equivalent to the endomorphism algebra* $\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)}(T)$.

For each vertex i of Q, we denote by e_i the corresponding primitive idempotent of A.

We recall a complex given by Okuyama and Rickard [O, R2], which is a special case of tilting mutation (see [AI]).

Definition-Theorem 13. [O] Fix a vertex i of Q. We define a complex by

$$T_j := \begin{cases} \begin{array}{ccc} (0\text{th}) & & (1\text{st}) \\ P_j & \longrightarrow & 0 \\ P & \xrightarrow{\pi_i} & P_i \end{array} & \begin{array}{c} \\ (j \neq i) \\ (j = i) \end{array} \end{cases}$$

where $P \xrightarrow{\pi_i} P_i$ is a minimal projective presentation of $e_iA/e_iA(1-e_i)A$. Now we call $T(i) := \bigoplus_{j\in Q_0} T_j$ an *Okuyama-Rickard complex* with respect to i and put $\mu_i^+(A) := \mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)}(T(i))$. If A is symmetric, then T is tilting. In particular, $\mu_i^+(A)$ is derived equivalent to A.

3. Mutation of SB quivers

The aim of this paper is to give a purely combinatorial description of tilting mutation of symmetric special biserial algebras.

To do this, we introduce *mutation of SB quivers* by dividing to three cases, which is a new SB quiver $\mu_i^+(Q,\mathsf{C})$ made from a given one (Q,C).

Now, the main theorem in this paper is stated, which gives the compatibility between tilting mutation and mutation of SB quivers.

Theorem 14. *Let A be a symmetric special biserial algebra and take a SB quiver (Q,C) satisfying $A \simeq A_{(Q,\mathsf{C})}$. Let i be a vertex of Q. Then we have an isomorphism $A_{\mu_i^+(Q,\mathsf{C})} \simeq \mu_i^+(A)$. In particular, $A_{\mu_i^+(Q,\mathsf{C})}$ is derived equivalent to A.*

Let (Q,C) be a SB quiver and i be a vertex of Q. We say that Q is *multiplex* at i if there exists arrows $i \underset{\beta}{\overset{\alpha}{\rightleftarrows}} j$ with $\beta \neq \mathsf{na}(\alpha)$ and $\alpha \neq \mathsf{na}(\beta)$.

3.1. **Non-multiplex case.** We introduce mutation of SB quivers at non-multiplex vertices.

Let (Q,C) be a SB quiver and fix a vertex i of Q. We define a new SB quiver $\mu_i^+(Q,\mathsf{C}) = (Q',\mathsf{C}')$ as follows.

3.1.1. *Mutation rules.*

Definition 15. Suppose that Q is non-multiplex at i. We define a quiver Q' as the following three steps:

(QM1) Consider any path

$$h \xrightarrow{\alpha} i \xrightarrow{\beta} j \text{ with } \beta = \mathsf{na}(\alpha) \qquad \text{or} \qquad \begin{tikzcd} h \arrow[r, "\alpha"] & i \arrow[loop above, "\gamma"] \arrow[r, "\beta"] & j \end{tikzcd} \text{ with } \gamma = \mathsf{na}(\alpha), \beta = \mathsf{na}(\gamma)$$

for $h \neq i \neq j$. Then draw a new arrow $h \xrightarrow{x} j$

(QM1-1) if $h \neq j$ or

(QM1-2) if $h = j, \alpha = \mathsf{na}(\beta)$ and $\mathsf{mult}(C_\alpha) > 1$.

(QM2) Remove all arrows $h \longrightarrow i$ for $h \neq i$.

(QM3) Consider any arrow $i \xrightarrow{\alpha} h$ for $h \neq i$.

(QM3-1) If there exists a path $i \xrightarrow{\alpha} h \xrightarrow{\beta} j$ with $\beta \neq \mathsf{na}(\alpha)$, then replace it by a new path $h \xrightarrow{x} i \xrightarrow{y} j$.

(QM3-2) Otherwise, add a new arrow $h \xrightarrow{x} i$.

It is easy to see that the new quiver Q' is again special.

3.1.2. *Cycle-decompositions.* We give a cycle-decomposition C' of Q'.

Definition 16. We use the notation of Definition 15.

(1) We define a cycle containing a new arrow x in (QM1) as follows:
  (a) In the case (QM1-1), C_x is obtained by replacing $\alpha\beta$ or $\alpha\gamma\beta$ in C_α by x.
  (b) In the case (QM1-2), C_x is a new cycle $\begin{tikzcd} h \arrow[loop right, "x"] \end{tikzcd}$ with multiplicity $\mathsf{mult}(C_\alpha)$.

(2) We define a cycle containing a new arrow x and y in (QM3) as follows:
  (a) In the case (QM3-1), $C_x = C_y$ and replace β in C_β by xy.
  (b) In the case (QM3-2),
    (i) if there exists an arrow $h \xrightarrow{\beta} i$ of Q, then C_x is defined by replacing β in C_β by x.
    (ii) Otherwise, C_x is a new cycle

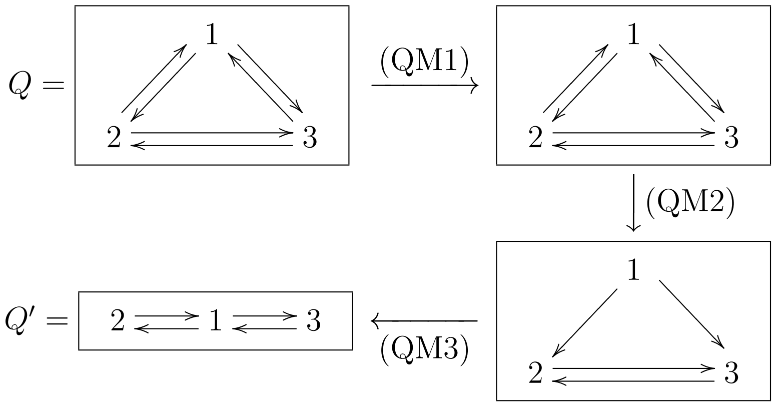
$$\begin{cases} \begin{tikzcd} i \arrow[r, shift left, "\alpha"] & h \arrow[l, shift left, "x"] \end{tikzcd} & \text{if there is no loop at } i \text{ belonging to } C_\alpha \\ \begin{tikzcd} & i \arrow[dr, "\alpha"] & \\ i \arrow[ur, "\beta"] & & h \arrow[ll, "x"] \end{tikzcd} & \text{if there is a loop } \beta \text{ at } i \text{ belonging to } C_\alpha \end{cases}$$

with multiplicity 1.

Then we obtain a cycle-decomposition C' of Q'.

Thus, we get a new SB quiver $\mu_i^+(Q, \mathsf{C}) := (Q', \mathsf{C}')$, called *right mutation* of (Q, C) at i.
Dually, we define the *left mutation* $\mu_i^-(Q, \mathsf{C})$ of (Q, C) at i by $\mu_i^-(Q, \mathsf{C}) := \mu_i^+(Q^{\mathrm{op}}, \mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$, where Q^{op} is the opposite quiver of Q and C^{op} is the cycle-decomposition of Q^{op} corresponding to C.

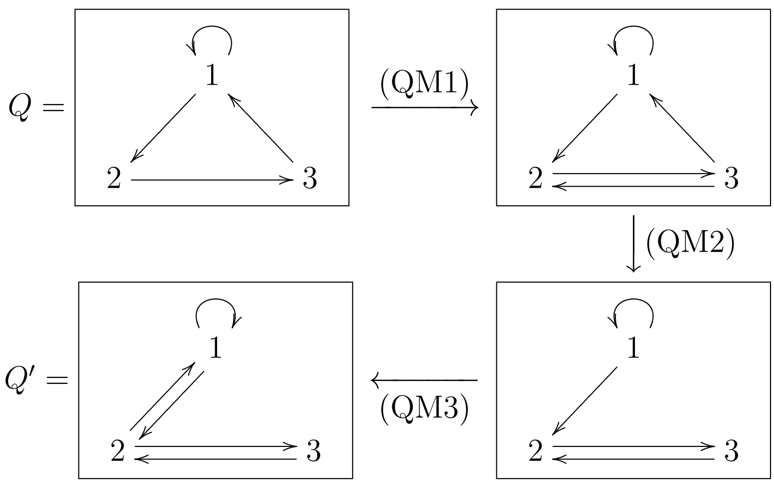
Example 17. (1) Let (Q, C) be the SB quiver as in Example 9 (1). Then we have the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 as follows:



and

$$\mathsf{C}' = \left\{ \left(\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 3 \arrow[u] & 1 \arrow[l] \end{tikzcd} \right) \right\}$$

(2) Let (Q, C) be the SB quiver of Example 9 (2). Then the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 is obtained as follows:

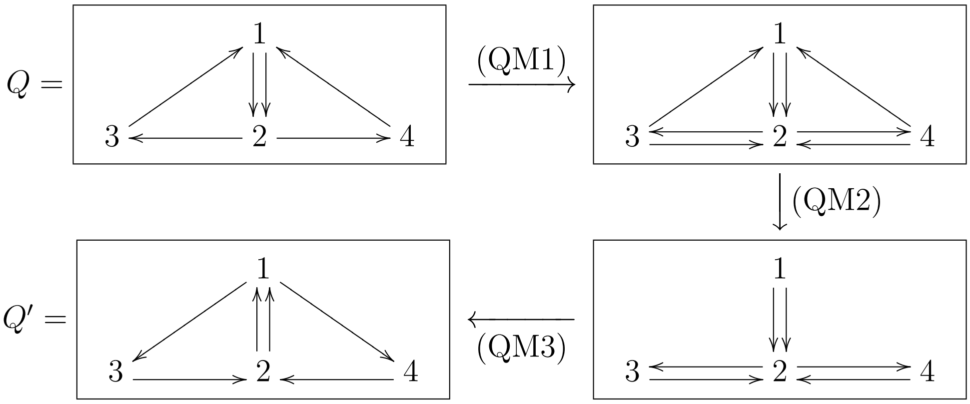


and

$$\mathsf{C}' = \left\{ \left(\begin{tikzcd} & 1 \arrow[dr] & \\ 1 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd} \right), \left(\begin{tikzcd} 2 \arrow[r, shift left] & 3 \arrow[l, shift left] \end{tikzcd} \right) \right\}$$

where the first and the second cycles have multiplicity 1 and 2, respectively.

(3) Let (Q, C) be the SB quiver as in Example 9 (3). Then we get the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 as follows:



and

$$\mathsf{C}' = \left\{ \left(\begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \\ 3 \arrow[r] & 2 \arrow[u] \end{tikzcd} \right), \left(\begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & \\ 2 \arrow[u] & 4 \arrow[l] \end{tikzcd} \right) \right\}$$

3.2. **Multiplex case (1).** Next, we introduce mutation at multiplex vertices and its cycle-decomposition. They are defined by making minor alterations to mutation at non-multiplex vertices.

Let (Q, C) be a SB quiver and fix a vertex i of Q. We consider the following situation:

$$\begin{tikzcd} j' \arrow[dr, "\alpha'"] & & \\ & i \arrow[r, shift left, "\alpha"] \arrow[dl, "\beta'"] & j \arrow[l, shift left, "\beta"] \\ h & & \end{tikzcd}$$

with $\beta \neq \mathsf{na}(\alpha)$ and $\alpha \neq \mathsf{na}(\beta)$: in this case, it is observed that $\alpha = \mathsf{na}(\alpha')$ and $\beta' = \mathsf{na}(\beta)$.

We define a new SB quiver $\mu_i^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ as follows.

3.2.1. *Mutation rules.*

Definition 18. We assume that $j' \neq h$. A quiver Q' of Q at i is defined by the following three steps:

(QM1)' Draw a new arrow $j' \xrightarrow{x} j$

(QM1-1)' if $j' \neq j$ or

(QM1-2)' if $j' = j$ and $\mathsf{mult}(C_\alpha) > 1$.

(QM2)' Remove two arrows α and α'.

(QM3)' Add new arrows in the following way:

(QM3-1)' If there is an arrow $\gamma : h \longrightarrow h'$ with $\gamma \neq \mathsf{na}(\beta')$, then remove γ and add new arrows $h \xrightarrow{x} i \xrightarrow{y} h'$.

(QM3-2)' Otherwise, add new arrows $h \underset{y}{\overset{x}{\rightleftarrows}} i$.

We can easily check that the new quiver Q' is again special.

3.2.2. *Cycle-decompositions.* We give a cycle-decomposition C' of Q'.

Definition 19. Assume that $j' \neq h$. We use the notation of Definition 18.

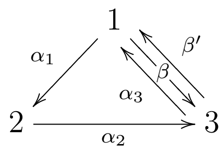
(1) We define a cycle containing a new arrow x in (QM1)' as follows:
  (a) In the case (QM1-1)', C_x is obtained by replacing $\alpha'\alpha$ in C_α by x.
  (b) In the case (QM1-2)', C_x is a new cycle $j \circlearrowright x$ with multiplicity $\mathsf{mult}(C_\alpha)$.

(2) We define a cycle containing a new arrow x and y in (QM3)' as follows:
  (a) In the case (QM3-1)', $C_x = C_y$ and replace γ in C_γ by xy.
  (b) In the case (QM3-2)', C_x and C_y are new cycles satisfying $C_x = C_y = \left(h \underset{y}{\overset{x}{\rightleftarrows}} i \right)$ with multiplicity 1.

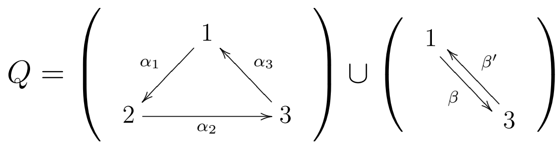
Then we have a cycle-decomposition C' of Q'.

Thus, we get a new SB quiver $\mu_i^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$, called *right mutation* of (Q, C) at i. Dually, we define the *left mutation* $\mu_i^-(Q, \mathsf{C})$ of (Q, C) at i by $\mu_i^-(Q, \mathsf{C}) := \mu_i^+(Q^{\mathrm{op}}, \mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$.

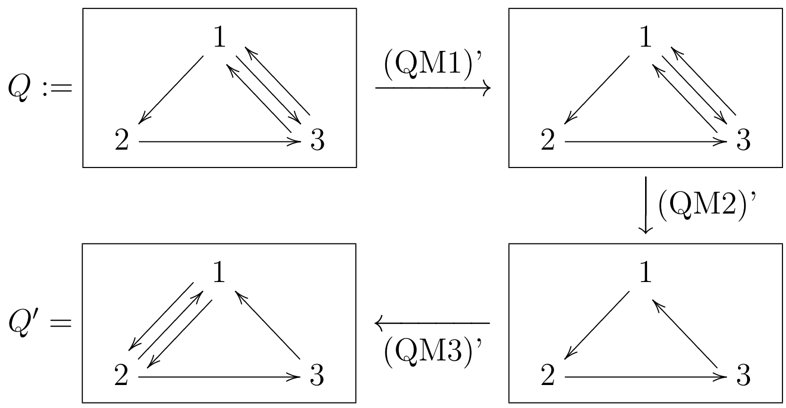
Example 20. Let Q be the quiver



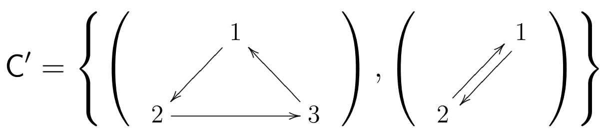
with cycle-decomposition C:



such that the multiplicity of each cycle is 1. Then we see that the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 is



and



3.3. **Multiplex case (2).** Finally, we introduce the last case of mutation of SB quivers.

Let (Q, C) be a SB quiver and fix a vertex i of Q. Suppose that the subquiver of Q around the vertex i is

$$\begin{tikzcd} j' \arrow[r, shift left, "\alpha'"] & i \arrow[l, shift left, "\beta'"] \arrow[r, shift left, "\alpha"] & j \arrow[l, shift left, "\beta"] \end{tikzcd}$$

with $\beta \neq \mathsf{na}(\alpha)$ and $\alpha \neq \mathsf{na}(\beta)$: i.e., the case of $j' = h$ in Multiplex case (1).

Definition 21. We define the *right mutation* $\mu_i^+(Q, \mathsf{C})$ of (Q, C) at i by $\mu_i^+(Q, \mathsf{C}) = (Q, \mathsf{C})$.

Dually, the *left mutation* $\mu_i^-(Q, \mathsf{C})$ of (Q, C) at i is also defined by $\mu_i^-(Q, \mathsf{C}) = (Q, \mathsf{C})$.

4. Brauer graph algebras

In this section, we introduce *flip* of Brauer graphs and show that it is compatible with tilting mutation of Brauer graph algebras. Note that any Brauer graph algebra is symmetric special biserial (see Proposition 26).

We recall the definition of Brauer graphs.

Definition 22. A *Brauer graph* G is a graph with the following data:

(1) There exists a cyclic ordering of the edges adjacent to each vertex, usually described by the clockwise ordering.
(2) For every vertex v, there exists a positive integer m_v assigned to v, called the *multiplicity* of v. We say that a vertex v is *exceptional* if $m_v > 1$

4.1. **Flip of Brauer graphs.** Let G be a Brauer graph. For a cyclic ordering $(\cdots, i, j, \cdots)$ adjacent to a vertex v with $j \neq i$, we write j by $e_v(i)$ and denote by $v_j(i)$ (simply, $v(i)$) the vertex of j distinct from v if it exists, otherwise $v(i) := v$.

We say that an edge i of G is *external* if it has a vertex with cyclic ordering which consists of only i, otherwise it is said to be *internal*.

We now introduce flip of Brauer graphs.

Definition 23. Let G be a Brauer graph and fix an edge i of G. We define the *flip* $\mu_i^+(G)$ of G as follows:

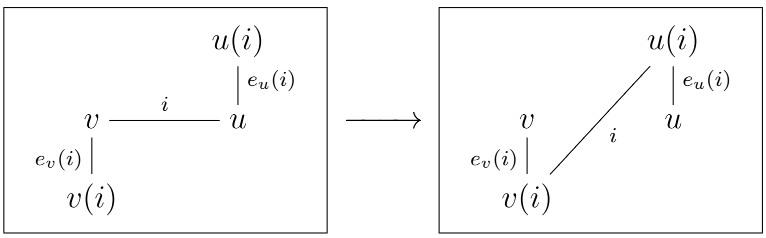
Case (1) The edge i has the distinct two vertices v and u:

- If i is internal, then

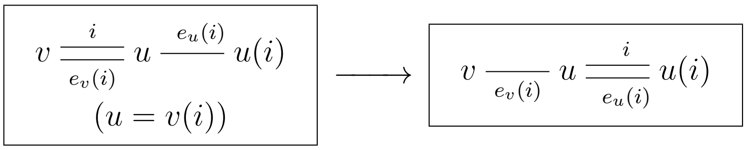
(Step 1) detach i from v and u;
(Step 2) attach it to $v(i)$ and $u(i)$ by $e_{v(i)}(e_v(i)) = i$ and $e_{u(i)}(e_u(i)) = i$, respectively.

Locally there are the following three cases:

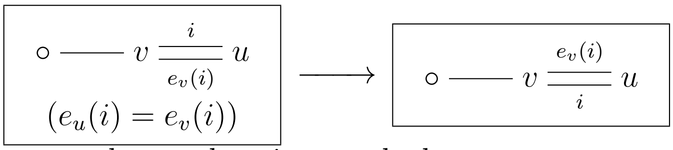
(i)



(ii)



(iii)

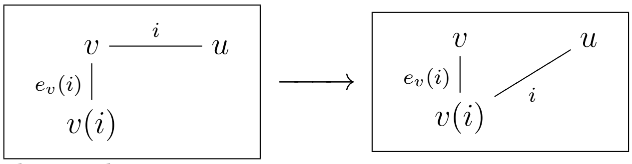


- If i is external, namely u is at end, then

(Step 1) detach i from v;
(Step 2) attach it to $v(i)$ by $e_{v(i)}(e_v(i)) = i$.

The local picture is the following:

(iv)



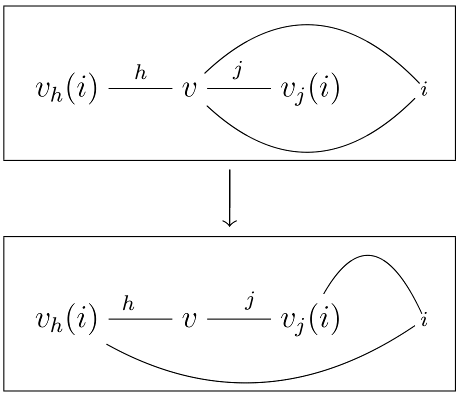
Case (2) The edge i has only one vertex v:

- If there exists the distinct two edges h and j written by $e_v(i)$, then

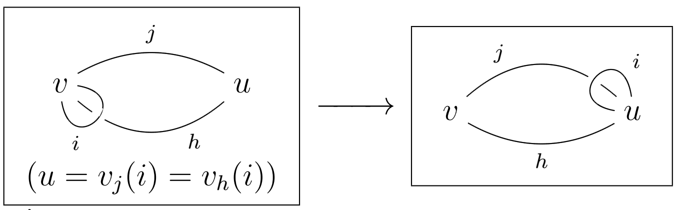
(Step 1) detach i from v;
(Step 2) attach it to $v_h(i)$ and $v_j(i)$ by $e_{v_h(i)}(h) = i$ and $e_{v_j(i)}(j) = i$.

Locally there are the following two cases:

(v)



(vi)

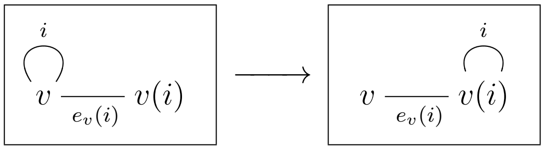


- Otherwise,

(Step 1) detach i from v;
(Step 2) attach it to the only one vertex $v(i)$ by $e_{v(i)}(e_v(i)) = i$.

The local picture is the following:

(vii)



In all cases, the multiplicity of any vertex does not change.

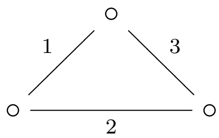
Dually, we define $\mu_i^-(G)$ by $\mu_i^-(G) := \big(\mu_i^+(G^{\mathrm{op}})\big)^{\mathrm{op}}$ where the opposite Brauer graph, namely its cyclic ordering is described by counter-clockwise, is denoted by G^{op}.

Every case of flip of Brauer graphs is covered in Definition 23.

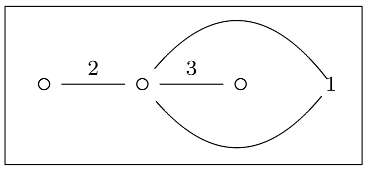
We also point out that our flip of Brauer graphs can be regarded as a generalization of flip of triangulations of surfaces [FST].

Example 24. For a Brauer graph, we denote by $\bullet$ an exceptional vertex and by $\circ$ a non-exceptional vertex.

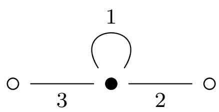
(1) Let G be the Brauer graph



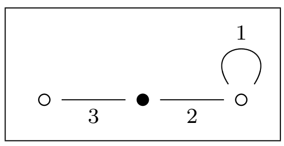
Then the flip of G at 1 is

$\mu_1^+(G) =$ 

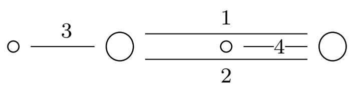
(2) Let G be the Brauer graph



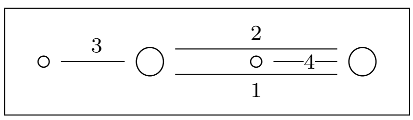
such that the multiplicity of the exceptional vertex $\bullet$ is 2. Then we have the flip of G at 1:

$\mu_1^+(G) =$ 

(3) Let G be the Brauer graph



Then the flip of G at 1 is observed by

$\mu_1^+(G) =$ 

(4) Let G be the Brauer graph

$$\circ \underset{3}{\overset{1}{=\!=\!=}} \circ \overset{2}{\text{———}} \circ$$

Then the flip of G at 1 is:

$$\mu_1^+(G) = \boxed{\circ \overset{3}{\text{———}} \circ \underset{2}{\overset{1}{=\!=\!=}} \circ}$$

4.2. **Compatibility of flip and tilting mutation.** For a Brauer graph G, we denote by $\mathsf{vx}(G)$ the set of the vertices of G.

We construct a SB quiver from a Brauer graph.

Definition 25. Let G be a Brauer graph. A *Brauer quiver* $Q = Q_G$ is a finite quiver given by a Brauer graph G as follows:

(1) There exists a one-to-one correspondence between vertices of Q and edges of G.
(2) For two distinct edges i and j, an arrow $i \longrightarrow j$ of Q is drawn if there exists a cyclic ordering of the form $(\cdots, i, j, \cdots)$.
(3) For an edge i of G, we draw a loop at i if it has an exceptional vertex which is at end.

Then Q is special.

For each vertex v of G, let $(i_1, i_2, \cdots, i_s, i_1)$ be a cyclic ordering at v. Then we define a cycle C_v by

$$\begin{tikzcd} i_1 \arrow[r] & i_2 \arrow[r] & \cdots \arrow[r] & i_s \arrow[lll, bend right] \end{tikzcd}$$

with multiplicity m_v if $s \neq 1$, otherwise by an empty set. We have a cycle-decomposition $\mathsf{C} = \mathsf{C}_G = \{C_v \mid v \in \mathsf{vx}(G)\}$.

Thus we obtain a SB quiver (Q, C).

For a Brauer graph G, a *Brauer graph algebra* $A = A_G$ is a symmetric special biserial algebra associated with the SB quiver (Q_G, C_G).

It is known that the notion of Brauer graph algebras is nothing but that of symmetric special biserial algebras. The following result is obtained.

Proposition 26. (1) [Ro] *An algebra is a Brauer graph algebra if and only if it is symmetric special biserial.*
(2) *The property of being a Brauer graph algebra is derived invariant.*

Proof. The second assertion follows from the first assertion and Proposition 10. $\square$

For an edge i of a Brauer graph G, we say that G *has multi-edges* at i if there exists a subgraph $\overset{v}{\circ} \underset{j}{\overset{i}{=\!=\!=}} \overset{u}{\circ}$ of G such that the cyclic orderings at v and u are $(\cdots, i, j, \cdots)$ and $(\cdots, j, i, \cdots)$, respectively; it is allowed that $u = v$.

We have the following easy observation.

Proposition 27. *Let G be a Brauer graph and i be an edge of G. Then Q_G is multiplex at i if and only if G has multi-edges at i.*

It is not difficult to see that flip of each Brauer graph G coincides with right mutation of the corresponding SB quiver (Q_G, C_G), that is, we have:

Proposition 28. *Let G be a Brauer graph and i be an edge of G. Then one has $(Q_{\mu_i^+(G)}, \mathsf{C}_{\mu_i^+(G)}) = \mu_i^+(Q_G, \mathsf{C}_G)$.*

We observe that Example 17 (1)–(3) and Example 20 coincide with Example 24 (1)–(3) and (4), respectively.

Applying Theorem 14 to Brauer graph algebras, we figure out that flip of Brauer graph is compatible with tilting mutation of Brauer graph algebras.

Theorem 29. *Let G be a Brauer graph and i be an edge of G.*

(1) *We have an isomorphism $A_{\mu_i^+(G)} \simeq \mu_i^+(A_G)$.*
(2) *The algebra $\mu_i^+(A_G)$ has the Brauer graph $\mu_i^+(G)$.*
(3) *The algebra $A_{\mu_i^+(G)}$ is derived equivalent to A_G.*

Remark 30. Special cases of this theorem were given in [K], where he considered the cases (i)(iv) and (vii) in Definition 23.

4.3. **(Double-) Star theorem.** In this subsection, we generalize Rickard's star theorem.

Let G be a Brauer graph. We denote by m_G the sequence $(m_{v_1}, \cdots, m_{v_\ell})$ of the multiplicities of all vertices satisfying $m_{v_1} \geq \cdots \geq m_{v_\ell}$.

For a Brauer graph algebra A, the Brauer graph of A is denoted by G_A.

A Brauer graph G is said to be *double-star* if there exist two vertices v and u of G such that any edge is either of the following:

- It is external having the vertex v:
- It has both the vertices v and u:
- It has only the vertex v, that is, it is of the form v.

We call v and u *center* and *vice-center*, respectively.

We say that a Brauer double-star G satisfies *multiplicity condition* if the multiplicities of the center and the vice-center are the first and the second greatest among them of all vertices of G, respectively.

The following theorem is obtained.

Theorem 31. *Any Brauer graph algebra A is derived equivalent to a Brauer graph algebra with Brauer double-star G satisfying multiplicity condition such that*

(1) *the number of the edges of G coincides with that of G_A and*
(2) *$\mathsf{m}_G = \mathsf{m}_{G_A}$, in particular G and G_A have the same number of exceptional vertices.*

We raise a question on classification of derived equivalence classes of Brauer graph algebras.

Question 32. *For a given Brauer graph G, is a Brauer double-star algebra satisfying multiplicity condition which is derived equivalent to A_G unique, up to isomorphism and opposite isomorphism?*

It is well-known that this question has a positive answer if G is a tree as a graph. Such a Brauer graph is said to be a *generalized Brauer tree*. It is called *Brauer tree* if it has at most one exceptional vertex. A *(generalized) Brauer star* is a (generalized) Brauer tree and a Brauer double-star. Note that any edge of a generalized Brauer star is external and every vertex can be a vice-center.

From Theorem 31, we deduce star theorem for generalized Brauer tree algebras.

Corollary 33. [R2, M]

(1) *Any generalized Brauer tree algebra A is derived equivalent to a generalized Brauer star algebra B with $\mathsf{m}_{G_B} = \mathsf{m}_{G_A}$ such that the multiplicity of the center is maximal.*
(2) *Derived equivalence classes of generalized Brauer tree algebras are determined by the number of the edges and the multiplicities of the vertices.*

5. Tilting quivers of Brauer graph algebras

This section is based on joint work with Adachi and Chan [AAC].

In this section, we discuss tilting quivers which were introduced in [AI] to observe the behavior of tilting mutation.

We denote by $\mathsf{tilt}\,\Lambda$ the set of non-isomorphic basic tilting complexes of a finite dimensional algebra Λ.

Let us recall the definition of tilting quivers (see [AI] for the details).

Definition 34. Let Λ be a symmetric algebra. The *tilting quiver* of Λ is defined as follows:

- The set of vertices is $\mathsf{tilt}\,\Lambda$.
- An arrow $T \to U$ is drawn if T corresponds to an Okuyama-Rickard complex of the endomorphism algebra $\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)}(U)$ under the derived equivalence induced by U.

We naturally ask whether the tilting quiver of Λ is always connected or not. The answer of this question is No. An example of symmetric algebras whose tilting quivers are not connected was found by Grant, Iyama and the author.

A symmetric algebra is said to be *tilting-connected* if its tilting quiver is connected.

We raise the following question.

Question 35. *When is a symmetric algebra tilting-connected?*

We introduce a partial order on $\mathsf{tilt}\,\Lambda$.

Definition-Theorem 36. [AI] Let Λ be a symmetric algebra. For $T, U \in \mathsf{tilt}\,\Lambda$, we write $T \geq U$ if it satisfies $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)}(T, U[i]) = 0$ for any $i > 0$. Then $\geq$ gives a partial order on $\mathsf{tilt}\,\Lambda$.

We say that a symmetric algebra Λ is *tilting-discrete* if for any $\ell > 0$, there exist only finitely many tilting complexes T in $\mathsf{tilt}\,\Lambda$ satisfying $\Lambda \geq T \geq \Lambda[\ell]$. It is seen that a tilting-discrete symmetric algebra is tilting-connected [A1].

Two examples of tilting-discrete symmetric algebras are well-known.

Example 37. [AI, A1] A symmetric algebra is tilting-discrete if it is either local or representation-finite.

We refer to [AI] for more general examples of tilting/silting-connected algebras.

The aim of this section is to give a partial answer to Question 35 for Brauer graph algebras.

We say that a Brauer graph is of *type odd* if it has at most one cycle of odd length and none of even length. For example, any (generalized) Brauer tree is of type odd.

We easily observe the following result, which says that flip of Brauer graphs preserves the property to be of type odd.

Lemma 38. *Let G be a Brauer graph and i be an edge of G. If G is of type odd, then so is $\mu_i^+(G)$.*

Now we state the main theorem of this section.

Theorem 39. *Let G be a Brauer graph. Then G is of type odd if and only if A_G is tilting-discrete.*

Example 40. Any generalized Brauer graph algebra is tilting-discrete, and hence is tilting-connected.

The following corollary is an immediate consequence of Theorem 39.

Corollary 41. *Let G be a Brauer graph of type odd and B a derived equivalent algebra to A_G. Then the following hold:*

(1) *The algebra B is a Brauer graph algebra whose Brauer graph can be obtained from G by iterated flip.*
(2) *The Brauer graph of B is of type odd.*

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QUANTUM PLANES AND ITERATED ORE EXTENSIONS

RYO AKIYAMA

Abstract. Quantum projective planes are well studied in noncommutative algebraic geometry. However, there has never been a precise definition of a quantum affine plane. In this paper, we define a quantum affine plane, and classify quantum affine planes by using 3-iterated quadratic Ore extensions of k.

1. Preliminaries

Throughout this paper, we fix an algebraically closed field k of characteristic 0, and we assume that all vector spaces and algebras are over k. In this paper, a graded algebra means a connected graded algebra finitely generated over k. A connected graded algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ such that $A_0 = k$. We denote by $\mathrm{GrMod}A$ the category of graded right A-modules. An AS-regular algebra defined below is one of the main objects of study in noncommutative algebraic geometry.

Definition 1 ([1]). A noetherian connected graded algebra A is called a d-dimensional AS-regular algebra if

- $\mathrm{gl.dim}A = d < \infty$, and
- $\mathrm{Ext}^i_A(k, A) \cong \begin{cases} k & i = d \\ 0 & i \neq d. \end{cases}$

One of the first achievements of noncommutative algebraic geometry was classifying all 3-dimensional AS-regular algebras by Artin, Tate and Van den Bergh using geometric techniques [2]. In this paper, we will use their classification only in the quadratic case.

Let $T(V)$ be the tensor algebra on V over k where V is a finite dimensional vector space. We say that A is a quadratic algebra if A is a graded algebra of the form $T(V)/(I)$ where $I \subseteq V \otimes_k V$ is a subspace and (I) is the two-sided ideal of $T(V)$ generated by I. For a quadratic algebra $A = T(V)/(I)$, we define

$$\mathcal{V}(I) = \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) | \ f(p, q) = 0 \text{ for all } f \in I\}.$$

Definition 2 ([5]). A quadratic algebra $A = T(V)/(I)$ is called geometric if there exists a geometric pair (E, τ) where $E \subseteq \mathbb{P}(V^*)$ is a closed k-subscheme and τ is a k-automorphism of E such that

(G1) $\mathcal{V}(I) = \{(p, \tau(p) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) | \ p \in E\}$, and
(G2) $I = \{f \in V \otimes_k V | \ f(p, \tau(p)) = 0 \text{ for all } p \in E\}$.

Let $A = T(V)/(I)$ be a quadratic algebra. If A satisfies the condition **(G1)**, then A determines a geometric pair (E, τ). If A satisfies the condition **(G2)**, then A is determined by a geometric pair (E, τ), so we will write $A = \mathcal{A}(E, \tau)$. All 3-dimensional quadratic

The detailed version of this paper will be submitted for publication elsewhere.

AS-regular algebras are geometric by [2]. Moreover, it follows that they can be classified in terms of geometric pairs (E, τ), where E is either $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$ by [2].

2. Ore extensions.

Ore extensions are defined as follows:

Definition 3 ([4]). Let R be an algebra, σ an automorphism of R and δ a σ-derivation (i.e., $\delta : R \to R$ is a linear map such that $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in R$). Then σ, δ uniquely determine an algebra S satisfying the following two properties;

- $S = R[z]$ as a left R-module .
- For any $a \in R$, $za = \sigma(a)z + \delta(a)$.

The algebra S is denoted by $R[z; \sigma, \delta]$ and is called the Ore extension of R associated to σ and δ. Then we define an n-iterated Ore extension of k by

$$k[z_1; \sigma_1, \delta_1][z_2; \sigma_2, \delta_2] \cdots [z_n; \sigma_n, \delta_n].$$

Iterated graded Ore extensions of k are defined bellow.

Definition 4. Let A be a graded algebra, σ a graded automorphism of A and δ a graded σ-derivation (i.e., $\delta : A \to A$ is a linear map of degree ℓ for some $\ell \in \mathbb{N}$ such that $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in A$). Then σ, δ uniquely determine a graded algebra B satisfying the following two properties

- $B = A[z]$ with $\deg(z) = \ell$ as a graded left A-module.
- For any $a \in A$, $za = \sigma(a)z + \delta(a)$.

The graded algebra B is denoted by $A[z; \sigma, \delta]$ and is called the graded Ore extension of A associated to σ and δ. Then we define an n-iterated graded Ore extension of k by

$$k[z_1; \sigma_1, \delta_1][z_2; \sigma_2, \delta_2] \cdots [z_n; \sigma_n, \delta_n].$$

If $\deg(x_i) = 1$ for any $i \in \{1, 2, \cdots, n\}$, the above algebra is a quadratic algebra. Then we call it an n-iterated quadratic Ore extension of k.

It is known that n-iterated quadratic Ore extensions of k are n-dimensional quadratic AS-regular algebras. Moreover, if $n \leqslant 2$, then n-dimensional AS-regular algebras are n-iterated graded Ore extensions of k by [6]. In this paper, we answer the question which 3-dimensional quadratic AS-regular algebras are 3-iterated quadratic Ore extensions of k.

Theorem 5. *Let $A = \mathcal{A}(E, \tau)$ be a 3-dimensional quadratic AS-regular algebra. Then there exists a 3-iterated quadratic Ore extension B such that* $\mathrm{GrMod}A \cong \mathrm{GrMod}B$ *if and only if E is not a elliptic curve.*

3. Quantum planes

Definition 6 ([3]). Let R be an algebra. We denote by $\mathrm{Mod}R$ the category of right R-modules. We define the noncommutative affine scheme $\mathrm{Spec}_{\mathrm{nc}}R$ associated to R by the pair $(\mathrm{Mod}R, R)$.

Let $\mathrm{Tails}A$ be the the quotient category $\mathrm{GrMod}A/\mathrm{Tors}A$ where $\mathrm{Tors}A$ is the full subcategory of $\mathrm{GrMod}A$ consisting of direct limits of modules finite dimensional over k, and let π be the canonical functor $\mathrm{GrMod}A \to \mathrm{Tails}A$.

Definition 7 ([3])**.** Let A be a graded algebra. We define the noncommutative projective scheme $\mathrm{Proj}_{\mathrm{nc}}A$ associated to A by the pair $(\mathrm{Tails}A, \pi A)$.

The simplest surface in algebraic geometry is the affine plane, which is Spec $k[x, y]$, so the simplest noncommutative surface must be a quantum affine plane, which should be $\mathrm{Spec}_{\mathrm{nc}}R$, where R is a noncommutative analogue of $k[x, y]$. Since a skew polynomial algebra $R = k\langle x, y\rangle/(xy - \lambda yx)$ is the simplest example of a noncommutative analogue of $k[x, y]$ in noncommutative algebraic geometry, it can be regarded as a coordinate ring of a quantum affine plane. However, there has never been a precise definition of quantum affine plane. In the projective case, if A is a $(d + 1)$-dimensional quadratic AS-regular algebra, then we call $\mathrm{Proj}_{\mathrm{nc}}A$ a d-dimensional quantum projective space (q-$\mathbb{P}^d$). In particular, if A is a 3-dimensional quadratic AS-regular algebra, then we call $\mathrm{Proj}_{\mathrm{nc}}A$ a quantum projective plane (q-$\mathbb{P}^2$).

In algebraic geometry, the following result is well known. If A is a polynomial algebra $k[x, y, z]$ and $u \in A_1$, then

$$\begin{array}{ccccc} \mathrm{Proj}\,A & = & \mathrm{Proj}\,A/(u) & \cup & \mathrm{Spec}\,A[u^{-1}]_0 \\ \wr\Vert & & \wr\Vert & & \wr\Vert \\ \mathbb{P}^2 & & \mathbb{P}^1 & & \mathbb{A}^2 \end{array}.$$

Meanwhile, if A be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element (i.e., $uA = Au$), then $\mathrm{Proj}_{\mathrm{nc}}A$ is a q-$\mathbb{P}^2$ and $\mathrm{Proj}_{\mathrm{nc}}A/(u)$ is a q-$\mathbb{P}^1$. Following the above facts, we define a quantum affine plane as follows.

Definition 8. Let A be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element (i.e., $uA = Au$), then we define a quantum affine plane by

$$\mathrm{Spec}_{\mathrm{nc}}A[u^{-1}]_0$$

where $A[u^{-1}]_0$ is the degree zero part of the noncommutative graded localization of A.

Example 9. The algebra $A = k\langle x, y, z\rangle/(yz-\alpha zy, zx-\beta xz, xy-\gamma yx)$ where $0 \neq \alpha, \beta, \gamma \in k$ is a 3-dimensional quadratic AS-regular algebra. Then A has a normal element $x \in A_1$, and one can show that $A[x^{-1}]_0 \cong k\langle s, t\rangle/(st - \alpha\beta\gamma ts)$.

4. Classification of quantum affine planes

In this section, we will classify quantum affine planes. We define $\mathrm{Spec}_{\mathrm{nc}}R$ and $\mathrm{Spec}_{\mathrm{nc}}R'$ are isomorphic if there exists an equivalence functor $F : \mathrm{Mod}R \to \mathrm{Mod}R'$ such that $F(R) \cong R'$. Since $\mathrm{Spec}_{\mathrm{nc}}R$ and $\mathrm{Spec}_{\mathrm{nc}}R'$ are isomorphic if and only if $R \cong R'$, we call the coordinate ring $A[u^{-1}]_0$ a quantum affine plane.

Although it is difficult to find normal elements of a given algebra in general, we can find normal elements of 3-dimensional quadratic AS-regular algebras by using geometric pairs.

Lemma 10. *Let $A = \mathcal{A}(E, \tau)$ be a 3-dimensional quadratic AS-regular algebra, and let $u \in A_1$. Then $u \in A_1$ is a normal element if and only if*

(1) $\mathcal{V}(u) \subset E$,
(2) $\tau(\mathcal{V}(u)) = \mathcal{V}(u)$.

In addition, the following lemma is very useful to classify quantum affine planes.

Lemma 11. *Let A be a 3-dimensional quadratic AS-regular algebra and $u \in A_1$ a normal element. Then there exist a 3-iterated quadratic Ore extension B and a normal element $v \in B_1$ which satisfy*

$$\mathrm{GrMod}A \cong \mathrm{GrMod}B \text{ and } A[u^{-1}]_0 \cong B[v^{-1}]_0.$$

By using the above lemmas,

Theorem 12. *Every quantum affine plane is isomorphic to exactly one of the following:*

$$\begin{aligned}
&k\langle s,t\rangle/(st-\lambda ts) =: S_\lambda \ (0 \neq \lambda \in k)\\
&k\langle s,t\rangle/(st-\lambda ts+1) =: T_\lambda \ (0 \neq \lambda \in k)\\
&k\langle s,t\rangle/(ts-st+t)\\
&k\langle s,t\rangle/(ts-st+t^2)\\
&k\langle s,t\rangle/(ts-st+t^2+1)
\end{aligned}$$

where

$$S_\lambda \cong S_{\lambda'} \Leftrightarrow \lambda' = \lambda^{\pm 1}, \quad T_\lambda \cong T_{\lambda'} \Leftrightarrow \lambda' = \lambda^{\pm 1}.$$

All of the above algebras are 2-iterated (ungraded) Ore extensions of k. Hence, we see that quantum affine planes have nice properties like a polynomial algebras. For example, they are noetherian domains and have finite global dimension.

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ON SOME FINITENESS QUESTIONS ABOUT HOCHSCHILD COHOMOLOGY OF FINITE-DIMENSIONAL ALGEBRAS

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Abstract. In this article, we give the dimensions of the Hochschild cohomology groups of certain finite-dimensional algebras $A = A_T(q_0, q_1, q_2, q_3)$ ($T \geq 0$; $q_0, q_1, q_2, q_3 \in K^\times$) and Λ_t ($t \geq 1$). Moreover, we study the Hochschild cohomology rings modulo nilpotence of A and Λ_t. We then see that A gives a negative answer to Happle's question whereas Λ_t is a counterexample to Snashall-Solberg's conjecture.

1. Introduction

Throughout this article, let K be an algebraically closed field. Let B be a finite-dimensional K-algebra. We denote by B^e the enveloping algebra $B^{\mathrm{op}} \otimes_K B$ of B. Here, note that there is a natural one-to-one correspondence between the family of right B^{e}-modules and that of B-B-bimodules. Recall that the ith *Hochschild cohomology group* $\mathrm{HH}^i(B)$ of B is defined to be the K-space $\mathrm{HH}^i(B) := \mathrm{Ext}^i_{B^{\mathrm{e}}}(B, B)$ for $i \geq 0$. Also, the *Hochschild cohomology ring* $\mathrm{HH}^*(B)$ of B is defined to be the graded ring $\mathrm{HH}^*(B) := \bigoplus_{i \geq 0} \mathrm{HH}^i(B) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{B^{\mathrm{e}}}(B, B)$, where the product is given by the Yoneda product. It is known that $\mathrm{HH}^*(B)$ is a graded commutative K-algebra. Let $\mathcal{N}_B$ be an ideal in $\mathrm{HH}^*(B)$ generated by all homogeneous nilpotent elements. Note that $\mathcal{N}_B$ is a homogeneous ideal. Then the graded K-algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$ is called the *Hochschild cohomology ring modulo nilpotence* of B. It is known that $\mathrm{HH}^*(B)/\mathcal{N}_B$ is a commutative K-algebra.

In this article, we consider the following question and conjecture:

(1) Happel's question ([9]). For a finite-dimensional algebra B, if $\mathrm{HH}^i(B) = 0$ for all $i \gg 0$, then is the global dimension of B finite?

(2) Snashall-Solberg's conjecture ([12]). For any finite-dimensional algebra B, the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}_B$ is finitely generated as an algebra.

In the papers [2, 3, 10], a negative answer to the question **(1)** was obtained, where the authors studied the Hochschild cohomology groups for several weakly symmetric algebras. On the other hand, a counterexample to the conjecture **(2)** was recently given by Xu ([14]) and Snashall ([11]).

In this article, we study the Hochschild cohomology groups and rings for two finite-dimensional algebras $A_T(q_0, q_1, q_2, q_3)$ ($T \geq 0$; $q_0, q_1, q_2, q_3 \in K^\times$) and Λ_t ($t \geq 1$) (see Section 2), and then see that the algebra $A_T(q_0, q_1, q_2, q_3)$ also gives a negative answer to **(1)** and that the algebra Λ_t is also a counterexample to **(2)**.

The detailed version of this paper will be submitted for publication elsewhere.

In Section 2, we define two finite-dimensional algebras $A_T(q_0, q_1, q_2, q_3)$ and Λ_t by using some finite quivers, where $T \geq 0$ and $t \geq 1$ are integers and q_i are elements in $K^\times$ for $i = 0, 1, 2, 3$. In Section 3, we give the dimensions of the Hochschild cohomology groups of $A_T(q_0, q_1, q_2, q_3)$, where the product $q_0q_1q_2q_3$ is not a roof of unity, and Λ_t for $t \geq 3$ (Theorems 1 and 3). In Section 4, we describe the structures of the Hochschild cohomology rings modulo nilpotence of $A_T(q_0, q_1, q_2, q_3)$ and Λ_t (Theorems 4 and 5).

2. Algebras $A_T(q_0, q_1, q_2, q_3)$ and Λ_t

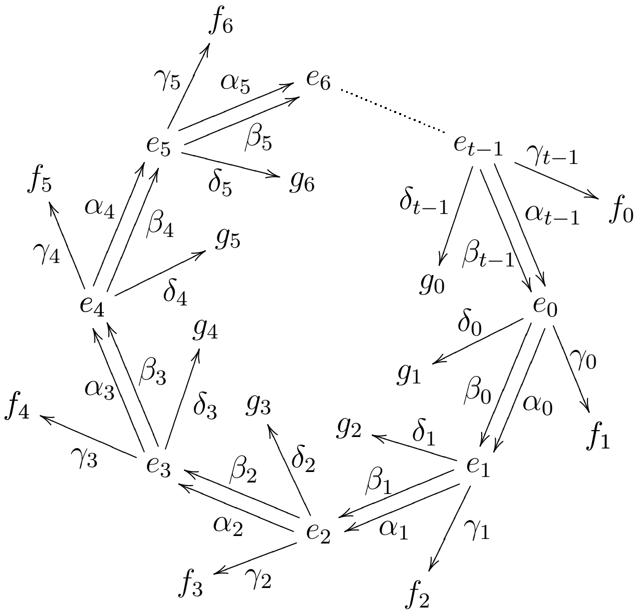
Let $T \geq 0$ and $t \geq 1$ be integers, and let q_0, q_1, q_2, q_3 be elements in $K^\times$. We definite the algebras $A = A_T(q_0, q_1, q_2, q_3)$ and Λ_t as follows:

(i) Let Γ be the following quiver with four vertices e_0, e_1, e_2, e_3 and eight arrows $a_{l,m}$ for $l = 0, 1$ and $m = 0, 1, 2, 3$:

$$\begin{tikzcd} e_0 \arrow[r, shift left, "a_{1,0}"] \arrow[r, shift right, "a_{0,0}"'] & e_1 \arrow[d, shift right, "a_{0,1}"'] \arrow[d, shift left, "a_{1,1}"] \\ e_3 \arrow[u, shift left, "a_{1,3}"] \arrow[u, shift right, "a_{0,3}"'] & e_2 \arrow[l, shift right, "a_{0,2}"'] \arrow[l, shift left, "a_{1,2}"] \end{tikzcd}$$

Let $K\Gamma$ be the path algebra, and set $x_l := \sum_{m=0}^{3} a_{l,m} \in K\Gamma$ for $l = 0, 1$. Let I denote the ideal in $K\Gamma$ generated by the uniform elements $e_ix_0x_1$, $e_ix_1x_0$, $e_j(q_jx_0^{4T+2} + x_1^{4T+2})$, and $e_k(q_kx_1^{4T+2} + x_0^{4T+2})$ for $0 \leq i \leq 3$, $j = 0, 2$ and $k = 1, 3$. We then define the algebra $A = A_T(q_0, q_1, q_2, q_3)$ by $A = A_T(q_0, q_1, q_2, q_3) := K\Gamma/I$.

(ii) Let Δ be the following quiver with $3t$ vertices e_i, f_i, g_i for $i = 0, \ldots, t-1$ and $4t$ arrows $\alpha_i, \beta_i, \gamma_i, \delta_i$ for $i = 0, \ldots, t-1$:



Let I be the ideal in the path algebra $K\Delta$ generated by the elements $\alpha_i\alpha_{i+1}$, $\beta_i\beta_{i+1}$, $\alpha_i\delta_{i+1}$, $\beta_i\gamma_{i+1}$, $\alpha_i\beta_{i+1}+\beta_i\alpha_{i+1}$ for $i=0,\dots,t-1$ (where $\alpha_t:=\alpha_0$, $\beta_t:=\beta_0$, $\gamma_t:=\gamma_0$ and $\delta_t:=\delta_0$). Define the algebra Λ_t by $\Lambda_t:=K\Delta/I$.

We see that $A=A_T(q_0,q_1,q_2,q_3)$ is a self-injective special biserial algebra for all $T\geq 0$ and $q_i \in K^\times$ $(i=0,1,2,3)$. In particular, if $T=0$, then $A_0(q_0,q_1,q_2,q_3)$ is a Koszul algebra for all $q_i\in K^\times$ $(i=0,1,2,3)$. Also, Λ_t is a Koszul algebra for $t\geq 1$.

3. The Hochschild cohomology groups for A and Λ_t

In this section, we give dimensions of the Hochschild cohomology groups $\mathrm{HH}^n(A)$ $(n\geq 0)$, where the product $q_0q_1q_2q_3\in K^\times$ is not a roof of unity, and $\mathrm{HH}^n(\Lambda_t)$ $(n\geq 0)$ for $t\geq 3$.

Theorem 1 ([5])**.** *Suppose that the product $q_0q_1q_2q_3\in K^\times$ is not a root of unity. Then*

(a) *For $m\geq 0$ and $0\leq r\leq 3$,*

$$\dim_K \mathrm{HH}^{4m+r}(A)=\begin{cases} 2T+1 & \text{if } m=r=0\\ 2T+3 & \text{if } m=0,\ r=1 \text{ and } \operatorname{char}K\mid 2T+1\\ 2T+2 & \text{if } m=0,\ r=1 \text{ and } \operatorname{char}K\nmid 2T+1\\ 2T+2 & \text{if } m=0,\ r=2 \text{ and } \operatorname{char}K\mid 2T+1\\ 2T+1 & \text{if } m=0,\ r=2 \text{ and } \operatorname{char}K\nmid 2T+1\\ 2T+2 & \text{if } m\geq 1,\ r=0 \text{ and } \operatorname{char}K\mid 2T+1,\\ & \text{or if } m\geq 1,\ r=1 \text{ and } \operatorname{char}K\mid 2T+1\\ 2T & \text{if } m\geq 1,\ r=0 \text{ and } \operatorname{char}K\nmid 2T+1,\\ & \text{or if } m\geq 1,\ r=1 \text{ and } \operatorname{char}K\nmid 2T+1\\ 2T & \text{if } m\geq 0 \text{ and } r=2,\\ & \text{or if } m\geq 0 \text{ and } r=3.\end{cases}$$

(b) $\mathrm{HH}^n(A)=0$ *for all $n\geq 3$ if and only if $T=0$.*

Remark 2. If $T=0$, then since the global dimension of $A_0(q_0,q_1,q_2,q_3)$ is infinite for all $q_i\in K^\times$ $(i=0,1,2,3)$, by Theorem 1 (b) we have got a negative answer to Happel's question **(1)**.

Theorem 3 ([6])**.** *Let $t\geq 3$. Then,*

(1) $\dim_K \mathrm{HH}^0(\Lambda_t)=1$ *and* $\dim_K\mathrm{HH}^1(\Lambda_t)=2$.
(2) *For an integer $n\geq 2$, write $n=mt+r$ for integers $m\geq 0$ and r with $0\leq r\leq t-1$.*
  (a) *If t is even, m is even, or $\operatorname{char}K=2$, then*

$$\dim_K \mathrm{HH}^n(\Lambda_t)=\begin{cases} mt-1 & \text{if } r=0\\ 2mt+2t & \text{if } r=1\\ 2mt+2t+1 & \text{if } r=2\\ 0 & \text{if } 3\leq r\leq t-1\end{cases}$$

(b) *If t is odd, m is odd and* $\operatorname{char} K \neq 2$*, then*

$$\dim_K \mathrm{HH}^n(\Lambda_t) = \begin{cases} 0 & \text{if } r = 0, \text{ or } 3 \leq r \leq t-1 \\ 2t & \text{if } r = 1, \text{ or } r = 2. \end{cases}$$

4. The Hochschild cohomology rings modulo nilpotence of A and Λ_t

In this section, we describe the structures of the Hochschild cohomology rings modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}_A$, where $T = 0$ and the product $q_0q_1q_2q_3 \in K^\times$ is not a roof of unity, and $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ for $t \geq 3$.

Theorem 4 ([5])**.** *Let $T = 0$ and $q_i \in K^\times$ for $0 \leq i \leq 3$. Suppose that the product $q_0q_1q_2q_3$ is not a root of unity. Then $\mathrm{HH}^*(A)$ is a 4-dimensional local algebra, and $\mathrm{HH}^*(A)/\mathcal{N}_A$ is isomorphic to K.*

Let m be an integer, and denote by C_m the quotient ring

$$K[z_0, \ldots, z_m]/\left\langle z_iz_j - z_kz_l \mid 0 \leq i,j,k,l \leq m;\ i+j = k+l \right\rangle$$

of the polynomial ring $K[z_0, \ldots, z_m]$ in $m+1$ variables. Then we have the following:

Theorem 5 ([6])**.** *Let $t \geq 3$.*

(a) *If t is even or* $\operatorname{char} K = 2$*, then $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ is isomorphic to the graded subalgebra of C_t with a K-basis*

$$\{1_K\} \cup \{z_i^j \mid j \geq 1;\ 1 \leq i \leq t-1\} \cup \{z_i^k z_{i+1}^{l-k} \mid l \geq 2;\ 1 \leq k \leq l-1;\ 0 \leq i \leq t-1\},$$

where $\deg z_i = t$ $(i = 0, \ldots, t)$.

(b) *If t is odd and* $\operatorname{char} K \neq 2$*, then $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ is isomorphic to the graded subalgebra of C_{2t} with a K-basis*

$$\{1_K\} \cup \{z_i^j \mid j \geq 1;\ 1 \leq i \leq 2t-1\} \cup \{z_i^k z_{i+1}^{l-k} \mid l \geq 2;\ 1 \leq k \leq l-1;\ 0 \leq i \leq 2t-1\},$$

where $\deg z_i = 2t$ $(i = 0, \ldots, 2t)$.

By Theorem 5, we easily have the following corollary, which tells us that Λ_t $(t \geq 3)$ is a counterexample to the conjecture **(2)**:

Corollary 6 ([6])**.** *For $t \geq 3$, $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ is not finitely generated as an algebra.*

Remark 7. Since Λ_t is a Koszul algebra for $t \geq 1$, the graded centre $Z_{\mathrm{gr}}(E(\Lambda_t))$ of the Ext algebra $E(\Lambda_t) := \bigoplus_{i\geq 0} \mathrm{Ext}^i_{\Lambda_t}(\Lambda_t/\operatorname{rad}\Lambda_t, \Lambda_t/\operatorname{rad}\Lambda_t)$ of Λ_t is isomorphic to $\mathrm{HH}^*(\Lambda_t)/\mathcal{N}_{\Lambda_t}$ as graded algebras (see [1]). Hence, for $t \geq 3$, $Z_{\mathrm{gr}}(E(\Lambda_t))$ is also isomorphic to the algebra described in Theorem 5 and is not finitely generated as an algebra.

In [11], Snashall gave the following new question:

Snashall's question ([11]). Can we give necessary and sufficient conditions on a finite-dimensional algebra for its Hochschild cohomology ring modulo nilpotence to be finitely generated as an algebra?

It is known that the Hochschild cohomology rings modulo nilpotence for following finite-dimensional algebras are finitely generated:

- group algebras ([4, 13]) • self-injective algebras of finite representation type ([7])
- monomial algebras ([8]) • algebras with finite global dimension ([9])

However, a definitive answer to the question above has not been obtained yet.

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SERRE SUBCATEGORIES OF ARTINIAN MODULES

NAOYA HIRAMATSU

Abstract. We study subcategories of the category of artinian modules. We prove that all wide subcategories of artinian modules are Serre subcategories. We also provide the bijection between the set of Serre subcategories of artinian modules and the set of specialization closed subsets of the set of closed prime ideals of some completed ring.

1. Introduction

Let R be a commutative noetherian ring and M be an R-module. We denote by $\mathrm{Mod}(R)$ the category of R-modules and R-homomorphisms and by $\mathrm{mod}(R)$ the full subcategory consisting of finitely generated R-modules. We also denote by $\mathrm{Spec}\ R$ the set of prime ideals of R and by $\mathrm{Ass}_R M$ the set of associated prime ideals of M. A subcategory of an abelian category is said to be a wide subcategory if it is closed under kernels, cokernels and extensions. We also say that a subcategory is a Serre subcategory if it is a wide subcategory which is closed under subobjects.

Classifying subcategories of a module category also has been studied by many authors. Classically, Gabriel [2] gives a bijection between the set of Serre subcategories of $\mathrm{mod}(R)$ and the set of specialization closed subsets of $\mathrm{Spec}\ R$. Recently, Takahashi [10] and Krause [4] proved the following.

Theorem 1. [5, Theorem 4.1][3, Corollary 2.6] *Let R be a noetherian ring. Then we have the following 1-1 correspondences;*

$$\begin{aligned}&\{\ \textit{subcategories of } \mathrm{mod}(R) \textit{ closed under submodules and extensions}\ \}\\ &\cong \{\ \textit{subsets of }\ \mathrm{Spec}\ R\ \}.\end{aligned}$$

Moreover this induces the bijection

$$\{\ \textit{Serre subcategories of } \mathrm{mod}(R)\ \} \cong \left\{\ \textit{specialization closed subsets of } \mathrm{Spec}\ R\ \right\}.$$

In addition, Takahashi [10] pointed out a property concerning wide subcategories of $\mathrm{mod}(R)$.

Theorem 2. [10, Theorem 3.1, Corollary 3.2] *Let R be a noetherian ring. Then every wide subcategory of $\mathrm{mod}(R)$ is a Serre subcategory of $\mathrm{mod}(R)$.*

In this note, we want to consider the artinian analogue of these results. We prove that all wide subcategories of artinian modules are Serre subcategories (Theorem 9). We also provide the bijection between the set of Serre subcategories and the set of specialization closed subsets of the set of closed prime ideals of some completed ring (Theorem 26). We

We refer to [3] for more details on the present article.

consider some completion of a ring (see Proposition 16), so that all of artinian modules can be regarded as modules over it.

Throughout the note, we always assume that R is a commutative ring with identity, and by a subcategory we mean a nonempty full subcategory which is closed under isomorphism.

2. WIDE SUBCATEGORIES OF ARTINIAN MODULES

In this section, we investigate wide subcategories of artinian modules.

Let M be an artinian R-module. We denote by $\mathrm{Soc}(M)$ the sum of simple submodules of M. Since $\mathrm{Soc}(M)$ is also artinian, there exist only finitely many maximal ideals $\mathfrak{m}$ of R for which $\mathrm{Soc}(M)$ has a submodule isomorphic to $R/\mathfrak{m}$. Let the distinct such maximal ideals be $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$. Set $J_M = \bigcap_{i=1}^{s} \mathfrak{m}$ and $\hat{R}^{(M)} = \varprojlim R/J_M^n$.

Lemma 3. [9, Lemma 2.2] *Each non-zero element $m \in M$ is annihilated by some power of J_M. Hence M has the natural structure of a module over $\hat{R}^{(M)}$ in such a way that a subset of M is an R-submodule if and only if it is an $\hat{R}^{(M)}$-submodule.*

Proof. We need in the present note how the $\hat{R}^{(M)}$-module structure is defined for an artinian module M. For this reason we briefly recall the proof of the lemma.

Since $\mathrm{Soc}(M) = \oplus_{i=1}^{s}(R/\mathfrak{m}_i)^{n_i}$, M can be embedded in $\oplus_{i=1}^{s}(E_R(R/\mathfrak{m}_i))^{n_i}$ where $E_R(R/\mathfrak{m})$ is an injective hull of $R/\mathfrak{m}$. Note that an element of $E_R(R/\mathfrak{m})$ is annihilated by some power of $\mathfrak{m}$. Hence one can show that each element of M is annihilated by some power of $\mathfrak{m}_1 \cdots \mathfrak{m}_s = J_M$.

Let $x \in M$ and $\hat{r} = (r_n + J_M^n)_{n \in \mathbb{N}} \in \hat{R}^{(M)}$. Suppose that $J_M^k x = 0$. It is straightforward to check that M has the structure of an $\hat{R}^{(M)}$-module such that $\hat{r}x = r_k x$.

□

By virtue of Lemma 3, each artinian R-module can be regarded as a module over some complete semi-local ring. We note that the Matlis duality theorem holds over a noetherian complete semi-local ring (cf. [6, Theorem 1.6]). It is the strategy of the note that we replace the categorical property on a subcategory of finitely generated (namely, noetherian) modules with that of artinian modules by using Matlis duality. We denote by $\mathrm{Art}(R)$ the subcategory consisting of artinian R-modules. The following lemma holds from the Matlis duality theorem.

Lemma 4. *Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_s)$ be a noetherian complete semi-local ring and set $E = \oplus_{i=1}^{s} E_R(R/\mathfrak{m}_i)$. For each subcategory $\mathcal{X}$ of $\mathrm{Mod}(R)$, we denote by $\mathcal{X}^\vee = \{M^\vee \mid M \in \mathcal{X}\}$ where $(-)^\vee = \mathrm{Hom}_R(-, E)$. Then the following assertions hold.*

1. *If $\mathcal{X}$ is a subcategory of $\mathrm{Art}(R)$ (resp. $\mathrm{mod}(R)$) which is closed under quotient modules (resp. submodules) and extensions, then $\mathcal{X}^\vee$ is a subcategory of $\mathrm{mod}(R)$ (resp. $\mathrm{Art}(R)$) which is closed under submodules (resp. quotient modules) and extensions.*
2. *If $\mathcal{X}$ is a wide subcategory of $\mathrm{Art}(R)$ (resp. $\mathrm{mod}(R)$), then $\mathcal{X}^\vee$ is also a wide subcategory of $\mathrm{mod}(R)$ (resp. $\mathrm{Art}(R)$).*
3. *If $\mathcal{X}$ is a Serre subcategory of $\mathrm{Art}(R)$ (resp. $\mathrm{mod}(R)$), then $\mathcal{X}^\vee$ is also a Serre subcategory of $\mathrm{mod}(R)$ (resp. $\mathrm{Art}(R)$).*

Definition 5. Let M be an R-module. For a nonnegative integer n, we inductively define a subcategory $\mathrm{Wid}_R^n(M)$ of $\mathrm{Mod}(R)$ as follows:

(1) Set $\mathrm{Wid}_R^0(M) = \{M\}$.
(2) For $n \geq 1$, let $\mathrm{Wid}_R^n(M)$ be a subcategory of $\mathrm{Mod}(R)$ consisting of all R-modules X having an exact sequence of either of the following three forms:

$$\begin{array}{l} A \to B \to X \to 0, \\ 0 \to X \to A \to B, \\ 0 \to A \to X \to B \to 0 \end{array}$$

where $A, B \in \mathrm{Wid}_R^{n-1}(M)$.

Remark 6. Let M be an R-module. Then the following hold.

(1) There is an ascending chain $\{M\} = \mathrm{Wid}_R^0(M) \subseteq \mathrm{Wid}_R^1(M) \subseteq \cdots \subseteq \mathrm{Wid}_R^n(M) \subseteq \cdots \subseteq \mathrm{Wid}_R(M)$ of subcategories $\mathrm{Mod}(R)$. Here we denote by $\mathrm{Wid}_R(M)$ the smallest wide subcategory of $\mathrm{Mod}(R)$ which contains M.
(2) $\bigcup_{n\geq 0} \mathrm{Wid}_R^n(M)$ is wide and the equality $\mathrm{Wid}_R(M) = \bigcup_{n\geq 0} \mathrm{Wid}_R^n(M)$ holds.
(3) If M is artinian, then $\bigcup_{n\geq 0} \mathrm{Wid}_R^n(M)$, hence $\mathrm{Wid}_R(M)$, is a subcategory of $\mathrm{Art}(R)$.

Definition 7. Let J be an ideal of R. For each R-module M, we denote by $\Gamma_J(M)$ the set of elements of M which are annihilated by some power of J, namely $\Gamma_J(M) = \bigcup_{n\in\mathbb{N}}(0 :_M J^n)$. An R-module M is said to be J-torsion if $M = \Gamma_J(M)$. We denote by $\mathrm{Art}_J(R)$ the subcategory consisting of artinian J-torsion R-modules.

Proposition 8. *Let M be an artinian R-module. Then $\mathrm{Wid}_R(M)$ and $\mathrm{Wid}_{\hat{R}^{(M)}}(M)$ are equivalent as subcategories of $\mathrm{Art}(\hat{R}^{(M)})$.*

Proof. Since a J-torsioness is closed under taking kernels, cockerels and extension, we can naturally identify $\mathrm{Wid}_R(M)$ with a subcategory of $\mathrm{Art}_{J_M}(R)$. $\square$

Theorem 9. *Let R be a noetherian ring. Then every wide subcategory of $\mathrm{Art}(R)$ is a Serre subcategory of $\mathrm{Art}(R)$.*

Proof. Let $\mathcal{X}$ be a wide subcategory of $\mathrm{Art}(R)$. It is sufficiently to show that $\mathcal{X}$ is closed under submodules. Assume that $\mathcal{X}$ is not closed under submodules. Then there exists an R-module X in $\mathcal{X}$ and R-submodule M of X such that M does not belong to $\mathcal{X}$. Applying Lemma 3 to X, X is a module over the complete semi-local ring $\hat{R} := \hat{R}^{(J_X)}$ and M is an $\hat{R}$-submodule of X. Now we consider the wide subcategory $\mathrm{Wid}_R(X)$. By virtue of Proposition 8, $\mathrm{Wid}_R(X) = \mathrm{Wid}_{\hat{R}}(X)$ as a subcategory of $\mathrm{Art}(\hat{R})$. Since $\hat{R}$ is a complete semi-local ring, by Matlis duality, we have the equivalence of the categories $\mathrm{Wid}_{\hat{R}}(X) \cong \{\mathrm{Wid}_{\hat{R}}(X)^{\vee}\}^{op} \cong \mathrm{Wid}_{\hat{R}}(X^{\vee})^{op}$ where $(-)^{\vee} = \mathrm{Hom}_{\hat{R}}(-, E_{\hat{R}}(\hat{R}/J_X\hat{R}))$. Since $\mathrm{Wid}_{\hat{R}}(X^{\vee})$ is a wide subcategory of finitely generated $\hat{R}$-modules, it follows from Theorem 2 that $\mathrm{Wid}_{\hat{R}}(X^{\vee})$ is a Serre subcategory. Thus $M^{\vee}$ is contained in $\mathrm{Wid}_{\hat{R}}(X^{\vee})$. Using Matlis duality again, we conclude that M must be contained in $\mathrm{Wid}_{\hat{R}}(X) = \mathrm{Wid}_R(X)$, hence also in $\mathcal{X}$. This is a contradiction, so that $\mathcal{X}$ is closed under submodules.

$\square$

3. Classifying subcategories of artinian modules

In this section, we shall give the artinian analogue of the classification theorem of subcategories of finitely generated modules (Theorem 26). First, we state the notion and the basic properties of attached prime ideals which play a key role of our theorem. For the detail, we recommend the reader to look at [8, 9] and [5, §6 Appendix.] .

Definition 10. Let M be an R-module. We say that M is secondary if for each $a \in R$ the endomorphism of M defined by the multiplication map by a is either surjective or nilpotent.

Remark 11. If M is secondary then $\mathfrak{p} = \sqrt{\mathrm{ann}_R(M)}$ is a prime ideal and M is said to be $\mathfrak{p}$-secondary.

Definition 12. Let M be an R-module.

(1) $M = S_1 + \cdots + S_r$ is said to be a secondary representation if S_i is a secondary submodule of M for all i. And we also say that the representation is minimal if the prime ideals $\mathfrak{p}_i = \sqrt{\mathrm{ann}_R(S_i)}$ are all distinct, and none of the S_i is redundant

(2) A prime ideal $\mathfrak{p}$ is said to be an attached prime ideal of M if M has a $\mathfrak{p}$-secondary quotient. We denote by $\mathrm{Att}_R M$ the set of the attached prime ideals of M.

Remark 13. Let M be an R-module.

(1) If $M = S_1 + \cdots + S_r$ is a minimal representation and $p_i = \sqrt{\mathrm{ann}_R(S_i)}$. then $\mathrm{Att}_R M = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}$. See [5, Theorem 6.9] .

(2) Given a submodule $N \subseteq M$, we have

$$\mathrm{Att}_R M/N \subseteq \mathrm{Att}_R M \subseteq \mathrm{Att}_R(N) \cup \mathrm{Att}_R M/N.$$

See [5, Theorem 6.10] .

(3) It is known that if M is artinian then M has a secondary representation. Thus it has a minimal one. See [5, Theorem 6.11] .

In the rest of this section, we always assume that R is a noetherian ring. The following observation tells us that we should consider a larger set than Spec R to classify subcategories of artinian modules.

Example 14. Let $(R, \mathfrak{m})$ be a noetherian local ring and $\mathcal{X}$ a Serre subcategory of $\mathrm{Art}(R)$. By virtue of Lemma 3, $\mathrm{Art}(R)$ is equivalent to $\mathrm{Art}(\hat{R})$ where $\hat{R}$ is an $\mathfrak{m}$-adic completion of R. Now we consider $\mathcal{X}$ as a subcategory of $\mathrm{Art}(\hat{R})$. Since $\mathcal{X}^\vee$ is a Serre subcategory of $\mathrm{mod}(\hat{R})$ (Lemma 4), $\mathcal{X}^\vee$, hence $\mathcal{X}$, corresponds to the specialization closed subset of Spec $\hat{R}$ by Theorem 1. That is, there is the bijection between the set of Serre subcategories of $\mathrm{Art}(R)$ and the set of specialization closed subsets of Spec $\hat{R}$.

As mentioned in Lemma 3, we can determine some complete semi-local rings for each artinian module respectively, so that the artinian module has the module structure over such a completed ring. Now we attempt to treat all the artinian R-modules as modules over the same completed ring. For this, we consider the following set of ideals of R:

$$\mathcal{T} = \{\ I \mid \text{the length of } R/I \text{ is finite }\}.$$

The set $\mathcal{T}$ forms a directed set ordered by inclusion. Then we can consider the inverse system $\{R/I, f_{I,I'}\}$ where $f_{I,I'}$ are natural surjections. We denote $\varprojlim_{I\in\mathcal{T}} R/I$ by $\hat{R}_{\mathcal{T}}$.

The proof of the following lemma will go through similarly to the proof of Lemma 3.

Lemma 15. *Every artinian R-module has the structure of an $\hat{R}_{\mathcal{T}}$-module in such a way that a subset of an artinian R-module M is an R-submodule if and only if it is an $\hat{R}_{\mathcal{T}}$-submodule. Consequently, we have an equivalence of categories* $\mathrm{Art}(R) \cong \mathrm{Art}(\hat{R}_{\mathcal{T}})$.

We set another family of ideals of R as

$$\mathcal{J} = \{\ \mathfrak{m}_1^{k_1}\cdots\mathfrak{m}_s^{k_s} \mid \mathfrak{m}_i \text{ is a maximal ideal of } R,\ k_i \in \mathbb{N}\}.$$

It is also a directed set ordered by inclusion and we denote by $\hat{R}_{\mathcal{J}}$ its inverse limit on the system via natural surjections.

Next we consider a direct product of rings

$$\prod_{\mathfrak{n}\in\max(R)} \hat{R}_{\mathfrak{n}}$$

where $\max(R)$ is the set of maximal ideals of R and $\hat{R}_{\mathfrak{m}}$ is an $\mathfrak{m}$-adic completion of R. We regard the ring as a topological ring by a product topology, namely the linear topology defined by ideals which are of the form $\mathfrak{m}_1^{k_1}\hat{R}_{\mathfrak{m}_1}\times\cdots\times\mathfrak{m}_s^{k_s}\hat{R}_{\mathfrak{m}_s}\times\prod_{\mathfrak{n}\neq\mathfrak{m}_1,\cdots,\mathfrak{m}_s}\hat{R}_{\mathfrak{n}}$ for some $\mathfrak{m}_i\in\max(R)$ and $k_i\in\mathbb{N}$. For the rings $\hat{R}_{\mathcal{T}}$, $\hat{R}_{\mathcal{J}}$ and $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$, we have the following.

Proposition 16. [1, §2.13. Proposition 17] *There is an isomorphism of topological rings*

$$\hat{R}_{\mathcal{T}} \cong \hat{R}_{\mathcal{J}} \cong \prod_{\mathfrak{n}\in\max(R)} \hat{R}_{\mathfrak{n}}.$$

Remark 17. Let M be an artinian R-module. It follows from Lemma 15 that M is also an artinian $\hat{R}$-module. Then the radical of $\mathrm{ann}_{\hat{R}}(M)$ is a closed ideal. To show this, it suffices to prove that the inclusion $\sqrt{\mathrm{ann}_{\hat{R}}(M)} \supseteq \cap_{I\in\mathcal{T}}(\sqrt{\mathrm{ann}_{\hat{R}}(M)}+I)$ holds. Take an arbitrary element $\hat{a}\in\cap_{I\in\mathcal{T}}(\sqrt{\mathrm{ann}_{\hat{R}}(M)}+I)$. Then there exist some elements $\hat{b}_I\in\mathrm{ann}_{\hat{R}}(M)$ and $\hat{c}_I\in I$ such that $\hat{a}=\hat{b}_I+\hat{c}_I$ for all I. Let $x\in M$ and suppose that $Ix=0$ for some $I\in\mathcal{T}$. Since $\hat{b}_I\in\sqrt{\mathrm{ann}_{\hat{R}}(M)}$, $\hat{b}_I^k\in\mathrm{ann}_{\hat{R}}(M)$. Hence we see that $\hat{a}^k x=(\hat{b}_I+\hat{c}_I)^k x=0$ holds, so that $\hat{a}\in\sqrt{\mathrm{ann}_{\hat{R}}(M)}$. Consequently, $\mathrm{Att}_{\hat{R}}M$ is a subset of the set of closed prime ideals of $\hat{R}$.

For closed prime ideals of $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$, we have the following result.

Proposition 18. *Every proper closed prime ideal of $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$ is of the form $\mathfrak{p}\times\prod_{\mathfrak{n}\in\max(R),\mathfrak{m}\neq\mathfrak{n}}\hat{R}_{\mathfrak{n}}$ for some prime ideal $\mathfrak{p}\in\operatorname{Spec}\hat{R}_{\mathfrak{m}}$. Hence we can identify the set of closed prime ideals of $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$ with the disjoint union of* $\operatorname{Spec}\hat{R}_{\mathfrak{m}}$, *i.e.* $\coprod_{\mathfrak{n}\in\max(R)}\operatorname{Spec}\hat{R}_{\mathfrak{n}}$.

We can equate the rings $\hat{R}_{\mathcal{T}}$, $\hat{R}_{\mathcal{J}}$ and $\prod_{\mathfrak{n}\in\max(R)}\hat{R}_{\mathfrak{n}}$ by virtue of Proposition 16. In the rest of this note we always denote them by $\hat{R}$ and identify the set of closed prime ideals of $\hat{R}$ with $\coprod_{\mathfrak{n}\in\max(R)}\operatorname{Spec}\hat{R}_{\mathfrak{n}}$.

Lemma 19. [7, Exercise 8.49] *Let M be an artinian R-module. Assume* $\mathrm{Ass}_R M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}$. *Then M is the direct sums of the submodules $\Gamma_{\mathfrak{m}_i}(M)$, that is* $M = \oplus_{i=1}^{s} \Gamma_{\mathfrak{m}_i}(M)$. *Here we denote by $\Gamma_{\mathfrak{m}_i}(M)$ the $\mathfrak{m}_i$-torsion submodule of M.*

Remark 20. Let M be an $\mathfrak{m}$-torsion R-module. Then M has the structure of an $\hat{R}$-module and an $\hat{R_{\mathfrak{m}}}$-module. Note that the $\hat{R_{\mathfrak{m}}}$-module action on M is identical with the action by means of the natural inclusion $\hat{R_{\mathfrak{m}}} \to \prod_{\mathfrak{n} \in \max(R)} \hat{R_{\mathfrak{n}}} \cong \hat{R}$. We also note from Lemma 15 or Lemma 3 that N is an $\hat{R}$-submodule (resp. a quotient $\hat{R}$-module) of M if and only if it is an $\hat{R_{\mathfrak{m}}}$-submodule (resp. a quotient $\hat{R_{\mathfrak{m}}}$-module) of M.

Proposition 21. *Let M be an $\mathfrak{m}$-torsion R-module. Then*

$$\mathrm{Att}_{\hat{R}} M = \mathrm{Att}_{\hat{R_{\mathfrak{m}}}} M$$

as a subset of $\coprod_{\mathfrak{n} \in \max(R)} \mathrm{Spec}\, \hat{R_{\mathfrak{n}}}$.

Combing Proposition 21 with Lemma 19, we have the following corollary.

Corollary 22. *Let M be an artinian R-module. Then*

$$\mathrm{Att}_{\hat{R}} M = \coprod_{\mathfrak{m} \in \mathrm{Ass}_R M} \mathrm{Att}_{\hat{R_{\mathfrak{m}}}} \Gamma_{\mathfrak{m}}(M)$$

as a subset of $\coprod_{\mathfrak{n} \in \max(R)} \mathrm{Spec}\, \hat{R_{\mathfrak{n}}}$.

Let us state the result which is a key to classify the subcategory of the category of noetherian modules.

Theorem 23. [10, Corollary 4.4][4, Corollary 2.6] *Let M and N be finitely generated R-modules. Then M can be generated from N via taking submodules and extension if and only if* $\mathrm{Ass}_R M \subseteq \mathrm{Ass}_R N$.

The following lemma is due to Sharp [8].

Lemma 24. [8, 3.5.] *Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_s)$ be a commutative noetherian complete semi-local ring and set $E = \oplus_{i=1}^{s} E_R(R/\mathfrak{m}_i)$. For an artinian R-module M, we have*

$$\mathrm{Att}_R M = \mathrm{Ass}_R \mathrm{Hom}_R(M, E).$$

The next claim is reasonable as the artinian analogue of Theorem 23.

Theorem 25. *Let M and N be artinian R-modules. Then M can be generated from N via taking quotient modules and extensions as R-modules if and only if* $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N$.

Proof. Suppose that M is contained in quot-ext$_R(N)$. It is clear from the property of attached prime ideals (Remark 13) that $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N$ holds.

Conversely, suppose that $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} N$. First, we shall show that we may assume that M and N are $\mathfrak{m}$-torsion R-modules for some maximal ideal $\mathfrak{m}$. In fact, M (resp. N) can be decomposed as $M = \oplus_{\mathfrak{m} \in \mathrm{Ass}_R M} \Gamma_{\mathfrak{m}}(M)$ (resp. $N = \oplus_{\mathfrak{n} \in \mathrm{Ass}_R N} \Gamma_{\mathfrak{n}}(N)$) and the assumption implies that $\mathrm{Att}_{\hat{R_{\mathfrak{m}}}} \Gamma_{\mathfrak{m}}(M) \subseteq \mathrm{Att}_{\hat{R_{\mathfrak{m}}}} \Gamma_{\mathfrak{m}}(N)$ for all $\mathfrak{m} \in \mathrm{Ass}_R M$ by Corollary 22. If we show that $\Gamma_{\mathfrak{m}}(M)$ is contained in quot-ext$_R(\Gamma_{\mathfrak{m}}(N))$, we can get the assertion since quot-ext$_R(N)$ is closed under direct sums and direct summands.

Let M and N be $\mathfrak{m}$-torsion R-modules and E be an injective hull of $\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}}$ as an $\hat{R_\mathfrak{m}}$-module. Since M and N are also artinian $\hat{R_\mathfrak{m}}$-modules, $M^\vee$ and $N^\vee$ are finitely generated $\hat{R_\mathfrak{m}}$-modules by Matlis duality, where $(-)^\vee = \mathrm{Hom}_{\hat{R_\mathfrak{m}}}(-, E)$. Since $\mathrm{Att}_{\hat{R_\mathfrak{m}}} M$ (resp. $\mathrm{Att}_{\hat{R_\mathfrak{m}}} N$) is equal to $\mathrm{Ass}_{\hat{R_\mathfrak{m}}} M^\vee$ (resp. $\mathrm{Ass}_{\hat{R_\mathfrak{m}}} N^\vee$) (Lemma 24), the inclusion

$$\mathrm{Ass}_{\hat{R_\mathfrak{m}}} M^\vee \subseteq \mathrm{Ass}_{\hat{R_\mathfrak{m}}} N^\vee$$

holds. By virtue of Theorem 23, we conclude that $M^\vee$ can be generated from $N^\vee$ via taking submodules and extensions, i.e. $M^\vee \in \text{sub-ext}_{\hat{R_\mathfrak{m}}}(N^\vee)$. Hence it follows from Matlis duality and Lemma 4 that

$$M^{\vee\vee} \cong M \in \text{sub-ext}_{\hat{R_\mathfrak{m}}}(N^\vee)^\vee = \text{quot-ext}_{\hat{R_\mathfrak{m}}}(N).$$

Since artinian $\hat{R_\mathfrak{m}}$-modules are also artinian R-modules (cf. Lemma 3), we conclude that $M \in \text{quot-ext}_R(N)$.

□

We define by Ψ the map sending a subcategory $\mathcal{X}$ of $\mathrm{Art}(R)$ to

$$\mathrm{Att}\mathcal{X} = \cup_{M\in\mathcal{X}} \mathrm{Att}_{\hat{R}} M$$

and by Φ the map sending a subset S of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$ to

$$\{M \in \mathrm{Art}(R)\ |\mathrm{Att}_{\hat{R}} M \subseteq S\}.$$

Note from Corollary 22 that $\Psi(\mathcal{X})$ is a subset of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$. On the other hand, it follows from Remark 13 (2) that $\Phi(S)$ is closed under quotient modules and extensions.

Now we state the main theorem of this note.

Theorem 26. *Let R be a commutative noetherian ring. Then Ψ and Φ induce an inclusion preserving bijection between the set of subcategories of* $\mathrm{Art}(R)$ *which are closed under quotient modules and extensions and the set of subsets of* $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$.

Moreover, they also induce an inclusion preserving bijection between the set of Serre subcategories of $\mathrm{Art}(R)$ *and the set of specialization closed subsets of* $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$.

Proof. We show the first assertion of the theorem.

Let $\mathcal{X}$ be a subcategory of $\mathrm{Art}(R)$ which is closed under quotient modules and extensions. The subcategory $\Phi\Psi(\mathcal{X})$ consists of all artinian R-modules M with $\mathrm{Att}_{\hat{R}} M \subseteq \cup_{X\in\mathcal{X}} \mathrm{Att}_{\hat{R}} X$. It is clear that $\mathcal{X}$ is a subcategory of $\Phi\Psi(\mathcal{X})$. Let M be an artinian R-module with $\mathrm{Att}_{\hat{R}} M \subseteq \cup_{X\in\mathcal{X}} \mathrm{Att}_{\hat{R}} X$. For each ideal $\mathfrak{P} \in \mathrm{Att}_{\hat{R}} M$, there exists $X^{(\mathfrak{P})} \in \mathcal{X}$ such that $\mathfrak{P} \in \mathrm{Att}_{\hat{R}} X^{(\mathfrak{P})}$. Take the direct sums of such objects, that is $X = \oplus_{\mathfrak{P}\in\mathrm{Att}_{\hat{R}} M} X^{(\mathfrak{P})}$. X is also an object of $\mathcal{X}$, since $\mathrm{Att}_{\hat{R}} M$ is a finite set and $\mathcal{X}$ is closed under finite direct sums. It follows from the definition of X that $\mathrm{Att}_{\hat{R}} M \subseteq \mathrm{Att}_{\hat{R}} X$. By virtue of Theorem 25, M is contained in $\text{quot-ext}_R(X)$, so that M in $\mathcal{X}$. Hence we have the equality $\mathcal{X} = \Phi\Psi(\mathcal{X})$.

Let S be a subset of $\coprod_{\mathfrak{n}\in\max(R)} \mathrm{Spec}\ \hat{R_\mathfrak{n}}$. It is trivial that the set $\Psi\Phi(S)$ is contained in S. Let $\mathfrak{p}$ be a prime ideal in S. Take a maximal ideal $\mathfrak{m}$ so that $\mathfrak{p}$ is a prime ideal of $\hat{R_\mathfrak{m}}$. We consider an $\hat{R_\mathfrak{m}}$-module $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$. Then we have the equality:

$$\mathrm{Att}_{\hat{R_\mathfrak{m}}} E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}}) = \mathrm{Ass}_{\hat{R_\mathfrak{m}}} \hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}} = \{\mathfrak{p}\}.$$

Note that $E_{\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ is artinian as an R-module. Indeed, we have the equality $E_{\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}}) = E_R(R/\mathfrak{m}R)$ as R-modules ([5, Theorem 18.6 (iii)]), so that it is an artinian R-module since $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ is an $\hat{R_\mathfrak{m}}$-submodule (thus an R-submodule) of $E_{\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$. Hence $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ is an artinian R-module which is a $\mathfrak{p}$-secondary $\hat{R_\mathfrak{m}}$-module. Consequently, $E_{\hat{R_\mathfrak{m}}/\mathfrak{p}\hat{R_\mathfrak{m}}}(\hat{R_\mathfrak{m}}/\mathfrak{m}\hat{R_\mathfrak{m}})$ belongs to $\Phi(S)$, so that $\mathfrak{p} \in \Psi\Phi(S)$. $\square$

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ON THE RING OF COMPLEX-VALUED FUNCTIONS ON A FINITE RING

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Abstract. Greferath, Fadden and Zumbrägel [3] consider rings of complex-valued functions on some finite rings R. In this paper we consider the structure of such rings. First we consider the case when R is a finite local ring. Next we consider the case when R is a finite semisimple ring. Finally we consider rings of complex-valued invariant functions on some finite rings.

1. Rings of functions from a finite ring to the field of complex numbers

Greferath, Fadden and Zumbrägel [3] consider rings of complex-valued functions on some finite rings. First we give some definitions.

Definition 1. Let R be a ring and let $\mathbf{C}$ denote the field of complex numbers. A function $f : R \to \mathbf{C}$ is said to be *finite* if $\{r \in R \mid f(r) \neq 0\}$ is a finite set.

Consider the set $\mathbf{C}^R = \{f \mid f : R \to \mathbf{C} \text{ is finite}\}$ of all finite functions from R to $\mathbf{C}$.

Definition 2. For $f,\ g \in \mathbf{C}^R$ and for $\lambda \in \mathbf{C}$ we define addition and scalar multiplication by

$$(f + g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x).$$

Then $\mathbf{C}^R$ is a $\mathbf{C}$-vector space. We define multiplication by

$$(f * g)(x) = \sum_{\substack{a, b \in R \\ ab = x}} f(a)g(b).$$

Then $\mathbf{C}^R$ is a $\mathbf{C}$-algebra.

For any element $r \in R$, we define the function δ_r by

$$\delta_r(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{otherwise.} \end{cases}$$

Then δ_1 is is the identity of the $\mathbf{C}$-algebra $\mathbf{C}^R$. We can easily see that $\delta_r * \delta_s = \delta_{rs}$ for each $r,\ s \in R$. Also we can see that the set $\{\delta_r \mid r \in R\}$ forms a $\mathbf{C}$-basis of vector space.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 3. Let R be a ring with addition $+$ and multiplication $\cdot$. We denote the semigroup algebra of the semigroup $(R,\ \cdot)$ over $\mathbf{C}$ by $\mathbf{C}[R]$. Any element of $\mathbf{C}[R]$ can be written as a finite sum of the form $\sum_{r\in R} a_r\hat{r}$. For the definition of semigroup algebras, see [4, Page 33].

Other definitions and some basic results in this paper can be found in [2]. We start with the following fundamental proposition.

Proposition 4. *Let R be a finite ring. Then the $\mathbf{C}$-algebra $\mathbf{C}^R$ is isomorphic to the semigroup algebra $\mathbf{C}[R]$ of multiplicative semigroup $(R,\ \cdot\ )$ of the ring R.*

Proof. Recall that the semigroup algebra $\mathbf{C}[R]$ is a $\mathbf{C}$-vector space with the $\mathbf{C}$-basis $\{\hat{r} \mid r \in R\}$ and multiplication defined by

$$\hat{r}\hat{s} = \widehat{rs}$$

for each $r,\ s \in R$. We define a mapping $\phi : \mathbf{C}^R \to \mathbf{C}[R]$ by $\phi(\delta_r) = \hat{r}$ for each $r \in R$. Then ϕ is an isomorphism of $\mathbf{C}$-algebras. $\square$

2. Semigroup rings

First we give some basic results on semigroup rings of the semigroups of some rings.

Proposition 5. *Let R be a ring, let I be an ideal of R and let $\mathbf{Z}$ denote the ring of rational integers.*

(1) *Let A denote the ideal of $\mathbf{Z}[R]$ generated by $\{\widehat{r+s} - \hat{r} - \hat{s} \mid r, s \in R\}$. Then $\mathbf{Z}[R]/A \cong R$.*

(2) *Let B denote the ideal of $\mathbf{C}[R]$ generated by $\{\hat{r} - \hat{s} \mid r - s \in I\}$. Then $\mathbf{C}[R]/B \cong \mathbf{C}[R/I]$.*

Proposition 6. *Let R be a finite ring, let R^* denote the group of units in R and let C denote the ideal of $\mathbf{C}[R]$ generated by $\{\hat{r} \mid r \in R - R^*\}$. Then $\mathbf{C}[R]/C \cong \mathbf{C}[R^*]$.*

For a ring R, $J(R)$ denotes the Jacobson radical of R.

Proposition 7. *Let R be a finite local ring and let R^* denote the group of units in R. Then $J(\mathbf{C}[R]) = \sum_{r\in J(R)} \mathbf{C}(\hat{r} - \hat{0})$ and $\mathbf{C}[R]/J(\mathbf{C}[R]) \cong \mathbf{C}\hat{0} \oplus \mathbf{C}[R^*]$.*

Example 8. Consider the ring $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a,\ b,\ c \in GF(2) \right\}$. Then $|R| = 2^4 = 16$. Then $R^* = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a \neq 0,\ b,\ c \in GF(2) \right\}$. We can easily see that R^* is the dihedral group D_8 of order 8 and so $\mathbf{C}[R^*] \cong \mathbf{C}^4 \oplus M_2(\mathbf{C})$. Therefore $\mathbf{C}[R]/J(\mathbf{C}[R]) \cong \mathbf{C}^5 \oplus M_2(\mathbf{C})$.

By Proposition 7 we have the following corollaries.

Corollary 9. *Let R be a finite commutative local ring and let n denote the number of elements of R^*. Then $\mathbf{C}[R]$ is the direct sum of $n+1$ finite commutative local rings.*

Corollary 10. *Let R be a finite commutative ring and suppose that $R = R_1 \oplus \cdots \oplus R_d$, where $R_1, \cdots R_d$ are finite commutative local rings. Then $\mathbf{C}^R$ is the direct sum of $(|R_1^*| + 1) \times \cdots \times (|R_d^*| + 1)$ finite commutative local rings.*

Corollary 11. *Consider the Galois field $G(n,p)$. Then the $\mathbf{C}$-algebra $\mathbf{C}^{G(n,p)}$ is isomorphic to $\mathbf{C}^{p^n}$.*

Let V be a vector space over $\mathbf{C}$. We define a multiplication on the $\mathbf{C}$-linear space $\mathbf{C} \oplus V$ by the formula $(a, v) \cdot (b, w) = (ab, aw + bv)$ for any $a,\ b \in \mathbf{C},\ v,\ w \in V$. Then $\mathbf{C} \oplus V$ becomes an $\mathbf{C}$-algebra. We denote this algebra by $\mathbf{C} \ltimes V$.

Example 12. Consider the ring $\mathbf{C}[\mathbf{Z}/8\mathbf{Z}]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1, 2, \cdots, 7$. Then $\mathbf{C}[\mathbf{Z}/8\mathbf{Z}]$ is the direct sum of two sided ideals $\mathbf{C}\hat{0}$ and $S = \mathbf{C}g_1 + \mathbf{C}g_2 + \cdots + \mathbf{C}g_7$. The identity of the ring S is g_1. Let us set $e_1 = (1/4)(g_1 - g_3 - g_5 + g_7)$, $e_2 = (1/4)(g_1 + g_3 - g_5 - g_7)$, $e_3 = (1/4)(g_1 - g_3 + g_5 - g_7)$ and $e_4 = (1/4)(g_1 + g_3 + g_5 + g_7)$. Then $e_1,\ e_2,\ e_3,\ e_4$ are orthogonal central primitive idempotents of S and $g_1 = e_1 + e_2 + e_3 + e_4$. We can easily see $e_1 S \cong \mathbf{C}$, $e_2 S \cong \mathbf{C}$, $e_3 S \cong \mathbf{C} \ltimes \mathbf{C}$ and $e_4 S \cong \mathbf{C} \ltimes (\mathbf{C} \oplus \mathbf{C})$. Therefore $\mathbf{C}[\mathbf{Z}/8\mathbf{Z}] \cong \mathbf{C}^3 \oplus \{\mathbf{C} \ltimes \mathbf{C}\} \oplus \{\mathbf{C} \ltimes (\mathbf{C} \oplus \mathbf{C})\}$.

3. Rings of functions from a finite semisimple ring to the field of complex numbers

In this section we consider the case when the ring R is a finite simple ring.

Theorem 13. *Let R be a finite ring. Then the following are equivalent:*
(i) *R is a finite semisimple ring.*
(ii) *$\mathbf{C}^R$ is a semisimple Artinian ring.*

A von Neumann regular ring is a ring R such that for every a in R there exists an x in R such that $a = axa$. Since a semisimple Artinian ring is von Neumann regular, we have the following corollary.

Corollary 14. *Let R be a finite semisimple ring. Then, for any function $f \in \mathbf{C}^R$, there exists a function $g \in \mathbf{C}^R$ such that $a = a * x * a$.*

Example 15. Let $M_2(GF(2))$ denote the ring of 2×2 matrices over the field $GF(2)$. Then we can prove that $\mathbf{C}[M_2(GF(2))]$ is a semiprime ring. Let us set $H = M_2(GF(2)) - GL_2(GF(2))$. Then we can see that $\mathbf{C}[H] \cong \mathbf{C} \oplus M_3(\mathbf{C})$. In fact, let
$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $e_6 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $e_7 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e_8 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $e_9 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $E = \hat{O}, F_1 = \hat{e_1} - \hat{O}, F_2 = \hat{e_2} - \hat{O}, F_3 = (\hat{e_1} - \hat{O}) + (\hat{e_2} - \hat{O}) + (\hat{e_3} - \hat{O}) + (\hat{e_4} - \hat{O}) - (\hat{e_5} - \hat{O}) - (\hat{e_6} - \hat{O}) - (\hat{e_7} - \hat{O}) - (\hat{e_8} - \hat{O}) + (\hat{e_9} - \hat{O})$ are primitive orthogaonal idempotents, and $\mathbf{C}[H] = \mathbf{C}E \oplus \mathbf{C}[H](F_1 + F_2 + F_3) \cong \mathbf{C} \oplus M_3\mathbf{C})$. Since $GL_2(GF(2)) \cong S_3$, we have $\mathbf{C}[M_2(GF(2))]/\mathbf{C}[H] \cong \mathbf{C}[S_3]$. It is easily see that $\mathbf{C}[S_3] \cong \mathbf{C} \oplus \mathbf{C} \oplus M_2(\mathbf{C})$. Hence $\mathbf{C}[M_2(GF(2))]$ is isomorphic to the semisimple Artinian ring $\mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus M_2(\mathbf{C}) \oplus M_3(\mathbf{C})$.

Question 16. Let R be a (not necessarily finite) semiprime ring. Is $\mathbf{C}[R]$ semiprime?

Question 17. Let R be a finite semisimple ring and let C be a commutative ring. What conditions on C imply that $C[R]$ is a separable algebra over C? The definition of separability can be found in [1, Page 40].

4. Invariant functions

Let R ba a finite ring. A function $f : R \to \mathbf{C}$ is said to be *right invariant* if $f(ax) = f(a)$ for all $a \in R$ and all $x \in R^*$. Similarly we define a *left invariant function.* A right and left invariant function is called an *invariant function.*

Theorem 18. *Let R ba a finite ring.*

(1) *If we set $e = (1/|R^*|)\sum_{x\in R^*} \hat{x} \in \mathbf{C}[R]$, then e is an idempotent.*
(2) *The set of right invariant functions becomes the right ideal $\mathbf{C}[R]e$.*
(3) *The set of invariant functions becomes the ring $e\mathbf{C}[R]e$.*

Corollary 19. *Let R be a finite semisimple ring. The ring of invariant functions from R to $\mathbf{C}$ is a semisimple Artinian ring.*

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CLIFFORD EXTENSIONS

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Abstract. In this note, we generalize the construction of Clifford algebras and introduce the notion of Clifford extensions. Clifford extensions are constructed as Frobenius extensions which are Auslander-Gorenstein rings if so is a base ring.

Key Words: Auslander-Gorenstein ring, Clifford extension, Frobenius extension.

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Clifford algebras play important roles in various fields and the construction of Clifford algebras contains that of complex numbers, quaternions, and so on (see e.g. [6]). In this note, we generalize the construction of Clifford algebras and introduce the notion of Clifford extensions. Clifford extensions are constructed as Frobenius extensions, the notion of which we recall below, and we have already known that Frobenius extensions of Auslander-Gorenstein rings (see Definition 2) are also Auslander-Gorenstein rings. It should be noted that little is known about constructions of Auslander-Gorenstein rings although Auslander-Gorenstein rings appear in various fields of current research in mathematics including noncommutative algebraic geometry, Lie algebras, and so on (see e.g. [2], [3], [4] and [11]).

Recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10] which we modify as follows (cf. [1, Section 1]). We use the notation A/R to denote that a ring A contains a ring R as a subring. We say that A/R is a Frobenius extension if the following conditions are satisfied: (F1) A is finitely generated as a left R-module; (F2) A is finitely generated projective as a right R-module; (F3) there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A. Note that ϕ induces a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of R. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let A/R be a Frobenius extension. Then A is an Auslander-Gorenstein ring if so is R, and the converse holds true if A is projective as a left R-module, and if A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. Note that A is projective as a left R-module if A/R is of second kind.

Let $n \geq 2$ be an integer. We fix a set of integers $I = \{0, 1, \ldots, n-1\}$ and a ring R. First, we construct a split Frobenius extension Λ/R of second kind using a certain pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$. Namely, we define an appropriate multiplication on a free right R-module Λ with a basis $\{v_i\}_{i \in I}$. We show that this construction can be iterated

The detailed version of this paper will be submitted for publication elsewhere.

arbitrary times (Proposition 11). Then we deal with the case where $n = 2$ and study the iterated Frobenius extensions. For $m \geq 1$ we construct ring extensions Λ_m/R using the following data: a sequence of elements $c_1, c_2, \cdots$ in $\mathrm{Z}(R)$ and signs $\varepsilon(i,j)$ for $1 \leq i, j \leq m$. Namely, we define an appropriate multiplication on a free right R-module Λ_m with a basis $\{v_x\}_{x \in I^m}$. We show that Λ_m is obtained by iterating the construction above m times, that Λ_m/R is a split Frobenius extension of first kind, and that if $c_i \in \mathrm{rad}(R)$ for $1 \leq i \leq m$ then $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_m/\mathrm{rad}(\Lambda_m)$ (Theorem 13). We call Λ_m Clifford extensions of R because they have the following properties similar to Clifford algebras. For each $x = (x_1, \ldots, x_m) \in I^m$ we set $S(x) = \{i \mid x_i = 1\}$. Also we set $v_x = t_i$ for $x \in I^m$ with $S(x) = \{i\}$. Then the following hold: (C1) $t_i^2 = v_0 c_i$ for all $1 \leq i \leq m$; (C2) $t_i t_j + t_j t_i = 0$ unless $i = j$; (C3) $v_x = t_{i_1} \cdots t_{i_r}$ if $S(x) = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$.

1. Preliminaries

For a ring R we denote by $\mathrm{rad}(R)$ the Jacobson radical of R, by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. Usually, the identity element of a ring is simply denoted by 1. Sometimes, we use the notation 1_R to stress that it is the identity element of the ring R. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}). Sometimes, we use the notation X_R (resp., ${}_RX$) to stress that the module X considered is a right (resp., left) R-module.

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander)**.** *Let R be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

1. *In a minimal injective resolution $I^\bullet$ of R in* Mod-R, flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
2. *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}, flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.
3. *For any $1 \leq i \leq n+1$, any $M \in$* mod-R *and any submodule X of* $\mathrm{Ext}^i_R(M,R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R) = 0$ *for all* $0 \leq j < i$.
4. *For any $1 \leq i \leq n+1$, any $X \in$* mod-R^{op} *and any submodule M of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(M,R) = 0$ *for all* $0 \leq j < i$.

Proof. See e.g. [5, Theorem 3.7]. □

Definition 2 ([4])**.** A right and left noetherian ring R is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and inj dim $R =$ inj dim $R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring R we have inj dim $R =$ inj dim R^{op} whenever inj dim $R < \infty$ and inj dim $R^{\mathrm{op}} < \infty$ (see [12, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10], which we modify as follows.

Definition 3 ([7]). A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:

(F1) A is finitely generated as a left R-module;
(F2) A is finitely generated projective as a right R-module;
(F3) $A \cong \mathrm{Hom}_R(A, R)$ as right A-modules.

In case R is a right and left noetherian ring, for any Frobenius extension A/R the isomorphism $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A yields an Auslander-Gorenstein resolution of A over R in the sense of [8, Definition 3.5].

The next proposition is well-known and easily verified.

Proposition 4. *Let A/R be a ring extension and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in* Mod-A. *Then the following hold.*

(1) *There exists a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$.*
(2) *If $\phi' : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ is another isomorphism in* Mod-A, *then there exists $u \in A^\times$ such that $\phi'(1) = \phi(1)u$ and $\theta'(x) = u^{-1}\theta(x)u$ for all $x \in R$.*
(3) *ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*

Definition 5 (cf. [9, 10]). A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

Proposition 6 ([7, Proposition 1.6]). *If A/R is a Frobenius extension of second kind, then A is projective as a left R-module.*

Proposition 7 ([7, Proposition 1.7]). *For any Frobenius extensions Λ/A, A/R the following hold.*

(1) *Λ/R is a Frobenius extension.*
(2) *Assume Λ/A is of first kind. If A/R is of second (resp., first) kind, then so is Λ/R.*

Definition 8 ([1]). A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 9 ([7, Proposition 1.9]). *For any Frobenius extension A/R the following hold.*

(1) *If R is an Auslander-Gorenstein ring, then so is A with* $\mathrm{inj\ dim}\ A \leq \mathrm{inj\ dim}\ R$.
(2) *Assume A is projective as a left R-module and A/R is split. If A is an Auslander-Gorenstein ring, then so is R with* $\mathrm{inj\ dim}\ R = \mathrm{inj\ dim}\ A$.

2. Construction of Frobenius extensions

Throughout this section, we fix a set of integers $I = \{0, 1, \ldots, n-1\}$ with $n \geq 2$ arbitrary and a ring R together with a pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition:

$$(*) \quad \sigma^n = \mathrm{id}_R \quad \text{and} \quad c \in R^\sigma \cap \mathrm{Z}(R).$$

This is obviously satisfied if $\sigma = \mathrm{id}_R$ and $c \in \mathrm{Z}(R)$.

Let Λ be a free right R-module with a basis $\{v_i\}_{i \in I}$ and $\{\delta_i\}_{i \in I}$ the dual basis of $\{v_i\}_{i \in I}$ for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$, i.e., $\lambda = \sum_{i \in I} v_i \delta_i(\lambda)$ for all $\lambda \in \Lambda$. We set

$$v_{i+kn} = v_i c^k$$

for $i \in I$ and $k \in \mathbb{Z}_+$, the set of non-negative integers, and define a multiplication on Λ subject to the following axioms:

(L1) $v_i v_j = v_{i+j}$ for all $i, j \in I$;

(L2) $a v_i = v_i \sigma^i(a)$ for all $a \in R$ and $i \in I$.

Lemma 10. *The following hold.*

(1) $v_i v_j = v_j v_i$ *for all* $i, j \in I$ *and* $v_i^n = v_0 c^i$ *for all* $i \in I$.

(2) *For any* $\lambda, \mu \in \Lambda$ *we have* $\lambda\mu = \sum_{i,j \in I} v_{i+j} \sigma^j(\delta_i(\lambda))\delta_j(\mu)$ *and hence* $\delta_0(\lambda\mu) = \delta_0(\lambda)\delta_0(\mu) + \sum_{i \in I \setminus \{0\}} \sigma^{n-i}(\delta_i(\lambda))\delta_{n-i}(\mu)c$.

(3) *For any* $\lambda \in \Lambda$ *and* $i, j \in I$ *we have* $\delta_i(\lambda v_j) = \sigma^j(\delta_{i-j}(\lambda))$ *if* $i \geq j$ *and* $\delta_i(\lambda v_j) = \sigma^j(\delta_{i-j+n}(\lambda))c$ *if* $i < j$.

Proposition 11. *The following hold.*

(1) Λ *is an associative ring with* $1 = v_0$ *and contains* R *as a subring via the injective ring homomorphism* $R \to \Lambda, a \mapsto v_0 a$.

(2) Λ/R *is a split Frobenius extension of second kind.*

(3) *If* $c \in \mathrm{rad}(R)$, *then* $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda/\mathrm{rad}(\Lambda)$.

(4) *For any* $\varepsilon \in R^\sigma \cap \mathrm{Z}(R)$ *with* $\varepsilon^n = 1$ *there exists* $\tilde{\sigma} \in \mathrm{Aut}(\Lambda)$ *such that* $\delta_i(\tilde{\sigma}(\lambda)) = \sigma(\delta_i(\lambda))\varepsilon^i$ *for all* $\lambda \in \Lambda$ *and* $i \in I$, *and for any* $c' \in R^\sigma \cap \mathrm{Z}(R)$ *the pair* $(\tilde{\sigma}, c')$ *satisfies the condition (*).*

It should be noted that Proposition 11(4) enables us to iterate the construction above arbitrary times. For instance, one may start from $\sigma = \mathrm{id}_R$.

Remark 12. Let $R[t; \sigma]$ be a right skew polynomial ring with trivial derivation, i.e., $R[t; \sigma]$ consists of all polynomials in an indeterminate t with right-hand coefficients in R and the multiplication is defined by the following rule: $at = t\sigma(a)$ for all $a \in R$. Then $(t^n - c) = (t^n - c)R[t; \sigma]$ is a two-sided ideal and the residue ring $R[t; \sigma]/(t^n - c)$ is isomorphic to Λ.

In the next section, we will deal with the case where $n = 2$ and denote by $Cl_1(R; \sigma, c)$ the ring Λ constructed above.

3. Clifford extensions

In this section, we fix a set of integers $I = \{0, 1\}$ and a ring R together with a sequence of elements $c_1, c_2, \ldots$ in $\mathrm{Z}(R)$. Setting $0 + i = i + 0 = i$ for all $i \in I$ and $1 + 1 = 0$, we consider I as a cyclic group of order 2. For any $n \geq 1$ we denote by I^n the direct product of n copies of I and consider I^{n-1} as a subgroup of I^n via the injective group homomorphism

$$I^{n-1} \to I^n, (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 0),$$

where $I^0 = \{0\}$ is the trivial group. According to Proposition 11(4), one can construct inductively various I^n-graded rings which are Frobenius extensions of R. However, in this note we restrict ourselves to the following particular case.

Let $n \geq 1$. For each $x = (x_1, \dots, x_n) \in I^n$ we set $S(x) = \{i \mid x_i = 1\}$. Note that $S(x+y) = S(x) + S(y)$, the symmetric difference of $S(x)$ and $S(y)$, for all $x, y \in I^n$. We set

$$\varepsilon(i,j) = \begin{cases} +1 & \text{if } i \leq j, \\ -1 & \text{if } i > j \end{cases}$$

for $1 \leq i, j \leq n$ and

$$c(x,y) = \prod_{(i,j) \in S(x) \times S(y)} \varepsilon(i,j) \prod_{k \in S(x) \cap S(y)} c_k$$

for $x, y \in I^n$. We denote by s the element $x \in I^n$ with $S(x) = \{1, \dots, n\}$.

Let Λ_n be a free right R-module with a basis $\{v_x\}_{x \in I^n}$. We denote by $\{\delta_x\}_{x \in I^n}$ the dual basis of $\{v_x\}_{x \in I^n}$ for the free left R-module $\mathrm{Hom}_R(\Lambda_n, R)$, i.e., $\lambda = \sum_{x \in I^n} v_x \delta_x(\lambda)$ for all $\lambda \in \Lambda_n$. We define a multiplication on Λ_n subject to the following axioms:

(M1) $v_x v_y = v_{x+y} c(x,y)$ for all $x, y \in I^n$;

(M2) $a v_x = v_x a$ for all $x \in I^n$ and $a \in R$.

In the following, we set $v_x = t_i$ for $x \in I^n$ with $S(x) = \{i\}$. It is easy to see the following:

(C1) $t_i^2 = v_0 c_i$ for all $1 \leq i \leq n$;

(C2) $t_i t_j + t_j t_i = 0$ unless $i = j$;

(C3) $v_x = t_{i_1} \cdots t_{i_r}$ if $S(x) = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$.

Theorem 13. *For any $n \geq 1$ the following hold.*

1. *Λ_n is an associative ring with $1 = v_0$ and contains R as a subring via the injective ring homomorphism $R \to \Lambda_n, a \mapsto v_0 a$.*
2. *Λ_n/R is a split Frobenius extension of first kind.*
3. *If $c_i \in \mathrm{rad}(R)$ for all $1 \leq i \leq n$, then $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_n/\mathrm{rad}(\Lambda_n)$.*

Remark 14. If $d(x,y) = |S(x) \times S(y)| - |S(x) \cap S(y)|$ is even, then $v_x v_y = v_y v_x$. In particular, $v_s \in \mathrm{Z}(\Lambda_n)$ if n is odd.

Denote by J^n the subset of I^n consisting of all $x \in I^n$ with $|S(x)|$ even. Then J^n is a subgroup of I^n and $\Lambda_n^0 = \oplus_{x \in J^n} v_x R$ is a subring of Λ_n.

Proposition 15. *Assume n is even. Then $v_s \in \Lambda_n^0$ and the following hold.*

1. *Λ_n^0/R is a split Frobenius extension of first kind.*
2. *If $c_i \in \mathrm{rad}(R)$ for all $1 \leq i \leq n$, then $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_n^0/\mathrm{rad}(\Lambda_n^0)$.*

We denote by $Cl_n(R; c_1, \dots, c_n)$ (resp., $Cl_n^0(R; c_1, \dots, c_n)$) the ring Λ_n (resp., Λ_n^0) constructed above, which we call Clifford extensions of R.

Remark 16. If $c_i \in R^\times$ for some $1 \leq i \leq n$, then $Cl_n^0(R; c_1, \dots, c_n)/R$ is a split Frobenius extension of first kind.

Example 17. Let K be a commutative field and V a 3-dimensional K-space. Then $Cl_3^0(K;0,0,0) \cong K \ltimes V$, the trivial extension of K by V, which is not a Frobenius algebra.

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GROUP-GRADED AND GROUP-BIGRADED RINGS

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Abstract. Let I be a non-trivial finite multiplicative group with the unit element e and $A = \oplus_{x \in I} A_x$ an I-graded ring. We construct a Frobenius extension Λ of A and study when the ring extension A of A_e can be a Frobenius extension. Also, formulating the ring structure of Λ, we introduce the notion of I-bigraded rings and show that every I-bigraded ring is isomorphic to the I-bigraded ring Λ constructed above.

Introduction

Let I be a non-trivial finite multiplicative group with the unit element e and $A = \oplus_{x \in I} A_x$ an I-graded ring. In this note, assuming A_e is a local ring, we study when a ring extension A of A_e can be a Frobenius extension, the notion of which we recall below. Auslander-Gorenstein rings (see Definition 2) appear in various fields of current research in mathematics. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [5], [6] and [14], respectively). However, little is known about constructions of Auslander-Gorenstein rings. We have shown in [9, Section 3] that a left and right noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring. A Frobenius extension A of a left and right noetherian ring R is a typical example such that A admits an Auslander-Gorenstein resolution over R.

Now we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [11, 12] which we modify as follows (cf. [1, Section 1]). We use the notation A/R to denote that a ring A contains a ring R as a subring. We say that A/R is a Frobenius extension if the following conditions are satisfied: (F1) A is finitely generated as a left R-module; (F2) A is finitely generated projective as a right R-module; (F3) there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A. Note that ϕ induces a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of R. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let A/R be a Frobenius extension. Then A is an Auslander-Gorenstein ring if so is R, and the converse holds true if A is projective as a left R-module, and if A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. It should be noted that A is projective as a left R-module if A/R is of second kind.

The detailed version of this paper will be submitted for publication elsewhere.

To state our main theorem we have to construct a Frobenius extension Λ/A of first kind. Namely, we will define an appropriate multiplication on a free right A-module Λ with a basis $\{v_x\}_{x\in I}$ so that Λ/A is a Frobenius extension of first kind. Denote by $\{\gamma_x\}_{x\in I}$ the dual basis of $\{v_x\}_{x\in I}$ for the free left A-module $\mathrm{Hom}_A(\Lambda, A)$ and set $\gamma = \Sigma_{x\in I}\gamma_x$. Assume A_e is local, $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and A is reflexive as a right A_e-module. Our main theorem states that the following are equivalent: (1) $A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right A-modules; (2) There exist a unique $s \in I$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that $\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$ for all $x \in I$; (3) There exist a unique $s \in I$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in I$ (Theorem 18). Assume A/A_e is a Frobenius extension. We show that it is of second kind (Corollary 20), and that A is an Auslander-Gorenstein ring if and only if so is Λ (Theorem 21).

As we saw above, the ring Λ plays an essential role in our argument. Formulating the ring structure of Λ, we introduce the notion of group-bigraded rings as follows. A ring Λ together with a group homomorphism $\eta : I^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$ is said to be an I-bigraded ring, denoted by (Λ, η), if $1 = \sum_{x\in I} v_x$ with the v_x orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in I$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in I$. We conclude that every I-bigraded ring is isomorphic to the I-bigraded ring Λ constructed above (Proposition 24).

This note is organized as follows. In Section 1, we recall basic facts on Auslander-Gorenstein rings and Frobenius extensions. In Section 2, we construct a Frobenius extension Λ/A of first kind and study the ring structure of Λ. In Section 3, we prove the main theorem. In Section 4, we introduce the notion of group-bigraded rings and study the structure of such rings.

1. Preliminaries

For a ring R we denote by $\mathrm{rad}(R)$ the Jacobson radical of R, by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. Usually, the identity element of a ring is simply denoted by 1. Sometimes, we use the notation 1_R to stress that it is the identity element of the ring R. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}). Sometimes, we use the notation X_R (resp., ${}_RX$) to stress that the module X considered is a right (resp., left) R-module.

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander)**.** *Let R be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

(1) *In a minimal injective resolution $I^\bullet$ of R in* Mod-R*,* flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
(2) *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}*,* flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.

(3) *For any* $1 \le i \le n+1$*, any* $M \in \text{mod-}R$ *and any submodule* X *of* $\mathrm{Ext}^i_R(M,R) \in \text{mod-}R^{\mathrm{op}}$ *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R) = 0$ *for all* $0 \le j < i$*.*
(4) *For any* $1 \le i \le n+1$*, any* $X \in \text{mod-}R^{\mathrm{op}}$ *and any submodule* M *of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in \text{mod-}R$ *we have* $\mathrm{Ext}^j_R(M,R) = 0$ *for all* $0 \le j < i$*.*

Proof. See e.g. [7, Theorem 3.7]. □

Definition 2 ([6])**.** A right and left noetherian ring R is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \ge 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and $\text{inj dim } R = \text{inj dim } R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring R we have $\text{inj dim } R = \text{inj dim } R^{\mathrm{op}}$ whenever $\text{inj dim } R < \infty$ and $\text{inj dim } R^{\mathrm{op}} < \infty$ (see [15, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [11, 12], which we modify as follows (cf. [1, Section 1]).

Definition 3. A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:
(F1) A is finitely generated as a left R-module;
(F2) A is finitely generated projective as a right R-module;
(F3) $A \cong \mathrm{Hom}_R(A,R)$ as right A-modules.

In case R is a right and left noetherian ring, for any Frobenius extension A/R the isomorphism $A \xrightarrow{\sim} \mathrm{Hom}_R(A,R)$ in Mod-A yields an Auslander-Gorenstein resolution of A over R in the sense of [9, Definition 3.5].

The next proposition is well-known and easily verified.

Proposition 4. *Let* A/R *be a ring extension and* $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A,R)$ *an isomorphism in* Mod-A*. Then the following hold.*
(1) *There exists a unique ring homomorphism* $\theta : R \to A$ *such that* $x\phi(1) = \phi(1)\theta(x)$ *for all* $x \in R$*.*
(2) *If* $\phi' : A \xrightarrow{\sim} \mathrm{Hom}_R(A,R)$ *is another isomorphism in* Mod-A*, then there exists* $u \in A^\times$ *such that* $\phi'(1) = \phi(1)u$ *and* $\theta'(x) = u^{-1}\theta(x)u$ *for all* $x \in R$*.*
(3) ϕ *is an isomorphism of* R*-*A*-bimodules if and only if* $\theta(x) = x$ *for all* $x \in R$*.*

Definition 5 (cf. [11, 12])**.** A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A,R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A,R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

Proposition 6. *If* A/R *is a Frobenius extension of second kind, then* A *is projective as a left* R*-module.*

Proof. Let $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A,R)$ be an isomorphism in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Then θ induces an equivalence $U_\theta : \text{Mod-}R^{\mathrm{op}} \xrightarrow{\sim} \text{Mod-}R^{\mathrm{op}}$ such that for any $M \in \text{Mod-}R^{\mathrm{op}}$ we have $U_\theta M = M$ as an additive group and the left R-module structure of $U_\theta M$ is given by the law of composition $R \times M \to M, (x, m) \mapsto \theta(x)m$. Since ϕ yields an isomorphism of R-A-bimodules $U_\theta A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$, and since $\mathrm{Hom}_R(A, R)$ is projective as a left R-module, it follows that $U_\theta A$ and hence A are projective as left R-modules. □

Proposition 7. *For any Frobenius extensions Λ/A, A/R the following hold.*

(1) *Λ/R is a Frobenius extension.*
(2) *Assume Λ/A is of first kind. If A/R is of second (resp., first) kind, then so is Λ/R.*

Proof. (1) Obviously, (F1) and (F2) are satisfied. Also, we have

$$\begin{aligned}\Lambda &\cong \mathrm{Hom}_A(\Lambda, A)\\ &\cong \mathrm{Hom}_A(\Lambda, \mathrm{Hom}_R(A, R))\\ &\cong \mathrm{Hom}_R(\Lambda \otimes_A A, R)\\ &\cong \mathrm{Hom}_R(\Lambda, R)\end{aligned}$$

in Mod-Λ.

(2) Let $\psi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A)$ be an isomorphism of A-Λ-bimodules and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Setting $\gamma = \psi(1)$ and $\alpha = \phi(1)$, as in (1), we have an isomorphism in Mod-Λ

$$\xi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, R), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

For any $x \in R$, we have

$$\begin{aligned}x\xi(1)(\mu) &= x\alpha(\gamma(\mu))\\ &= \alpha(\theta(x)\gamma(\mu))\\ &= \alpha(\gamma(\theta(x)\mu))\\ &= \xi(1)(\theta(x)\mu)\end{aligned}$$

for all $\mu \in \Lambda$ and $x\xi(1) = \xi(1)\theta(x)$. □

Definition 8 ([1])**.** A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 9 (cf. [1])**.** *For any Frobenius extension A/R the following hold.*

(1) *If R is an Auslander-Gorenstein ring, then so is A with* $\text{inj dim } A \le \text{inj dim } R$.
(2) *Assume A is projective as a left R-module and A/R is split. If A is an Auslander-Gorenstein ring, then so is R with* $\text{inj dim } R = \text{inj dim } A$.

Proof. (1) See [9, Theorem 3.6].

(2) It follows by [1, Proposition 1.7] that R is a right and left noetherian ring with $\text{inj dim } R = \text{inj dim } R^{\mathrm{op}} = \text{inj dim } A$. Let $A \to E^\bullet$ be a minimal injective resolution in Mod-A. For any $i \ge 0$, $\mathrm{Hom}_R(-, E^i) \cong \mathrm{Hom}_A(- \otimes_R A, E^i)$ as functors on Mod-R and E^i_R is injective, and $E^i \otimes_R - \cong E^i \otimes_A A \otimes_R -$ as functors on Mod-R^{op} and flat dim $E^i_R \le$ flat dim $E^i_A \le i$. Now, since R_R appears in A_R as a direct summand, it follows that R satisfies the Auslander condition. □

2. Graded rings

Throughout the rest of this note, I stands for a non-trivial finite multiplicative group with the unit element e.

Throughout this and the next sections, we fix a ring A together with a family $\{\delta_x\}_{x\in I}$ in $\mathrm{End}_{\mathbb{Z}}(A)$ satisfying the following conditions:

(D1) $\delta_x\delta_y = 0$ unless $x = y$ and $\sum_{x\in I}\delta_x = \mathrm{id}_A$;

(D2) $\delta_x(a)\delta_y(b) = \delta_{xy}(\delta_x(a)b)$ for all $a, b \in A$ and $x, y \in I$.

Namely, setting $A_x = \mathrm{Im}\,\delta_x$ for $x \in I$, $A = \oplus_{x\in I}A_x$ is an I-graded ring. In particular, A/A_e is a split ring extension.

To prove our main theorem (Theorem 18), we need an extension ring Λ of A such that Λ/A is a Frobenius extension of first kind. Let Λ be a free right A-module with a basis $\{v_x\}_{x\in I}$ and define a multiplication on Λ subject to the following axioms:

(M1) $v_xv_y = 0$ unless $x = y$ and $v_xv_x = v_x$ for all $x \in I$;

(M2) $av_x = \sum_{y\in I} v_y\delta_{yx^{-1}}(a)$ for all $a \in A$ and $x \in I$.

We denote by $\{\gamma_x\}_{x\in I}$ the dual basis of $\{v_x\}_{x\in I}$ for the free left A-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{x\in I} v_x\gamma_x(\lambda)$ for all $\lambda \in \Lambda$. It is not difficult to see that

$$\lambda\mu = \sum_{x,y\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)$$

for all $\lambda, \mu \in \Lambda$. Also, setting $\gamma = \sum_{x\in I}\gamma_x$, we define a mapping

$$\phi : \Lambda \to \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda.$$

Proposition 10. *The following hold.*

1. *Λ is an associative ring with $1 = \sum_{x\in I} v_x$ and contains A as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{x\in I} v_xa$.*
2. *ϕ is an isomorphism of A-Λ-bimodules, i.e., Λ/A is a Frobenius extension of first kind.*

Proof. (1) Let $\lambda \in \Lambda$. Obviously, $\sum_{x\in I} v_x \cdot \lambda = \lambda$. Also, by (D1) we have

$$\begin{aligned}\lambda\cdot\sum_{y\in I} v_y &= \sum_{x,y\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\\ &= \sum_{x\in I} v_x\gamma_x(\lambda)\\ &= \lambda.\end{aligned}$$

Next, for any $\lambda, \mu, \nu \in \Lambda$ by (D2) we have

$$\begin{aligned}(\lambda\mu)\nu &= \sum_{x,y,z\in I} v_x\delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu))\gamma_z(\nu)\\ &= \sum_{x,y,z\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu))\gamma_z(\nu)\\ &= \lambda(\mu\nu).\end{aligned}$$

The remaining assertions are obvious.

(2) Let $\lambda \in \operatorname{Ker} \phi$. For any $y \in I$ we have $0 = \gamma(\lambda v_y) = \sum_{x \in I} \delta_{xy^{-1}}(\gamma_x(\lambda))$ and $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $x \in I$. Thus for any $x \in I$ we have $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $y \in I$ and by (D1) $\gamma_x(\lambda) = 0$, so that $\lambda = 0$. Next, for any $f = \sum_{x \in I} a_x \gamma_x \in \operatorname{Hom}_A(\Lambda, A)$, setting $\lambda = \sum_{x,z \in I} v_x \delta_{xz^{-1}}(a_z)$, by (D1) we have

$$\begin{aligned}(\gamma\lambda)(v_y) &= \gamma(\lambda v_y) \\ &= \sum_{x \in I} \delta_{xy^{-1}}(\gamma_x(\lambda)) \\ &= \sum_{x,z \in I} \delta_{xy^{-1}}(\delta_{xz^{-1}}(a_z)) \\ &= a_y \\ &= f(v_y)\end{aligned}$$

for all $y \in I$ and $f = \gamma\lambda$. Finally, for any $a \in A$ by (D1) we have

$$\begin{aligned}(\gamma a)(\lambda) &= \gamma(a\lambda) \\ &= \sum_{x,y \in I} \delta_{yx^{-1}}(a)\gamma_x(\lambda) \\ &= a\gamma(\lambda)\end{aligned}$$

for all $\lambda \in \Lambda$ and $\gamma a = a\gamma$. □

Remark 11. Denote by $|I|$ the order of I. If $|I| \cdot 1_A \in A^\times$, then Λ/A is a split ring extension.

Lemma 12. *The following hold.*

(1) $v_x \lambda v_y = v_x \delta_{xy^{-1}}(\gamma_x(\lambda))$ *for all* $\lambda \in \Lambda$ *and* $x, y \in I$.
(2) $v_x \Lambda v_y = v_x A_{xy^{-1}}$ *for all* $x, y \in I$.
(3) $v_x a \cdot v_y b = v_x ab$ *for all* $x, y, z \in I$ *and* $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.

Proof. Immediate by the definition. □

Setting $\Lambda_{x,y} = v_x \Lambda v_y$ for $x, y \in I$, we have $\Lambda = \oplus_{x,y \in I} \Lambda_{x,y}$ with $\Lambda_{x,y}\Lambda_{z,w} = 0$ unless $y = z$ and $\Lambda_{x,y}\Lambda_{y,z} \subseteq \Lambda_{x,z}$ for all $x, y, z \in I$. Also, setting $\lambda_{x,y} = \delta_{xy^{-1}}(\gamma_x(\lambda)) \in A_{xy^{-1}}$ for $\lambda \in \Lambda$ and $x, y \in I$, we have a group homomorphism

$$\eta : I^{\mathrm{op}} \to \operatorname{Aut}(\Lambda), x \mapsto \eta_x$$

such that $\eta_x(\lambda)_{y,z} = \lambda_{yx^{-1},zx^{-1}}$ for all $\lambda \in \Lambda$ and $x, y, z \in I$. We denote by Λ^I the subring of Λ consisting of all λ such that $\eta_x(\lambda) = \lambda$ for all $x \in I$.

Proposition 13. *The following hold.*

(1) $\eta_y(v_x) = v_{xy}$ *for all* $x, y \in I$.
(2) $\Lambda^I = A$.
(3) $(\lambda\mu)_{x,z} = \sum_{y \in I} \lambda_{x,y}\mu_{y,z}$ *for all* $\lambda, \mu \in \Lambda$ *and* $x, z \in I$.

Proof. (1) Since $\eta_y(v_x)_{z,w} = \delta_{zw^{-1}}(\gamma_{zy^{-1}}(v_x))$ for all $z, w \in I$, we have

$$\eta_y(v_x)_{z,w} = \begin{cases} 1 & \text{if } z = w \text{ and } x = zy^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $a \in A$, since $\eta_x(a)_{y,z} = a_{yx^{-1},zx^{-1}} = \delta_{(yx^{-1})(zx^{-1})^{-1}}(a) = \delta_{yz^{-1}}(a) = a_{y,z}$ for all $x, y, z \in I$, we have $a \in \Lambda^I$. Conversely, for any $\lambda \in \Lambda^I$ we have $\delta_{y^{-1}}(\gamma_x(\lambda)) = \lambda_{x,yx} = \eta_{x^{-1}}(\lambda)_{e,y} = \lambda_{e,y} = \delta_{y^{-1}}(\gamma_e(\lambda))$ for all $x, y \in I$, so that $\gamma_x(\lambda) = \gamma_e(\lambda)$ for all $x \in I$.

(3) For any $\lambda, \mu \in \Lambda$ and $x, z \in I$ by (D2) we have

$$\begin{aligned}(\lambda\mu)_{x,z} &= \sum_{y\in I} \delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)) \\ &= \sum_{y\in I} \delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu)) \\ &= \sum_{y\in I} \lambda_{x,y}\mu_{y,z}.\end{aligned}$$

□

Remark 14. We have $\eta_y(v_x a_x)v_y b_y = v_{xy}a_x b_y$ for all $a_x \in A_x$ and $b_y \in A_y$.

Proposition 15. *The following hold.*

(1) $\mathrm{End}_\Lambda(v_x\Lambda) \cong A_e$ *as rings for all* $x \in I$.
(2) $v_x\Lambda \not\cong v_y\Lambda$ *in* Mod-Λ *for all* $x, y \in I$ *with* $A_{xy^{-1}}A_{yx^{-1}} \subsetneq \mathrm{rad}(A_e)$.

Proof. (1) We have $\mathrm{End}_\Lambda(v_x\Lambda) \cong v_x\Lambda v_x \cong A_e$ as rings.

(2) For any $f : v_x\Lambda \to v_y\Lambda$ and $g : v_y\Lambda \to v_x\Lambda$ in Mod-Λ, since $f(v_x) = v_y a$ with $a \in A_{yx^{-1}}$ and $g(v_y) = v_x b$ with $b \in A_{xy^{-1}}$, we have $g(f(v_x)) = v_x ba$ with $ba \in \mathrm{rad}(A_e)$. □

The proposition above asserts that if A_e is local and $A_x A_{x^{-1}} \subsetneq \mathrm{rad}(A_e)$ for all $x \neq e$ then Λ is semiperfect and basic. We refer to [3] for semiperfect rings.

3. Auslander-Gorenstein rings

In this section, we will ask when A/A_e is a Frobenius extension.

Lemma 16. *For any* $x \in I$ *the following hold.*

(1) $av_x = v_x a$ *for all* $a \in A_e$ *and* Λv_x *is a* Λ-A_e-*bimodule.*
(2) $\Lambda v_x = \sum_{y\in I} v_y A_{yx^{-1}}$.
(3) $A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y\in I} v_y \delta_{yx^{-1}}(a)$ *as* A-A_e-*bimodules.*
(4) *If* Λv_x *is reflexive as a right* A_e-*module, then* $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong A_e$ *as rings.*

Proof. (1) and (2) Immediate by the definition.

(3) By (2) we have a bijection $f_x : A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y\in I} v_y\delta_{yx^{-1}}(a)$. Since every $\delta_{yx^{-1}}$ is a homomorphism in Mod-A_e, so is f_x. Finally, for any $a, b \in A$ we have

$$\begin{aligned} a \cdot (\sum_{y\in I} v_y\delta_{yx^{-1}}(b)) &= \sum_{y,z\in I} v_z\delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b) \\ &= \sum_{z\in I} v_z(\sum_{y\in I} \delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b)) \\ &= \sum_{z\in I} v_z\delta_{zx^{-1}}(\sum_{y\in I} \delta_{zy^{-1}}(a)b) \\ &= \sum_{z\in I} v_z\delta_{zx^{-1}}(ab) \end{aligned}$$

and f_x is a homomorphism in Mod-A^{op}.

(4) Since the canonical homomorphism

$$\Lambda v_x \to \mathrm{Hom}_{A_e^{\mathrm{op}}}(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e), A_e), \lambda \mapsto (f \mapsto f(\lambda))$$

is an isomorphism, $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(\Lambda v_x)^{\mathrm{op}} \cong v_x\Lambda v_x \cong A_e$ as rings. □

It follows by Lemma 16(1) that $\delta_e\gamma_e : \Lambda \to A_e$ is a homomorphism of A_e-A_e-bimodules and Λ/A_e is a split ring extension.

Lemma 17. *For any $x, y \in I$ and $a, b \in A$ we have*

$$v_x a \cdot (\sum_{z\in I} v_z\delta_{zy^{-1}}(b)) = v_x(\sum_{z\in I} \delta_{xz^{-1}}(a)\delta_{zy^{-1}}(b))$$

Proof. Immediate by the definition. □

Theorem 18. *Assume A_e is local, $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and A is reflexive as a right A_e-module. Then the following are equivalent.*

(1) *$A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right A-modules.*

(2) *There exist a unique $s \in I$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that*

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

for all $x \in I$.

(3) *There exist a unique $s \in I$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that*

$$\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$$

for all $x \in I$.

Proof. (1) ⇒ (2). Let $A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), 1 \mapsto \alpha$ in Mod-A. Then, since by Proposition 10(2) $\Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda$ in Mod-Λ, by adjointness we have an isomorphism in Mod-Λ

$$\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

By Proposition 15(1) $\Lambda = \oplus_{x\in I} v_x\Lambda$ with the $\mathrm{End}_\Lambda(v_x\Lambda)$ local. Also, by (1) and (4) of Lemma 16

$$\mathrm{Hom}_{A_e}(\Lambda, A_e) \cong \oplus_{x\in I}\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)$$

with the $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e))$ local. Now, according to Proposition 15(2), it follows by the Krull-Schmidt theorem that there exists a unique $s \in I$ such that

$$\phi_{s,e} : v_s\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_e, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

Thus, setting $\alpha_s = \alpha|_{A_s}$, by Lemmas 16(2) and 17 we have

$$\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab))).$$

It then follows again by Lemmas 16(2) and 17 that

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

for all $x \in I$.

(2) $\Rightarrow$ (3). Since $A = \oplus_{x\in I} A_{sx} = \oplus_{x \in I} A_{x^{-1}}$, and since $A_{sx}A_{x^{-1}} \subseteq A_s$ for all $x \in I$, ψ induces $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in I$.

(3) $\Rightarrow$ (1). Setting $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for each $x \in I$, the ψ_x yields $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$. □

Remark 19. In the theorem above, α_s is an isomorphism and $A_e \xrightarrow{\sim} \mathrm{End}_{A_e}(A_s)$ canonically.

Proof. For any $b \in A_e$, setting $f : A_e \to A_e, 1 \mapsto b$, we have $f = \psi_e(a)$ and hence $b = \alpha_s(a)$ for some $a \in A_s$. Also, Ker α_s = Ker $\psi_s = 0$. Then, since the composite $A_e \to \mathrm{End}_{A_e}(A_s) \to \mathrm{Hom}_{A_e}(A_s, A_e)$ is an isomorphism, the last assertion follows. □

Corollary 20. *Assume A_e is local and $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$. If A/A_e is a Frobenius extension, then it is of second kind.*

Proof. Set $t = \alpha_s^{-1}(1) \in A_s$. Then for any $u \in A_s$ there exists $f \in \mathrm{End}_{A_e}(A_s)$ such that $u = f(t)$ and hence $u = at$ for some $a \in A_e$. Thus $A_et = A_s$ and there exists $\theta \in \mathrm{Aut}(A_e)$ such that $\theta(a)t = ta$ for all $a \in A_e$. Then $(\alpha_s\theta(a))(t) = \alpha_s(\theta(a)t) = \alpha_s(ta) = \alpha_s(t)a = a = (a\alpha_s)(t)$ and $\alpha_s\theta(a) = a\alpha_s$ for all $a \in A_e$. Now, setting $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$, we have $(a\psi(1))(b) = a\alpha_s(\delta_s(b)) = (a\alpha_s)(\delta_s(b)) = (\alpha_s\theta(a))(\delta_s(b)) = \alpha_s(\theta(a)\delta_s(b)) = \alpha_s(\delta_s(\theta(a)b)) = (\psi(1)\theta(a))(b)$ for all $a, b \in A$, so that $a\psi(1) = \psi(1)\theta(a)$ for all $a \in A$. □

Theorem 21. *Assume A_e is local, $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$, and A/A_e is a Frobenius extension. Then A is an Auslander-Gorenstein ring if and only if so is Λ.*

Proof. The "only if" part follows by Propositions 9(1) and 10(2). Assume Λ is an Auslander-Gorenstein ring. By Proposition 10(2) Λ/A is a Frobenius extension of first kind, and by Corollary 20 A/A_e is a Frobenius extension of second kind. Thus by Proposition 7 Λ/A_e is a Frobenius extension of second kind. Also, by Lemma 16(1) Λ/A_e is split. Hence by Propositions 6 and 9(2) A_e is an Auslander-Gorenstein ring and by Proposition 9(1) so is A. □

4. Bigraded rings

Formulating the ring structure of Λ constructed in Section 2, we make the following.

Definition 22. A ring Λ together with a group homomorphism

$$\eta : I^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$$

is said to be an I-bigraded ring, denoted by (Λ, η), if $1 = \sum_{x \in I} v_x$ with the v_x orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in I$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in I$.

Throughout this section, we fix an I-bigraded ring (Λ, η). Set $A_x = v_x\Lambda v_e$ for $x \in I$ and $A = \oplus_{x \in I} A_x$. Note that $\eta_y(A_x) = v_{xy}\Lambda v_y$ for all $x, y \in I$. For any $a_x \in A_x$ and $b_y \in A_y$ we define the multiplication $a_x \cdot b_y$ in A as the multiplication $\eta_y(a_x)b_y$ in Λ (cf. Remark 14).

Proposition 23. *The following hold.*

(1) *A is an associative ring with $1 = v_e$.*
(2) *A is an I-graded ring.*

Proof. (1) For any $a_x \in A_x$, $b_y \in A_y$ and $c_z \in A_z$ we have

$$\begin{aligned}(a_x \cdot b_y) \cdot c_z &= \eta_y(a_x)b_y \cdot c_z \\ &= \eta_z(\eta_y(a_x)b_y)c_z \\ &= \eta_{yz}(a_x)\eta_z(b_y)c_z \\ &= a_x \cdot (b_y \cdot c_z).\end{aligned}$$

Also, for any $a_x \in A_x$ we have $v_e \cdot a_x = \eta_x(v_e)a_x = v_xa_x = a_x$ and $a_x \cdot v_e = \eta_e(a_x)v_e = a_xv_e = a_x$.

(2) Obviously, $A_xA_y \subseteq A_{xy}$ for all $x, y \in I$. $\square$

In the following, for each $x \in I$ we denote by $\delta_x : A \to A_x$ the projection. Then, setting $\lambda_{x,y} = v_x\lambda v_y$ for $\lambda \in \Lambda$ and $x, y \in I$, we have a mapping $\varphi : A \to \Lambda$ such that $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a))$ for all $a \in A$ and $x, y \in I$.

Proposition 24. *The following hold.*

(1) *$\varphi : A \to \Lambda$ is an injective ring homomorphism with* $\mathrm{Im}\,\varphi = \Lambda^I$.
(2) *$v_x\Lambda v_y = v_x\varphi(A_{xy^{-1}})$ for all $x, y \in I$.*
(3) *$\{v_x\}_{x \in I}$ is a basis for the right A-module Λ.*
(4) *$\varphi(a)v_x = \sum_{y \in I} v_y\varphi(\delta_{yx^{-1}}(a))$ for all $a \in A$ and $x \in I$.*
(5) *$v_x\varphi(a)v_y\varphi(b) = v_x\varphi(ab)$ for all $x, y, z \in I$ and $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.*

Proof. (1) Obviously, φ is a monomorphism of additive groups. Also, we have

$$\varphi(v_e)_{x,y} = \begin{cases} v_x & \text{if } x = y, \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi(1_A) = 1_\Lambda$. Let $a_x \in A_x, b_y \in A_y$ and $z, w \in I$. Since $\varphi(a_x \cdot b_y)_{z,w} = \varphi(\eta_y(a_x)b_y)_{z,w} = \eta_w(\delta_{zw^{-1}}(\eta_y(a_x)b_y))$, $\varphi(a_x \cdot b_y)_{z,w} = 0$ unless $xy = zw^{-1}$. If $xy = zw^{-1}$, then $\eta_w(\delta_{zw^{-1}}(\eta_y(a_x)b_y)) =$

$\eta_{yw}(a_x)\eta_w(b_y)$. On the other hand,

$$(\varphi(a_x)\varphi(b_y))_{z,w} = \sum_{u\in I} \varphi(a_x)_{z,u}\varphi(b_y)_{u,w}$$
$$= \sum_{u\in I} \eta_u(\delta_{zu^{-1}}(a_x))\eta_w(\delta_{uw^{-1}}(b_y)).$$

Thus $(\varphi(a_x)\varphi(b_y))_{z,w} = 0$ unless $zu^{-1} = x$ and $uw^{-1} = y$, i.e., $zw^{-1} = xy$. If $zw^{-1} = xy$, then $\sum_{u\in I} \eta_u(\delta_{zu^{-1}}(a_x))\eta_w(\delta_{uw^{-1}}(b_y)) = \eta_{yw}(a_x)\eta_w(b_y)$. As a consequence, $\varphi(a_x \cdot b_y)_{z,w} = (\varphi(a_x)\varphi(b_y))_{z,w}$. The first assertion follows.

Next, for any $a \in A$ and $x, y, z \in I$ we have

$$\begin{aligned}\eta_x(\varphi(a))_{y,z} &= v_y\eta_x(\varphi(a))v_z \\ &= \eta_x(v_{yx^{-1}}\varphi(a)v_{zx^{-1}}) \\ &= \eta_x(\varphi(a)_{yx^{-1},zx^{-1}}) \\ &= \eta_x(\eta_{zx^{-1}}(\delta_{yz^{-1}}(a))) \\ &= \eta_z(\delta_{yz^{-1}}(a)) \\ &= \varphi(a)_{y,z},\end{aligned}$$

so that $\mathrm{Im}\, \varphi \subseteq \Lambda^I$. Conversely, let $\lambda \in \Lambda^I$. Then $\lambda_{x,y} = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e}$ for all $x, y \in I$. Thus, setting $a = \sum_{x\in I} \lambda_{x,e}$, we have $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a)) = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e} = \lambda_{x,y}$ for all $x, y \in I$ and $\varphi(a) = \lambda$.

(2) Let $x, y \in I$ and $a \in A_{xy^{-1}}$. For any $z \neq y$ we have $\delta_{xz^{-1}}(a) = 0$ and hence $v_x\varphi(a)v_z = \varphi(a)_{x,z} = \eta_z(\delta_{xz^{-1}}(a)) = 0$. Thus $v_x\varphi(a) = \varphi(a)_{x,y} = \eta_y(a)$. It follows that $v_x\Lambda v_y = \eta_y(v_{xy^{-1}}\Lambda v_e) = \eta_y(A_{xy^{-1}}) = v_x\varphi(A_{xy^{-1}})$.

(3) This follows by (2).

(4) Note that $\eta_x(\delta_{yx^{-1}}(a)) = v_y\eta_x(\delta_{yx^{-1}}(a))$ for all $y \in I$. Thus $\varphi(a)v_x = \sum_{y\in I} v_y\varphi(a)v_x = \sum_{y\in I} \eta_x(\delta_{yx^{-1}}(a)) = \sum_{y\in I} v_y\eta_x(\delta_{yx^{-1}}(a))$. Also,

$$\begin{aligned}v_y\varphi(\delta_{yx^{-1}}(a)) &= \sum_{z\in I} v_y\varphi(\delta_{yx^{-1}}(a))v_z \\ &= \sum_{z\in I} v_y\eta_z(\delta_{yz^{-1}}(\delta_{yx^{-1}}(a))) \\ &= v_y\eta_x(\delta_{yx^{-1}}(a))\end{aligned}$$

for all $y \in I$.

(5) This follows by (2) and (4). □

Let us call the I-bigraded ring constructed in Section 2 standard. Then the proposition above asserts that every I-bigraded ring is isomorphic to a standard one. Namely, according to Lemma 12, $\varphi : A \to \Lambda$ can be extended to an isomorphism of I-bigraded rings.

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THE DIMENSION FORMULA OF THE CYCLIC HOMOLOGY OF TRUNCATED QUIVER ALGEBRAS OVER A FIELD OF POSITIVE CHARACTERISTIC

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Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Tokyo University of Science, Japan, 10-12 October 2013. In this paper, we give the dimension formula of the cyclic homology of truncated quiver algebras over a field of positive characteristic. This is done by using a mixed complex due to Cibils.

1. Introduction

Let Δ be a finite quiver and K a field. We fix a positive integer $m \geq 2$. The truncated quiver algebra is defined by $K\Delta/R_\Delta^m$ where R_Δ^m is the two-sided ideal of $K\Delta$ generated by the all paths of length m.

In [8], Sköldberg computes the Hochschild homology of a truncated quiver algebra A over a commutative ring using an explicit description of the minimal left A^e-projective resolution $\boldsymbol{P}$ of A. He also computes the Hochschild homology of quadratic monomial algebras. On the other hand, Cibils gives a useful projective resolution $\boldsymbol{Q}$ for more general algebras in [3].

If A is a K-algebra with a decomposition $A = E \oplus r$, where E is a separable subalgebra of A and r a two-sided ideal of A, then Cibils ([4]) gives the *E-normalized mixed complex*. Sköldberg [9] gives the chain maps between the left A^e-projective resolution given in [8] and $\boldsymbol{Q}$ above for a quadratic monomial algebra A, and he obtains the module structure of the cyclic homology by computing the E^2-term of a spectral sequence determined by the above mixed complex due to Cibils.

In [1], Ames, Cagliero and Tirao give chain maps between the left A^e-projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ of a truncated quiver algebra A over commutative ring. In this paper, by means of these chain maps, we obtain the dimension formula of the cyclic homology of truncated quiver algebras over a field.

On the other hand, by means of [7, Theorem 4.1.13], Taillefer [10] gives a dimension formula for the cyclic homology of truncated quiver algebras over a field of characteristic zero. Our result generalizes the formula into the case of the field of any characteristic.

2. Preliminaries

Let Δ be a finite quiver and $m(\geq 2)$ a positive integer. For $\alpha \in \Delta_1$, its source and target are denoted by $s(\alpha)$ and $t(\alpha)$, respectively. A path in Δ is a sequence of arrows

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$\alpha_1\alpha_2\cdots\alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$. The set of all paths of length n is denoted by Δ_n.

By adjoining the element $\perp$, we will consider the following set (cf. [8], [9]):

$$\hat{\Delta} = \{\perp\} \cup \bigcup_{i=0}^{\infty} \Delta_i.$$

This set is a semigroup with the multiplication defined by

$$\delta \cdot \gamma = \begin{cases} \delta\gamma & \text{if} \quad t(\delta) = s(\gamma), \\ \perp & \text{otherwise}, \end{cases} \qquad \delta, \gamma \in \bigcup_{i=0}^{\infty} \Delta_i,$$

and

$$\perp \cdot \gamma = \gamma \cdot \perp = \perp, \quad \gamma \in \hat{\Delta}.$$

Let K be a commutative ring. Then $K\hat{\Delta}$ is a semigroup algebra and the path algebra $K\Delta$ is isomorphic to $K\hat{\Delta}/(\perp)$. So, $K\Delta$ is a $\hat{\Delta}$-graded algebra with a basis consisting of the paths in Δ. Moreover, $K\Delta$ is $\mathbb{N}$-graded, that is, $K\Delta = \bigoplus_{i=0}^{\infty} K\Delta_i$. In particular, R_Δ^m is $\hat{\Delta}$-graded and $\mathbb{N}$-graded, thus the truncated quiver algebra $A = K\Delta/R_\Delta^m$ is a $\hat{\Delta}$-graded and $\mathbb{N}$-graded algebra.

For an $\mathbb{N}$-graded vector space V, V_+ is defined by $V_+ = \bigoplus_{i\geq 1} V_i$.

Let Δ be a finite quiver. For a path γ, $|\gamma|$ denotes the length of γ. A path γ is said to be a cycle if $|\gamma| \geq 1$ and its source and target coincide. The period of a cycle γ is defined by the smallest integer i such that $\gamma = \delta^j$ $(j \geq 1)$ for a cycle δ of length i, which is denoted by $\operatorname{per}\gamma$. A cycle is said to be a basic cycle if the length of the cycle coincides with its period. It is also called a proper cycle [5]. Denote by Δ_n^{c} (respectively Δ_n^{b}) the set of cycles (respectively basic cycles) of length n. Let $G_n = \langle t_n \rangle$ be the cyclic group of order n and the path $\alpha_1 \cdots \alpha_{n-1}\alpha_n$ a cycle where α_i is an arrow in Δ. Then we define the action of G_n on Δ_n^{c} by $t_n \cdot (\alpha_1 \cdots \alpha_{n-1}\alpha_n) := \alpha_n\alpha_1 \cdots \alpha_{n-1}$, and $\Delta_n^{\mathrm{c}}/G_n$ denotes the set of all G_n-orbits on Δ_n^{c}. Similarly, G_n acts on Δ_n^{b}, and $\Delta_n^{\mathrm{b}}/G_n$ denotes the set of all G_n-orbits on Δ_n^{b}. For $\bar{\gamma} \in \Delta_n^{\mathrm{c}}/G_n$, we define the period $\operatorname{per}\bar{\gamma}$ of $\bar{\gamma}$ by $\operatorname{per}\gamma$. For convenience we use the notation $\Delta_0^{\mathrm{c}}/G_0$ for the set of vertices Δ_0. Throughout this paper, $\alpha_i (i \geq 0)$ denotes an arrow in Δ.

3. The Hochschild homology of truncated quiver algebras

In this section, we intoroduce the Hochschild homology of truncated quiver algebra in [8].

Theorem 1 ([8, Theorem 1]). *The following is a projective $\hat{\Delta}$-graded resolution of A as a left A^e-module*:

$$\boldsymbol{P} : \cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0.$$

Here the modules are defined by

$$P_i = A \otimes_{K\Delta_0} K\Gamma^{(i)} \otimes_{K\Delta_0} A,$$

where $\Gamma^{(i)}$ *is given by*

$$\Gamma^{(i)} = \begin{cases} \Delta_{cm} & \textit{if } i = 2c \ (c \geq 0), \\ \Delta_{cm+1} & \textit{if } i = 2c+1 \ (c \geq 0), \end{cases}$$

and the differentials are defined by

$$\begin{aligned} & d_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta) \\ & \quad = \sum_{j=0}^{m-1} \alpha\alpha_1 \cdots \alpha_j \otimes \alpha_{1+j} \cdots \alpha_{(c-1)m+1+j} \otimes \alpha_{(c-1)m+2+j} \cdots \alpha_{cm}\beta, \end{aligned}$$

and

$$d_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta) = \alpha\alpha_1 \otimes \alpha_2 \cdots \alpha_{cm+1} \otimes \beta - \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1}\beta.$$

The augmentation $\varepsilon\colon A \otimes_{K\Delta_0} K\Delta_0 \otimes_{K\Delta_0} A \cong A \otimes_{K\Delta_0} A \longrightarrow A$ *is defined by*

$$\varepsilon(\alpha \otimes \beta) = \alpha\beta.$$

Theorem 2 ([8, Theorem 2])**.** *Let* K *be a commutative ring and* A *a truncated quiver algebra* $K\Delta/R_\Delta^m$ *and* $q = cm + e$ *for* $0 \leq e \leq m-1$*. Then the degree* q *part of the pth Hochschild homology* $HH_p(A)$ *is given by*

$$HH_{p,q}(A) = \begin{cases} K^{a_q} & \textit{if } 1 \leq e \leq m-1 \textit{ and } 2c \leq p \leq 2c+1, \\ \displaystyle\bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \operatorname{Ker}\left(\cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \\ & \textit{if } e = 0 \textit{ and } 0 < 2c-1 = p, \\ \displaystyle\bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \operatorname{Coker}\left(\cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \\ & \textit{if } e = 0 \textit{ and } 0 < 2c = p, \\ K^{\#\Delta_0} & \textit{if } p = q = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Here we set $a_q := \#(\Delta_q^{\mathrm{c}}/G_q)$ *and* $b_r := \#(\Delta_r^{\mathrm{b}}/G_r)$*.*

4. Main result

In this section, by means of chain maps which are given by Ames, Cagliero, Tirao, we determine the dimension formula of the cyclic homology of truncated quiver algebra.

Lemma 3 ([3, Lemma 1.1])**.** *Let* Δ *be a finite quiver,* I *an admissible ideal,* $K\Delta_0$ *the subalgebra of* $A = K\Delta/I$ *generated by* Δ_0 *and* r *the Jacobson radical of* A*. The following* *is a projective resolution of A as a left A^e-module*:

$$\boldsymbol{Q} : \cdots \longrightarrow A \otimes_{K\Delta_0} r^{\otimes^i_{K\Delta_0}} \otimes_{K\Delta_0} A \xrightarrow{d_i} A \otimes_{K\Delta_0} r^{\otimes^{i-1}_{K\Delta_0}} \otimes_{K\Delta_0} A \longrightarrow \cdots$$
$$\longrightarrow A \otimes_{K\Delta_0} r \otimes_{K\Delta_0} A \xrightarrow{d_1} A \otimes_{K\Delta_0} A \xrightarrow{d_0} A \longrightarrow 0,$$

where

$$d_0(\lambda[\quad]\mu) = \lambda\mu,$$
$$d_i(\lambda[x_1|\cdots|x_i]\mu) = \lambda x_1[x_2|\cdots|x_i]\mu + \sum_{j=1}^{i-1}(-1)^i\lambda[x_1|\cdots|x_jx_{j+1}|\cdots|x_i]\mu$$
$$+ (-1)^i\lambda[x_1|\cdots|x_{i-1}]x_i\mu \quad \text{for } i \geq 1,$$

and we use the bar notation $\lambda[x_1|\cdots|x_i]\mu$ for $\lambda \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes \mu$.

Cibils constructs the following mixed complex.

Theorem 4 ([4], [9])**.** *Let Δ be a finite quiver, K a field, and $A = K\Delta/I$ for I a homogeneous ideal. Define the mixed complex $(C_{K\Delta_0}(A), b, B)$ by*

$$C_{K\Delta_0}(A)_n = A \otimes_{K\Delta_0^e} A_+^{\otimes^n_{K\Delta_0}},$$

and

$$b(x_0[x_1|\cdots|x_n]) = x_0x_1[x_2|\cdots|x_n]$$
$$+ \sum_{i=1}^{n-1}(-1)^i x_0[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$$
$$+ (-1)^n x_nx_0[x_1|\cdots|x_{n-1}],$$
$$B(x_0[x_1|\cdots|x_n]) = \sum_{i=0}^{n}(-1)^{in}[x_i|\cdots|x_n|x_0|\cdots|x_{i-1}].$$

Then $HH_n(C_{K\Delta_0}(A)) = HH_n(A)$ and $HC_n(C_{K\Delta_0}(A)) = HC_n(A)$.

In particular, if A is a truncated quiver algebra $K\Delta/R_\Delta^m (m \geq 2)$, then the map B in $(C_{K\Delta_0}(A), b, B)$ respects the Δ_*^c/G_*-grading (cf. [9]). Furthermore if we consider the double complex $\mathcal{B}C$ associate to this mixed complex and filter the total complex $\operatorname{Tot}\mathcal{B}C$ by the column filtration, then the resulting spectral sequence is Δ_*^c/G_*-graded. Thus $HC_n(A)$ is Δ_*^c/G_*-graded. Moreover, for $\bar{\gamma} \in \Delta_*^c/G_*$ the degree $\bar{\gamma}$ part of the E^1-term of this spectral sequence is $E^1_{p,q,\bar{\gamma}} = HH_{q-p,\bar{\gamma}}(A)$.

On the other hand, Ames, Cagliero and Tirao find the chain maps between the left A^e-projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ of a truncated quiver algebra A over an arbitrary field as follows:

Proposition 5 ([1]). *Define the map* $\iota : \boldsymbol{P} \longrightarrow \boldsymbol{Q}$ *as follows*:

$$\begin{aligned}
&\iota_0(\alpha \otimes \beta) = \alpha[\quad]\beta,\ \iota_1(\alpha \otimes \alpha_1 \otimes \beta) = \alpha[\alpha_1]\beta,\\
&\iota_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta)\\
&\quad = \sum_{0 \le j_1, \dots, j_c \le m-2} \alpha[\alpha_1 \cdots \alpha_{1+j_1}|\alpha_{2+j_1}|\alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2}|\alpha_{4+j_1+j_2}| \cdots\\
&\qquad\qquad |\alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c}|\alpha_{2c+j_1+\cdots+j_c}]\alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{cm}\beta,\\
&\iota_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta)\\
&\quad = \sum_{0 \le j_1, \dots, j_c \le m-2} \alpha[\alpha_1|\alpha_2 \cdots \alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1} \cdots \alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}| \cdots\\
&\qquad\qquad |\alpha_{2c+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c} \cdots \alpha_{cm+1}\beta.
\end{aligned}$$

Then, ι *is a chain map.*

Proposition 6 ([1]). *Let* m_i *be a positive integer for any* $i \ge 1$. *Suppose that* x_i *is the path* $\alpha_{m_1+\cdots+m_{i-1}+1} \cdots \alpha_{m_1+\cdots+m_i}$ *of length* m_i. *Define the map* $\pi : \boldsymbol{Q} \longrightarrow \boldsymbol{P}$ *as follows*:

$$\begin{aligned}
&\pi_0(\alpha[\quad]\beta) = \alpha \otimes \beta,\\
&\pi_1(\alpha[x_1]\beta) = \sum_{j=1}^{m_1} \alpha\alpha_1 \cdots \alpha_{j-1} \otimes \alpha_j \otimes \alpha_{j+1} \cdots \alpha_{m_1}\beta,\\
&\pi_{2c}(\alpha[x_1|x_2|\cdots|x_{2c}]\beta) = \begin{cases} \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1} \cdots \alpha_{m_1+\cdots+m_{2c}}\beta \\ \qquad\qquad\qquad\qquad \textit{if} \quad m_{2i-1} + m_{2i} \ge m \ (1 \le i \le c), \\ 0 \qquad \textit{otherwise}, \end{cases}\\
&\pi_{2c+1}(\alpha[x_1|x_2|\cdots|x_{2c+1}]\beta) = \begin{cases} \sum_{j=1}^{m_1} \alpha\alpha_1 \cdots \alpha_{j-1} \otimes \alpha_j \cdots \alpha_{j+cm} \otimes \\ \qquad\qquad \alpha_{j+cm+1} \cdots \alpha_{m_1+\cdots+m_{2c+1}}\beta \\ \qquad\qquad \textit{if} \quad m_{2i} + m_{2i+1} \ge m \ (1 \le i \le c), \\ 0 \qquad \textit{otherwise}. \end{cases}
\end{aligned}$$

Then, π *is a chain map and* $\pi\iota = \mathrm{id}_{\boldsymbol{P}}$.

By investigating the basis of the Hochschild homology and finding the chain maps between the projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$, we are able to compute $B : HH_{p,\bar{\gamma}}(A) \longrightarrow HH_{p+1,\bar{\gamma}}(A)$ induced by the differential of the Cibils' mixed complex. Moreover, for $\bar{\gamma} \in \Delta_t^{\mathrm{c}}/G_t$ we are able to determine the degree $\bar{\gamma}$ part of the E^2-term of the spectral sequence associated with the Cibils' mixed complex. Therefore we have the following result.

Theorem 7 ([6, Theorem 5.1]). *Suppose that* $m \ge 2$ *and* $A = K\Delta/R_\Delta^m$. *Then the dimension formula of the cyclic homology of* A *is given by, for* $c \ge 0$,

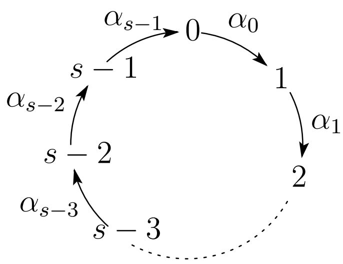
$$\dim_K HC_{2c}(A) = \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e} + \sum_{c'=0}^{c-1} \sum_{e=1}^{m-1} \sum_{\substack{r > 0 \\ \text{s.t. } r\zeta | c'm+e}} b_r$$

$$+\sum_{c'=1}^{c}\sum_{\substack{r>0\\ \text{s.t. } r|c'm,\\ \gcd(m,r)\zeta|m}} b_r+\sum_{c'=1}^{c}\sum_{\substack{r>0\\ \text{s.t.} r\zeta|\gcd(m,r)c'}}(\gcd(m,r)-1)b_r,$$

$$\dim_K HC_{2c+1}(A)=\sum_{\substack{r>0\\ \text{s.t. } r|(c+1)m}}(\gcd(m,r)-1)b_r+\sum_{c'=0}^{c}\sum_{e=1}^{m-1}\sum_{\substack{r>0\\ \text{s.t. } r\zeta|c'm+e}} b_r$$
$$+\sum_{c'=1}^{c+1}\sum_{\substack{r>0\\ \text{s.t. } r|c'm,\\ \gcd(m,r)\zeta|m}} b_r+\sum_{c'=1}^{c}\sum_{\substack{r>0\\ \text{s.t. } r\zeta|\gcd(m,r)c'}}(\gcd(m,r)-1)b_r.$$

Remark 8. If $\zeta=0$, then the above result coincides with the result of Taillefer in [10].

Example 9 ([6, Example 5.3])**.** Let K be a field of characteristic ζ and Δ the following quiver:



Suppose $m\geq 2$ and $A=K\Delta/R_\Delta^m$, which is called a truncated cycle algebra in [2].
Since

$$a_r=\begin{cases}1 & \text{if} \quad s|r,\\ 0 & \text{otherwise,}\end{cases}\qquad b_r=\begin{cases}1 & \text{if} \quad s=r,\\ 0 & \text{otherwise,}\end{cases}$$

we have, for $c\geq 0$,

$$\dim_K HC_{2c}(A)=s+\left[\frac{(c+1)m-1}{s}\right]-\left[\frac{cm}{s}\right]+\sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right]-\left[\frac{c'm}{s\zeta}\right]\right)$$
$$+\left(\left[\frac{m}{\gcd(m,s)\zeta}\right]-\left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c}\left(\left[\frac{c'm}{s}\right]-\left[\frac{c'm-1}{s}\right]\right)$$
$$+(\gcd(m,s)-1)\left[\frac{\gcd(m,s)c}{s\zeta}\right],$$

and

$$\dim_K HC_{2c+1}(A) = (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right]-\left[\frac{(c+1)m-1}{s}\right]+\left[\frac{\gcd(m,s)c}{s\zeta}\right]\right)$$
$$+\left(\left[\frac{m}{\gcd(m,s)\zeta}\right]-\left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c+1}\left(\left[\frac{c'm}{s}\right]-\left[\frac{c'm-1}{s}\right]\right)$$
$$+\sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right]-\left[\frac{c'm}{s\zeta}\right]\right).$$

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GEIGLE-LENZING PROJECTIVE SPACES AND d-CANONICAL ALGEBRAS

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Abstract. Following [4], we introduced Geigle-Lenzing projective spaces and d-canonical algebras.

重み付き射影直線は, 1987 年に Geigle-Lenzing [3] によって導入されたものであり, 射影直線 $\mathbb{P}^1$ を recollement により拡大したものであり, また $\mathbb{P}^1$ 上の特別な整環から与えられる [10, Appendix A]. 重み付き射影直線は, Ringel [11] が 1984 年に導入した標準多元環と導来圏同値であり, 今日では箙の道多元環とともに, 多元環の表現論における基本的対象となっている. 本稿では, 重み付き射影直線と標準多元環の高次元化である *Geigle-Lenzing* **射影空間**と d-**標準多元環**を導入して基本的な性質を調べ, 応用として高次元 Auslander-Reiten 理論において基本的な d-**無限表現型多元環**が得られることを観察する. 詳細は論文 [4][7] を参照されたい.

Remark 1. Geigle-Lenzing の導入した重み付き射影直線や, その一般化である Geigle-Lenzing 射影空間は, 後述する特別な完全交叉環 R から構成される. 一方, 非標準的に次数付けされた多項式環 S に付随する Proj$S$ を重み付き射影空間とよぶことがあるが, 両者は一般には異なるものである.

以下, 基礎体を k とする. 射影空間 $\mathbb{P}^d$ の斉次座標環を $k[T_0,\dots,T_d]$ とし, 各 $1 \le i \le n$ に対して, $\mathbb{P}^d$ 内の超平面 $L_1,\dots,L_n$ が一次式

$$\ell_i = \sum_{j=0}^{d} \lambda_{ij} T_j \in k[T_0,\dots,T_d].$$

で定義されるとする. 正整数の組 $(p_1,\dots,p_n)$ に対し, 可換 k-代数 R を

$$R := k[T_0,\dots,T_d,X_1,\dots,X_n]/(X_i^{p_i} - \ell_i \mid 1 \le i \le n)$$

と定める. 次に, $\vec{x}_i$ $(1 \le i \le n)$ と $\vec{c}$ で生成される自由アーベル群 $\langle \vec{x}_1,\dots,\vec{x}_n,\vec{c}\rangle$ の剰余群

$$\mathbb{L} := \langle \vec{x}_1,\dots,\vec{x}_n,\vec{c}\rangle / \langle p_i\vec{x}_i - \vec{c} \mid 1 \le i \le n\rangle$$

を考える. $\deg X_i := \vec{x}_i$, $\deg T_j := \vec{c}$ とおくことにより, R は $\mathbb{L}$-次数付き k-代数となる. R の基本的な性質を挙げる.

- R は Krull 次元 $d+1$ の完全交叉環である.
- R の a-**不変量**（*Gorenstein* **パラメータ**）は

$$\vec{\omega} := (n-d-1)\vec{c} - \sum_{i=1}^{n} \vec{x}_i \in \mathbb{L}$$

The detailed version of this paper will be submitted for publication elsewhere.

で与えられる．つまり $\mathbb{L}$-次数付き R-加群としての同型 $\mathrm{Ext}_R^{d+1}(k, R(\vec{\omega})) \simeq R$ が存在する．

以下, 本文を通して $L_1, \ldots, L_n$ が一般の位置にあると仮定する．$\mathbb{L}$-次数付き有限生成 R-加群の圏を $\mathsf{mod}^{\mathbb{L}} R$ で表し, 有限次元加群からなる充満部分圏を $\mathsf{mod}_0^{\mathbb{L}} R$ で表わす．$\mathsf{mod}_0^{\mathbb{L}} R$ は $\mathsf{mod}^{\mathbb{L}} R$ の Serre 部分圏となっており, 商圏

$$\mathsf{coh}\,\mathbb{X} := \mathsf{mod}^{\mathbb{L}} R / \mathsf{mod}_0^{\mathbb{L}} R$$

はアーベル圏となる．これを *Geigle-Lenzing* **射影空間** $\mathbb{X}$ 上の**連接層**の圏と呼ぶ．$d = 1$ の場合が, Geigle-Lenzing の導入した重み付き射影直線に他ならない．

$\mathsf{coh}\,\mathbb{X}$ の基本的な性質を挙げる．

- $\mathsf{coh}\,\mathbb{X}$ は大域次元 d のアーベル圏である．
- （*Serre* **双対性**）関手的同型 $\mathrm{Ext}_{\mathbb{X}}^d(X, Y) \simeq D\,\mathrm{Hom}_{\mathbb{X}}(Y, X(\vec{\omega}))$ $(X, Y \in \mathsf{coh}\,\mathbb{X})$ が存在する．

Geigle-Lenzing 射影空間の持つ重要な性質として, 傾対象の存在が挙げられる．まず傾対象の定義を復習する．

Definition 2. 三角圏 $\mathcal{T}$ の対象 $M \in \mathcal{T}$ が**傾対象**であるとは, 任意の整数 $i \neq 0$ に対して $\mathrm{Hom}_{\mathcal{T}}(M, M[i]) = 0$ が成立し, さらに M を含む $\mathcal{T}$ の最小の thick 部分圏（=直和因子で閉じた三角部分圏）が $\mathcal{T}$ となることである．

例えば, 環 B に対し有限生成射影 B-加群の有界ホモトピー圏 $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B)$ は傾対象を持つ．逆に, 三角圏 $\mathcal{T}$ が傾対象 T を持つとき, 若干の仮定の下（代数的かつ idempotent complete）で, $\mathcal{T}$ は $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathrm{End}_{\mathcal{T}}(T))$ と三角同値になる．

$\vec{x}_i$ $(1 \leq i \leq n)$ と $\vec{c}$ で生成される $\mathbb{L}$ の部分モノイドを, $\mathbb{L}_+$ で表わす．$\vec{x} - \vec{y} \in \mathbb{L}_+$ であるときに $\vec{x} \geq \vec{y}$ と表わすことにより, $\mathbb{L}$ は半順序集合となる．今,

$$[0, d\vec{c}] := \{\vec{x} \in \mathbb{L} \mid 0 \leq \vec{x} \leq d\vec{c}\}$$

とおく．

Theorem 3. (a) $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ は傾対象 $T := \bigoplus_{\vec{x} \in [0, d\vec{c}]} R(\vec{x})$ を持つ．

(b) T の自己準同型環 $A := \mathrm{End}_{\mathbb{X}}(T)$ に対し, 三角圏同値 $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X}) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$ が存在する．

A を d-**標準多元環**と呼ぶ．$n = 0$ の場合は Beilinson [2] による古典的な結果であり, この場合の A を *Beilinson* **多元環**と呼ぶ．$d = 1$ の場合が Geigle-Lenzing [3] によるものであり, この場合の A が Ringel [11] によって導入された標準多元環である．$n \leq d + 1$ の場合は Baer [1] によって知られており, 最近 $n = d + 2$ の場合が Ishii-Ueda [6] によって与えられた．

d-標準多元環の箙 (quiver) 表示を与えるために, 一般性を失うことなく以下を仮定する:

- $n \geq d + 1$.
- 各 $1 \leq i \leq d + 1$ に対して $\ell_i = T_{i-1}$.

このとき, R は以下で表示される:

$$R = k[X_1, \ldots, X_n] / (X_i^{p_i} - \ell_i(X_1^{p_1}, \ldots, X_{d+1}^{p_{d+1}}) \mid d + 2 \leq i \leq n)$$

Theorem 4. d-標準多元環 A は以下の箙と関係式で表示される:

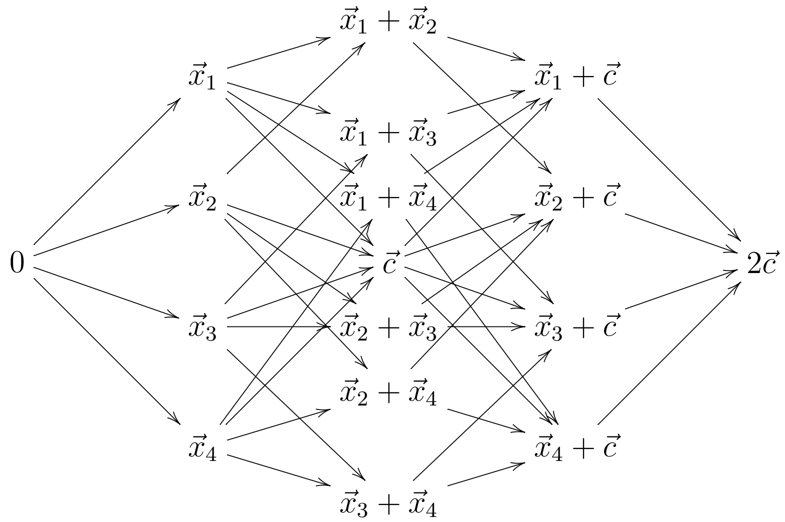
(i) 点は $Q_0 := [0, d\vec{c}]$.

(ii) 矢は$Q_1 := \{x_i : \vec{x} \to \vec{x} + \vec{x}_i \mid 1 \le i \le n, \quad \vec{x}, \vec{x} + \vec{x}_i \in [0, d\vec{c}]\}$.

(iii) 関係式は以下の *2* 種類*:*

- $x_i x_j - x_j x_i : \vec{x} \to \vec{x} + \vec{x}_i + \vec{x}_j$ *(*$1 \le i < j \le n$*,* $\vec{x}, \vec{x} + \vec{x}_i + \vec{x}_j \in [0, d\vec{c}]$*).*
- $x_i^{p_i} - \sum_{j=1}^{d+1} \lambda_{i,j-1} x_j^{p_j} : \vec{x} \to \vec{x} + \vec{c}$ *(*$d + 2 \le i \le n$*,* $\vec{x}, \vec{x} + \vec{c} \in [0, d\vec{c}]$*).*

例えば $d = 2$, $n = 4$, $p_1 = p_2 = p_3 = p_4 = 2$ の場合に, 箙 Q は以下で与えられる.



いま B を有限次元多元環とし, 大域次元が有限であると仮定する. 中山関手

$$\nu := - \otimes_B (DB) : \mathsf{D}^{\mathrm{b}}(\mathsf{mod} B) \to \mathsf{D}^{\mathrm{b}}(\mathsf{mod} B)$$

に対して $\nu_d := \nu \circ [-d]$ とおく. 次の概念は高次元 Auslander-Reiten 理論で基本的である.

Definition 5. [5] B が d-**無限表現型**であるとは, B の大域次元が d であり, かつ $\nu_d^{-i}(B) \in \mathsf{mod} B$ が任意の $i \ge 0$ に対して成立することである.

k が代数的閉体の場合, 1-無限表現型多元環とは, 非 Dynkin 型箙の道多元環に他ならない. d-無限表現型多元環は, d-**有限表現型**と呼ばれる多元環とともに, 大域次元 d の多元環の中で表現論的観点からもっとも基本的なクラスと考えられる.

d-標準多元環の次の性質は基本的である.

Theorem 6. 一般性を失うことなく $p_i \ge 2$ *(*$1 \le i \le n$*)* であると仮定する. このとき

$$\mathrm{gl.dim}\, A = \begin{cases} d & (n \le d + 1), \\ 2d & (n > d + 1). \end{cases}$$

さらに $n \le d + 1$ ならば*,* A は d-無限表現型である*.*

$n \le d+1$ の場合は, より詳しく [5] で $\widetilde{A}$ **型**と呼ばれる d-無限表現型多元環となっている.

いま R の a-不変量 $\vec{\omega}$ に対して,

$$\deg \vec{\omega} := n - d - 1 - \sum_{i=1}^{n} \frac{1}{p_i}$$

とおき, その符号によって Geigle-Lenzing 射影空間を以下の 3 通りに分類する.

$\deg \vec{\omega}$	< 0	$= 0$	> 0
$\mathbb{X}$	d-Fano	d-Calabi-Yau	d-anti-Fano

$d = 1$ の場合, これらは重み付き射影直線の domestic, tubular, wild の 3 つのタイプに相当する.

Geigle-Lenzing 射影空間の d-Fano 性は, d-標準多元環の Minamoto [8][9] の意味での d-Fano 性と同値である:

Proposition 7. *Geigle-Lenzing* 射影空間 $\mathbb{X}$ が *d-Fano* であることと, d-標準多元環 A が *d-Fano* 多元環であることは同値.

Geigle-Lenzing 射影空間のうち, d-Fano であるものはより基本的であると考えられる. 実際 $d = 1$ の場合, domestic 型の重み付き射影直線は, 拡大ディンキン型箙の道多元環と導来圏同値である. このことから Theorem 6 後半の一般化として, 次が自然に期待される.

Conjecture $\mathbb{X}$ が d-Fano ならば, ある d-無限表現型多元環と導来圏同値である.

この予想に対する部分的な回答として, 以下が成立する.

Theorem 8. $n = d + 2$ かつ $p_1 = p_2 = 2$ ならば, $\mathbb{X}$ はある d-無限表現型多元環と導来圏同値である.

証明は, R の Cohen-Macaulay 表現を調べることからはじまる. 詳細は割愛する.

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SPECIALIZATION ORDERS ON ATOM SPECTRUM OF GROTHENDIECK CATEGORIES

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Abstract. We introduce systematic methods to construct Grothendieck categories from colored quivers and develop a theory of the specialization orders on the atom spectra of Grothendieck categories. We showed that any partially ordered set is realized as the atom spectrum of some Grothendieck category, which is an analog of Hochster's result in commutative ring theory. In this paper, we explain techniques in the proof by using examples.

Key Words: Atom spectrum, Grothendieck category, Partially ordered set, Colored quiver.

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1. Introduction

The aim of this paper is to provide systematic methods to construct Grothendieck categories with certain structures and to establish a theory of the specialization orders on the spectra of Grothendieck categories. There are important Grothendieck categories appearing in representation theory of rings and algebraic geometry: the category $\operatorname{Mod}\Lambda$ of (right) modules over a ring Λ, the category $\operatorname{QCoh} X$ of quasi-coherent sheaves on a scheme X ([2, Lem 2.1.7]), and the category of quasi-coherent sheaves on a noncommutative projective scheme introduced by Verevkin [10] and Artin and Zhang [1]. Furthermore, by using the Gabriel-Popescu embedding ([7, Proposition]), it is shown that any Grothendieck category can be obtained as the quotient category of the category of modules over some ring by some localizing subcategory. In this sense, the notion of Grothendieck category is ubiquitous.

In commutative ring theory, Hochster characterized the topological spaces appearing as the prime spectra of commutative rings with Zariski topologies ([3, Theorem 6 and Proposition 10]). Speed [8] pointed out that Hochster's result gives the following characterization of the partially ordered sets appearing as the prime spectra of commutative rings.

Theorem 1 (Hochster [3, Proposition 10] and Speed [8, Corollary 1])**.** *Let P be a partially ordered set. Then P is isomorphic to the prime spectrum of some commutative ring with the inclusion relation if and only if P is an inverse limit of finite partially ordered sets in the category of partially ordered sets.*

The detailed version of this paper will be submitted for publication elsewhere.

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We showed a theorem of the same type for Grothendieck categories. In [4] and [5], we investigated Grothendieck categories by using the *atom spectrum* $\operatorname{ASpec}\mathcal{A}$ of a Grothendieck category $\mathcal{A}$. It is the set of equivalence classes of monoform objects, which generalizes the prime spectrum of a commutative ring.

In fact, our main result claims that any partially ordered set is realized as the atom spectrum of some Grothendieck categories.

Theorem 2. *Any partially ordered set is isomorphic to the atom spectrum of some Grothendieck category.*

In this paper, we explain key ideas to show this theorem by using examples. For more details, we refer the reader to [6].

2. Atom spectrum

In this section, we recall the definition of atom spectrum and fundamental properties. Throughout this paper, let $\mathcal{A}$ be a Grothendieck category.

Definition 3. A nonzero object H in $\mathcal{A}$ is called *monoform* if for any nonzero subobject L of H, there does not exist a nonzero subobject of H which is isomorphic to a subobject of H/L.

Monoform objects have the following properties.

Proposition 4. *Let H be a monoform object in $\mathcal{A}$. Then the following assertions hold.*

(1) *Any nonzero subobject of H is also monoform.*
(2) *H is uniform, that is, for any nonzero subobjects L_1 and L_2 of H, we have $L_1 \cap L_2 \neq 0$.*

Definition 5. For monoform objects H and H' in $\mathcal{A}$, we say that H is *atom-equivalent* to H' if there exists a nonzero subobject of H which is isomorphic to a subobject of H'.

Remark 6. The atom equivalence is an equivalence relation between monoform objects in $\mathcal{A}$ since any monoform object is uniform.

Now we define the notion of atoms, which was originally introduced by Storrer [9] in the case of module categories.

Definition 7. Denote by $\operatorname{ASpec}\mathcal{A}$ the quotient set of the set of monoform objects in $\mathcal{A}$ by the atom equivalence. We call it the *atom spectrum* of $\mathcal{A}$. Elements of $\operatorname{ASpec}\mathcal{A}$ are called *atoms* in $\mathcal{A}$. The equivalence class of a monoform object H in $\mathcal{A}$ is denoted by $\overline{H}$.

The following proposition shows that the atom spectrum of a Grothendieck category is a generalization of the prime spectrum of a commutative ring.

Proposition 8. *Let R be a commutative ring. Then the map* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *given by* $\mathfrak{p} \mapsto \overline{(R/\mathfrak{p})}$ *is a bijection.*

The notions of associated primes and support are also generalized as follows.

Definition 9. Let M be an object in $\mathcal{A}$.

(1) Define the *atom support* of M by

$$\operatorname{ASupp} M = \{\overline{H} \in \operatorname{ASpec} \mathcal{A} \mid H \text{ is a subquotient of } M\}.$$

(2) Define the set of *associated atoms* of M by

$$\operatorname{AAss} M = \{\overline{H} \in \operatorname{ASpec} \mathcal{A} \mid H \text{ is a subobject of } M\}.$$

The following proposition is a generalization of a proposition which is well known in the commutative ring theory.

Proposition 10. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in* $\mathcal{A}$*. Then the following assertions hold.*

(1) $\operatorname{ASupp} M = \operatorname{ASupp} L \cup \operatorname{ASupp} N$.
(2) $\operatorname{AAss} L \subset \operatorname{AAss} M \subset \operatorname{AAss} L \cup \operatorname{AAss} N$.

A partial order on the atom spectrum is defined by using atom support.

Definition 11. Let α and β be atoms in $\mathcal{A}$. We write $\alpha \leq \beta$ if for any object M in $\mathcal{A}$ satisfying $\alpha \in \operatorname{ASupp} M$ also satisfies $\beta \in \operatorname{ASupp} M$.

Proposition 12. *The relation* $\leq$ *on* $\operatorname{ASpec} \mathcal{A}$ *is a partial order.*

In the case where $\mathcal{A}$ is the category of modules over a commutative ring R, the notion of associated atoms, atom support, and the partial order on the atom spectrum coincide with associated primes, support, and the inclusion relation between prime ideals, respectively, through the bijection in Proposition 8.

3. Construction of Grothendieck categories

In order to construct Grothendieck categories, we use colored quivers.

Definition 13. (1) A *colored quiver* is a sextuple $\Gamma = (Q_0, Q_1, C, s, t, u)$, where Q_0, Q_1, and C are sets, and $s\colon Q_1 \to Q_0$, $t\colon Q_1 \to Q_0$, and $u\colon Q_1 \to C$ are maps. We regard the colored quiver Γ as the quiver (Q_0, Q_1, s, t) with the *color* $u(r)$ on each arrow $r \in Q_1$.

(2) We say that a colored quiver $\Gamma = (Q_0, Q_1, C, s, t, u)$ satisfies the *finite arrow condition* if for each $v \in Q_0$ and $c \in C$, the number of arrows r satisfying $s(r) = v$ and $u(r) = c$ is finite.

From now on, we fix a field K. From a colored quiver satisfying the finite arrow condition, we construct a Grothendieck category as follows.

Definition 14. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be a colored quiver satisfying the finite arrow condition. Denote a free K-algebra on C by $S_C = K\langle s_c \mid c \in C\rangle$. Define a K-vector space M_Γ by $M_\Gamma = \bigoplus_{v \in Q_0} F_v$, where $F_v = x_v K$ is a one-dimensional K-vector space generated by an element x_v. Regard M_Γ as a right S_C-module by defining the action of $s_c \in S_C$ as follows: for each vertex v in Q,

$$x_v \cdot s_c = \sum_r x_{t(r)},$$

where r runs over all the arrows $r \in Q_1$ with $s(r) = v$ and $u(r) = c$. The number of such arrows r is finite since Γ satisfies the finite arrow condition. Denote by $\mathcal{A}_\Gamma$ the smallest full subcategory of $\operatorname{Mod} S_C$ which contains M_Γ and is closed under submodules, quotient modules, and direct sums.

The category $\mathcal{A}_\Gamma$ defined above is a Grothendieck category. The following proposition is useful to describe the atom spectrum of $\mathcal{A}_\Gamma$.

Proposition 15. *Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be a colored quiver satisfying the finite arrow condition. Then $\operatorname{ASpec} \mathcal{A}_\Gamma$ is isomorphic to the subset $\operatorname{ASupp} M_\Gamma$ of $\operatorname{ASpec}(\operatorname{Mod} S_C)$ as a partially ordered set.*

Example 16. Define a colored quiver $\Gamma = (Q_0, Q_1, C, s, t, u)$ by $Q_0 = \{v, w\}$, $Q_1 = \{r\}$, $C = \{c\}$, $s(r) = v$, $t(r) = w$, and $u(r) = c$. This is illustrated as

$$\begin{tikzcd} v \arrow[d, "c"'] \\ w \end{tikzcd} .$$

Then we have $S_C = K\langle s_c \rangle = K[s_c]$, $M_\Gamma = x_v K \oplus x_w K$ as a K-vector space, and $x_v s_c = x_w$, $x_w s_c = 0$. The subspace $L = x_w K$ of M_Γ is a simple S_C-submodule, and L is isomorphic to M_Γ / L as an S_C-module. Hence we have

$$\operatorname{ASpec} \mathcal{A}_\Gamma = \operatorname{ASupp} M_\Gamma = \operatorname{ASupp} L \cup \operatorname{ASupp} \frac{M_\Gamma}{L} = \{\overline{L}\}.$$

The next example explains the way to distinguish simple modules corresponding different vertices.

Example 17. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be the colored quiver

$$\begin{tikzcd} v \arrow[loop right, "c_v"] \arrow[d, "c"'] \\ w \arrow[loop right, "c_w"] \end{tikzcd}$$

and let $N = x_v K$ and $L = x_w K$. Then we have an exact sequence

$$0 \to L \to M_\Gamma \to N \to 0$$

of K-vector spaces and this can be regarded as an exact sequence in $\operatorname{Mod} S_C$. Hence we have

$$\operatorname{ASpec} \mathcal{A}_\Gamma = \operatorname{ASupp} M_\Gamma = \operatorname{ASupp} L \cup \operatorname{ASupp} N = \{\overline{L}, \overline{N}\},$$

where $\overline{L} \neq \overline{N}$.

In order to realize a partially ordered set with nontrivial partial order, we use an infinite colored quiver.

Example 18. Let Γ be the colored quiver

$$v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \cdots .$$

Let L be the simple S_C-module defined by $L = K$ as a K-vector space and $L s_{c_i} = 0$ for each $i \in \mathbb{Z}_{\geq 0}$. Then we have $\operatorname{ASpec} \mathcal{A}_\Gamma = \{\overline{M_\Gamma}, \overline{L}\}$, where $\overline{M_\Gamma} < \overline{L}$.

We refer the reader to [6] for further techniques to show Theorem 2.

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ON THE POSET OF PRE-PROJECTIVE TILTING MODULES OVER PATH ALGEBRAS

RYOICHI KASE

Abstract. We study posets of pre-projective tilting modules over path algebras. We give a criterion for Ext-vanishing and an equivalent condition for a poset of pre-projective tilting modules $\mathcal{T}_p(Q)$ to be a distributive lattice. Moreover we realize $\mathcal{T}_p(Q)$ as an ideal-poset.

Introduction

Tilting theory first appeared in an article by Brenner and Butler [2]. In that article the notion of a tilting module for finite dimensional algebras was introduced. Tilting theory now appears in many areas of mathematics, for example algebraic geometry, theory of algebraic groups and algebraic topology. Let T be a tilting module for a finite dimensional algebra A and let $B = \mathrm{End}_A(T)$. Then Happel showed that the two bounded derived categories $\mathcal{D}^{\mathrm{b}}(A)$ and $\mathcal{D}^{\mathrm{b}}(B)$ are equivalent as triangulated category [4]. Therefore, classifying tilting modules is an important problem.

Theory of tilting-mutation introduced by Riedtmann and Schofield is an approach to this problem. They introduced a tilting quiver whose vertices are (isomorphism classes of) basic tilting modules and arrows correspond to mutations [9]. Happel and Unger defined a partial order on the set of basic tilting modules and showed that the tilting quiver coincides with the Hasse quiver of this poset [5]. This poset is now studied by many authors.

Notations. Let Q be a finite connected quiver without loops or oriented cycles. We denote by Q_0 (resp. Q_1) the set of vertices (resp. arrows) of Q. For any arrow $\alpha \in Q_1$ we denote by $s(\alpha)$ its starting point and denote by $t(\alpha)$ its target point (i.e. α is an arrow from $s(\alpha)$ to $t(\alpha)$). We call a vertex $x \in Q_0$ a source (resp. sink) if there is an arrow starting at x (resp. ending at x) and there is no arrow ending at x (resp. starting at x). Let kQ be the path algebra of Q over an algebraically closed field k. Denote by mod-kQ the category of finite dimensional right kQ-modules and by ind-kQ the full subcategory of indecomposable modules. For any module $M \in$ mod-kQ we denote by $|M|$ the number of pairwise non isomorphic indecomposable direct summands of M. Let $P(i)$ be the indecomposable projective module in mod-kQ associated with vertex $i \in Q_0$.

Aim. If Q is a non-Dynkin quiver, kQ is a representation-infinite algebra. In this case, to determine rigid modules is nearly impossible. However the pre-projective component of the Auslander-Reiten quiver of mod-kQ is completely determined. For example, there

The detailed version of this paper will be submitted for publication elsewhere.

is a bijection between the set of (isomorphism classes of) indecomposable pre-projective modules over kQ and $\mathbb{Z}_{\geq 0} \times Q_0$.

In this paper, we consider the set $\mathcal{T}_{\mathrm{p}}(Q)$ of basic pre-projective tilting modules and study its combinatorial structure in the case when Q is a non-Dynkin quiver. For the purpose we have to answer to the following problem:

• When does the Ext^1_{kQ}-group between two indecomposable pre-projective modules vanish?

We introduce a function $l_Q : Q_0 \times Q_0 \to \mathbb{Z}_{\geq 0}$ and, by using this function, we give an answer to this question for any quiver satisfying the following condition (C):

$$\text{(C)} \qquad \delta(a) := \#\{\alpha \in Q_1 \mid s(\alpha) = a \text{ or } t(\alpha) = a\} \geq 2, \quad \forall a \in Q_0.$$

By applying this result we have the following.

Theorem 1. *If Q satisfies the condition* (C)*, then for any $T \in \mathcal{T}_{\mathrm{p}}$ there exists $(r_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$ such that $T \simeq \bigoplus_{i \in Q_0} \tau_Q^{-r_i} P(i)$.*

Moreover, the map $\bigoplus_{i \in Q_0} \tau^{-r_i} P(i) \mapsto (r_i)_{i \in Q_0}$ induces a poset inclusion,

$$(\mathcal{T}_{\mathrm{p}}(Q), \leq) \to (\mathbb{Z}^{Q_0}, \leq^{\mathrm{op}}),$$

where $(r_i) \leq^{\mathrm{op}} (s_i) \overset{\mathrm{def}}{\Leftrightarrow} r_i \geq s_i$ for any $i \in Q_0$.

The above result says that if Q satisfies the condition (C), then study of the poset $\mathcal{T}_{\mathrm{p}}(Q)$ comes down to combinatorics on Z^{Q_0}.

As an application, we see a connection between the posets of tilting modules and distributive lattices. In particular we realize a poset of tilting modules as an ideal-poset.

1. Preliminary

1.1. **Tilting modules.** In this sub-section we will recall the definition of tilting modules and basic results for combinatorics of the set of tilting modules.

Definition 2. A module $T \in \text{mod-}kQ$ is tilting module if,
(1) $\mathrm{Ext}^1_{kQ}(T, T) = 0$,
(2) $|T| = \#Q_0$.

We denote by $\mathcal{T}(Q)$ the set of (isomorphism classes of) basic tilting modules in mod-kQ.

Proposition 3. [6, Lemma 2.1] *Let $T, T' \in \mathcal{T}(Q)$. Then the following relation $\leq$ define a partial order on $\mathcal{T}(Q)$,*

$$T \geq T' \overset{\mathrm{def}}{\Leftrightarrow} \mathrm{Ext}^1_{kQ}(T, T') = 0.$$

1.2. **Lattices and distributive lattices.** In this subsection we will recall definition of a lattice and a distributive lattice.

Definition 4. A poset $(L, \leq)$ is a lattice if for any $x, y \in L$ there is the minimum element of $\{z \in L \mid z \geq x, y\}$ and there is the maximum element of $\{z \in L \mid z \leq x, y\}$.

In this case we denote by $x \vee y$ the minimum element of $\{z \in L \mid z \geq x, y\}$ and call it join of x and y. We also denote by $x \wedge y$ the maximum element of $\{z \in L \mid z \leq x, y\}$ and call it meet of x and y.

Definition 5. A lattice L is a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ holds for any $x, y, z \in L$.

We note that any finite distributive lattice is realized as an ideal-poset of some finite poset. This fact is known as Birkhoff's representation theorem.

Theorem 6. (*Birkhoff's representation theorem,* cf. [1],[3]) *Let L be a finite distributive lattice and $J \subset L$ be the poset of join-irreducible elements of L. Then L is isomorphic to $\mathcal{I}(J)$.*

2. Pre-projective tilting modules

2.1. **Criterion for Ext-vanishing.** For any vertex $x \in Q_0$, we set

$$\delta(x) := \#\{\alpha \in Q_1 \mid s(\alpha = x) \text{or } t(\alpha = x)\}.$$

Now we consider the following condition:

$$\text{(C)} \qquad \delta(a) := \#\{\alpha \in Q_1 \mid s(\alpha) = a \text{ or } t(\alpha) = a\} \geq 2, \quad \forall a \in Q_0.$$

For a walk $w : x = x_0 \overset{\alpha_1}{-} x_1 \overset{\alpha_2}{\rightarrow} \cdots \overset{\alpha_r}{\rightarrow} x_r = y$ from x to y on Q, we set

$$c^+(w) := \#\{t \mid \text{there is an arrow from } x_{t-1} \text{ to } x_t\}.$$

Then we define

$$l_Q(i,j) := \begin{cases} \min\{c^+(w) \mid w : \text{walk from } i \text{ to } j \text{ on } Q\} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Then we have a criterion for Ext-vanishing.

Proposition 7. [7] *If Q satisfies the condition* (C)*, then*

$$\mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), \tau^{-s}P(j)) = 0 \Leftrightarrow r \leq s + l_Q(j,i)$$

Therefore $\mathcal{T}_{\mathrm{p}}(Q)$ may be embedded in $\mathbb{Z}$-lattice $\mathbb{Z}^Q_{\geq 0}$ as follows.

Proposition 8. [7] *If Q satisfies the condition* (C)*, then for any $T \in \mathcal{T}_{\mathrm{p}}$ there exists $(r_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}_{\geq 0}$ such that $T \simeq \bigoplus_{i \in Q_0} \tau_Q^{-r_i} P(i)$.*

Moreover, the map $\bigoplus_{i \in Q_0} \tau^{-r_i} P(i) \mapsto (r_i)_{i \in Q_0}$ induces a poset inclusion,

$$(\mathcal{T}_{\mathrm{p}}(Q), \leq) \to (\mathbb{Z}^{Q_0}, \leq^{\mathrm{op}}),$$

where $(r_i) \leq^{\mathrm{op}} (s_i) \overset{\mathrm{def}}{\Leftrightarrow} r_i \geq s_i$ for any $i \in Q_0$.

2.2. **Lattice theoretical aspects.** In this section we see a connection between posets of pre-projective tilting modules and distributive lattices.

Theorem 9. [8] *$\mathcal{T}_{\mathrm{p}}(Q)$ is an infinite distributive lattice if and only if Q satisfies the condition* (C)*.*

Example 10. Let Q be the following quiver:

$$Q: \ 1 \rightrightarrows 2 \overset{\alpha}{\rightrightarrows} 3$$

Thus l_Q is given by the following table:

$l_Q(a,b)$	$b=1$	$b=2$	$b=3$
$a=1$	0	1	2
$a=2$	0	0	1
$a=3$	0	0	0

We put $Q' := Q \setminus \{\alpha\}$. Then we can check that

$$\mathrm{Ext}^1_{kQ'}(\tau^{-r}P(a), \tau^{-s}t(b)) = 0 \Leftrightarrow r \leq s + l_Q(b,a) \text{ or } (a=b=3,\ r=s+2).$$

Therefore $\mathcal{T}_{\mathrm{p}}(Q)$ and $\mathcal{T}_{\mathrm{p}}(Q')$ is given by the following:

$\mathcal{T}_{\mathrm{p}}(Q)$

$$\begin{tikzcd}
 & & T_{0,0,0} \arrow[dl] \\
 & T_{0,0,1} \arrow[d] & \\
 & T_{0,1,1} \arrow[dl] \arrow[dr] & \\
T_{1,1,1} \arrow[dr] & & T_{0,1,2} \arrow[dl] \\
 & T_{1,1,2} \arrow[d] & \\
 & T_{1,2,2} \arrow[dl] \arrow[dr] & \\
T_{1,2,3} \arrow[dr] & & T_{2,2,2} \arrow[dl] \\
 & \vdots &
\end{tikzcd}$$

$\mathcal{T}_{\mathrm{p}}(Q')$

$$\begin{tikzcd}
 & & & T'_{0,0,0} \arrow[dl] \arrow[ddr] & \\
 & & T'_{0,0,1} \arrow[d] & & \\
 & & T'_{0,1,1} \arrow[dl] \arrow[dr] & & X \arrow[dl] \\
 & T'_{1,1,1} \arrow[dl] \arrow[dr] & & T'_{0,1,2} \arrow[dl] & \\
\tau^{-1}X \arrow[ddr] & & T'_{1,1,2} \arrow[d] & & \\
 & & T'_{1,2,2} \arrow[dl] \arrow[dr] & & \\
 & T'_{1,2,3} \arrow[dr] & & T'_{2,2,2} \arrow[dl] & \\
 & & \vdots & &
\end{tikzcd}$$

where

$$\begin{aligned}
T_{r,s,t} &:= \tau_Q^{-r}P(1) \oplus \tau_Q^{-s}P(2) \oplus \tau_Q^{-t}P(3), \\
T'_{r,s,t} &:= \tau_{Q'}^{-r}P'(1) \oplus \tau_{Q'}^{-s}P'(2) \oplus \tau_{Q'}^{-t}P'(3), \\
X &:= P(1) \oplus P(3) \oplus \tau_Q^{-2}P(3).
\end{aligned}$$

In particular $\mathcal{T}_{\mathrm{p}}(Q)$ is a distributive lattice and $\mathcal{T}_{\mathrm{p}}(Q')$ is not a distributive lattice.

If Q satisfies the condition (C), then Proposition 7 implies that a module

$$T(a) := \bigoplus_{x \in Q_0} \tau^{-l_Q(a,x)} P(x)$$

is a pre-projective tilting module, for any vertex $a \in Q_0$. In fact, by the definition of l_Q, we have

$$l_Q(a,x) \leq l_Q(a,y) + l_Q(y,x) \text{ for any } x, y \in Q_0.$$

Moreover, $\tau^{-r}T(a)$ is a minimal element of $\{T \in \mathcal{T}_{\mathrm{p}}(Q) \mid \tau^{-r}P(a) \in \operatorname{add} T\}$. Therefore we obtain the following.

Lemma 11. [8] *Assume that Q satisfies the condition* (C). *Then the set of join-irreducible elements of $\mathcal{T}_{\mathrm{p}}(Q)$ is $\{\tau^{-r}T(a) \mid a \in Q_0,\ r \in \mathbb{Z}_{\geq 0}^{Q_0}\}$.*

For any poset P, we denote by $\mathcal{I}(P)$ the ideal-poset of P. Now let $\mathcal{J}(Q) \subset \mathcal{T}_{\mathrm{p}}(Q)$ be the sub-poset of join-irreducible elements. Now we give Birkhoff's type result for the poset of pre-projective tilting modules $\mathcal{T}_{\mathrm{p}}(Q)$.

Proposition 12. [8] *Assume that Q satisfies the condition* (C). *Then a map*

$$\rho : \mathcal{I}(\mathcal{J}(Q)) \setminus \{\emptyset\} \ni I \mapsto \bigvee_{\mathbf{i} \in I} T(\mathbf{i}) \in \mathcal{T}_{\mathrm{p}}(Q)$$

induces a poset isomorphism

$$\mathcal{I}(\mathcal{J}(Q)) \setminus \{\emptyset\} \simeq \mathcal{T}_{\mathrm{p}}(Q).$$

Let $\Gamma_{\mathrm{p}}(Q)$ be the pre-projective component of Auslander-Reiten quiver of mod-kQ. We define a poset $\mathcal{P}(Q)$ as follows:
- $\mathcal{P}(Q) = \{$indecomposable pre-projective modules over $kQ\}/\simeq$ as a set.
- $X \geq Y$ if there is a path from X to Y in $\Gamma_{\mathrm{p}}(Q)$.

Then we have the following.

Theorem 13. [8] *Assume that Q satisfies the condition* (C). *Then there is a poset isomorphism*

$$\mathcal{J}(Q) \simeq \mathcal{P}(Q).$$

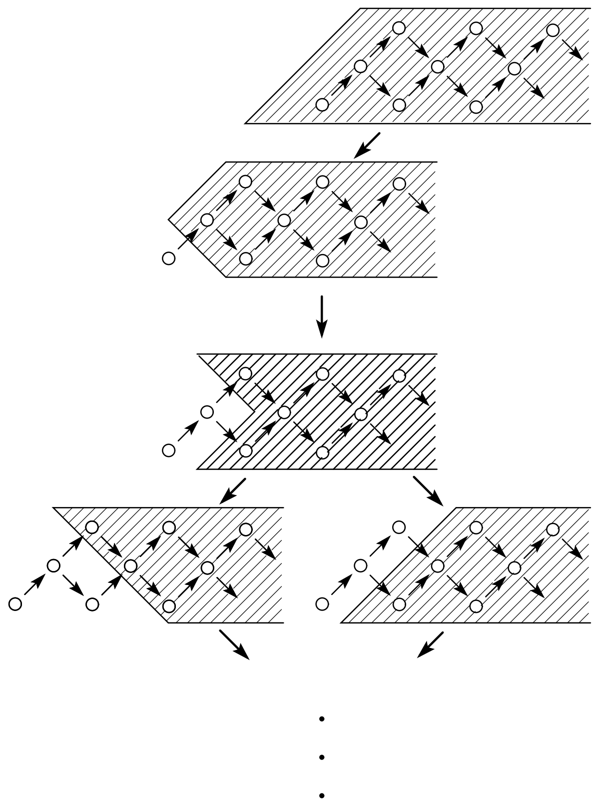
In particular, we have a poset isomorphism

$$\mathcal{T}_{\mathrm{p}}(Q) \simeq \mathcal{I}(\mathcal{P}(Q)) \setminus \{\emptyset\}.$$

Example 14. Let Q be the following quiver:

$$Q: \ 1 \Rrightarrow 2 \overset{\alpha}{\Rrightarrow} 3$$

Then $\mathcal{I}(\mathcal{P}(Q)) \setminus \{\emptyset\}$ is given by the following:



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TRIANGULATED SUBCATEGORIES OF EXTENSIONS AND TRIANGLES OF RECOLLEMENTS

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Abstract. Let T be a triangulated category with triangulated subcategories X and Y. We show that the subcategory of extensions $\mathsf{X} * \mathsf{Y}$ is triangulated if and only if every morphism from X to Y factors thorough $\mathsf{X} \cap \mathsf{Y}$.

In this situation, we show that there is a stable t-structure $\left(\frac{\mathsf{X}}{\mathsf{X}\cap\mathsf{Y}}, \frac{\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}\right)$ in $\frac{\mathsf{X}*\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}$. We use this to give a recipe for constructing triangles of recollements and recover some triangles of recollements from the literature.

This is joint work with Peter Jørgensen.

1. Introduction

Let T be a triangulated category. If X and Y are full subcategories of T, then the *subcategory of extensions* $\mathsf{X} * \mathsf{Y}$ is the full subcategory of objects e for which there is a distinguished triangle $x \to e \to y$ with $x \in \mathsf{X}$, $y \in \mathsf{Y}$. Subcategories of extensions have recently been of interest to a number of authors, see [1], [5], [6], [12].

We give necessary and sufficient conditions for $\mathsf{X} * \mathsf{Y}$ to be triangulated. It has been known that $\mathsf{X} * \mathsf{Y}$ is triangulated if there is no morphism from X to Y. Theorem 1 shows that this classical fact essentially gives the sufficient condition as well.

Theorem 1. *Let* X*,* Y *be triangulated subcategories of* T*. Then* $\mathsf{X}*\mathsf{Y}$ *is a triangulated subcategory of* $\mathsf{T} \Leftrightarrow \mathsf{Y} * \mathsf{X} \subseteq \mathsf{X} * \mathsf{Y} \Leftrightarrow$. $\mathrm{Hom}_{\mathsf{T}/\mathsf{X}\cap\mathsf{Y}}(\mathsf{X}/\mathsf{X} \cap \mathsf{Y}, \mathsf{Y}/\mathsf{X} \cap \mathsf{Y}) = 0$.

If this is the case, $\mathsf{X}/\mathsf{X} \cap \mathsf{Y}$ and $\mathsf{Y}/\mathsf{X} \cap \mathsf{Y}$ give a stable t-structure in $\mathsf{X} * \mathsf{Y}/\mathsf{X} \cap \mathsf{Y}$. Recall that a pair of triangulated subcategories (U, V) of T is called a stable t-structure if $\mathsf{U} * \mathsf{V} = \mathsf{T}$ and $\mathrm{Hom}_{\mathsf{T}}(\mathsf{U}, \mathsf{V}) = 0$, see [9, def. 9.14]. Indeed, for a given thick subcategory U of T, there is a one-to -one correspondence between stable t-structures of T/U and pairs of thick subcategories X, Y with $\mathsf{T} = \mathsf{X} * \mathsf{Y}$ and $\mathsf{X} \cap \mathsf{Y} = \mathsf{U}$, see [7] Lemma 4.6 .

Finally, under stronger assumptions, we show that a pair (or a triple) of triangulated subcategories of extensions induces a so-called (triangle of) recollements in a quotient category. A pair of stable t-structures (U, V), (V, W) is the equivalent notion to a recollement [8]. A triangle of recollements is a triple of stable t-structures (U, V), (V, W), (W, U). Triangles of recollements were introduced in [4, def. 0.3] and have a very high degree of symmetry; for instance, $\mathsf{U} \simeq \mathsf{V} \simeq \mathsf{W} \simeq \mathsf{T}/\mathsf{U} \simeq \mathsf{T}/\mathsf{V} \simeq \mathsf{T}/\mathsf{W}$. They have applications to the construction of triangle equivalences, see [4, prop. 1.16].

This is a preliminary report. The detailed version of this paper will be submitted for publication elsewhere.

2. Triangulated subcategory of extensions

Theorem 1 ([7] Theorem 4.1)**.** *Let* X*,* Y *be triangulated subcategories of* T *and let* $Q : \mathsf{T} \to \mathsf{T}/\mathsf{X} \cap \mathsf{Y}$ *be the quotient functor. Then the following are equivalent.*

(1) $\mathsf{X} * \mathsf{Y}$ *is a triangulated subcategory of* T*.*
(2) $\mathsf{Y} * \mathsf{X} \subseteq \mathsf{X} * \mathsf{Y}$*.*
(3) *Each morphism* $f : x \to y$ *with* $x \in \mathsf{X}$*,* $y \in \mathsf{Y}$ *factors through some object of* $\mathsf{X} \cap \mathsf{Y}$*.*
(4) $\mathrm{Hom}_{Q(\mathsf{T})}(Q(\mathsf{X}), Q(\mathsf{Y})) = 0$*.*
(5) $\mathsf{X} * \mathsf{Y}'$ *is a triangulated subcategory of* T *for every triangulated subcategory* Y' *of* Y *containing* $\mathsf{X} \cap \mathsf{Y}$*.*
(6) $\mathsf{X}' * \mathsf{Y}$ *is a triangulated subcategory of* T *for every triangulated subcategory* X' *of* X *containing* $\mathsf{X} \cap \mathsf{Y}$*.*

If $\mathsf{X} \cap \mathsf{Y} = 0$ in particular, we recover the following. This fact is well known but we have been unable to locate a reference.

Corollary 2. *Let* X*,* Y *be triangulated subcategories of* T*. If* $\mathrm{Hom}_{\mathsf{T}}(\mathsf{X}, \mathsf{Y}) = 0$ *then* $\mathsf{X} * \mathsf{Y}$ *is a triangulated subcategory of* T*.*

Lemma 3 ([7] Lemma 4.6)**.** *Let* U *and* V *be triangulated subcategories of* T *and assume that* $\mathsf{S} = \mathsf{U} * \mathsf{V}$ *is triangulated. Let* $Q : \mathsf{T} \to \mathsf{T}/\mathsf{U} \cap \mathsf{V}$ *and* $Q' : \mathsf{S} \to \mathsf{S}/\mathsf{U} \cap \mathsf{V}$ *be the quotient functors. We have the following.*

(1) $(Q'(\mathsf{U}), Q'(\mathsf{V}))$ *is a stable t-structure of* $Q'(\mathsf{S})$*.*
(2) *If* $\mathsf{U} \cap \mathsf{V}$ *is thick, then* $(Q(\mathsf{U}), Q(\mathsf{V}))$ *is a stable t-structure of* $Q(\mathsf{S})$*. In particular,* $\mathsf{S} = \mathsf{T}$ *if and only if* $Q(\mathsf{S}) = Q(\mathsf{T})$*.*

Remark 4. Yoshizawa gives the following example in [12, cor. 3.3]: If R is a commutative noetherian ring and S is a Serre subcategory of $\mathsf{Mod}\, R$, then $(\mathsf{mod}\, R) * \mathsf{S}$ is a Serre subcategory of $\mathsf{Mod}\, R$. Here $\mathsf{Mod}\, R$ is the category of R-modules and $\mathsf{mod}\, R$ is the full subcategory of finitely generated R-modules.

One might suspect a triangulated analogue to say that if T is compactly generated and U is a triangulated subcategory of T, then so is $\mathsf{T}^c * \mathsf{U}$ where T^c denotes the triangulated subcategory of compact objects. See [10, defs. 1.6 and 1.7]. However, this is false:

Set $\mathsf{T} = \mathsf{D}(\mathbb{Z})$ and $\mathsf{U} = \mathsf{D}(\mathbb{Q})$. Then T is compactly generated by $\{ \Sigma^i \mathbb{Z} \,|\, i \in \mathbb{Z} \}$. There is a homological epimorphism of rings $\mathbb{Z} \to \mathbb{Q}$ which induces an embedding of triangulated categories $\mathsf{U} \hookrightarrow \mathsf{T}$, see [2, def. 4.5]and [11, thm. 2.4]. Since $\mathbb{Q}$ is a field, each object of U has homology modules of the form $\coprod \mathbb{Q}$. This means that viewed in T, the only object of U which has finitely generated homology modules is 0. Hence 0 is the only object of U which is compact in T, see [10, cor. 2.3]. That is, $\mathsf{T}^c \cap \mathsf{U} = 0$.

If $\mathsf{T}^c * \mathsf{U}$ were a triangulated subcategory of T, then Theorem B would give that $(\mathsf{T}^c, \mathsf{U})$ was a stable t-structure in $\mathsf{T}^c * \mathsf{U}$, but this is false since the canonical map $\mathbb{Z} \to \mathbb{Q}$ is a non-zero morphism from an object of T^c to an object of U.

3. Recollements

In the previous section we see that a pair of triangulated subcategories induces a stable t-structure if the category of their extensions is triangulated. It is natural to ask whether a (triangle of) recollement(s) is induced by a triple of triangulated subcategories $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ with $\mathsf{X} * \mathsf{Y}$, $\mathsf{Y} * \mathsf{Z}$ (and $\mathsf{Z} * \mathsf{X}$) triangulated. Apparently we don't know which category the recollement lives in. However using "enlargement" and "restriction" of categories, we construct a subquotient category with desired recollement. Throughout this section, $\langle \mathsf{X}_1, \cdots, \mathsf{X}_n \rangle$ is the smallest triangulated subcategory containing $\mathsf{X}_1, \cdots, \mathsf{X}_n$.

Lemma 5 (restriction. [7] Lemma 6.1)**.** *Let U, V and W be triangulated subcategories of T.*

(1) *Assume both $\mathsf{U} * \mathsf{V}$ and $\mathsf{V} * \mathsf{W}$ are triangulated. Then $\mathsf{S} = (\mathsf{U} * \mathsf{V}) \cap (\mathsf{V} * \mathsf{W})$ is represented as $\mathsf{S} = \mathsf{U}_1 * \mathsf{V} = \mathsf{V} * \mathsf{W}_1$ where $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$ and $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$.*

(2) *Assume each of $\mathsf{U} * \mathsf{V}$, $\mathsf{V} * \mathsf{W}$ and $\mathsf{W} * \mathsf{U}$ is triangulated. Then $\mathsf{S} = (\mathsf{U} * \mathsf{V}) \cap (\mathsf{V} * \mathsf{W}) \cap (\mathsf{W} * \mathsf{U})$ is represented as $\mathsf{S} = \mathsf{U}_1 * \mathsf{V}_1 = \mathsf{V}_1 * \mathsf{W}_1 = \mathsf{W}_1 * \mathsf{U}_1$ where $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$, $\mathsf{V}_1 = \mathsf{V} \cap \mathsf{S}$ and $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$.*

Lemma 6 (enlargement. [7] Lemma5.1)**.** *Let U and V be triangulated subcategories of T. Assume $\mathsf{U} * \mathsf{V}$ is triangulated. For each triangulated subcategories $\mathsf{U}' \subset \mathsf{U}$ and $\mathsf{V}' \subset \mathsf{V}$, we have the following.*

(1) $\mathsf{U} * \mathsf{V} = \mathsf{U} * \langle \mathsf{V}, \mathsf{U}' \rangle$.
(2) $\langle \mathsf{V}, \mathsf{U}' \rangle \cap \mathsf{U} = \langle \mathsf{U} \cap \mathsf{V}, \mathsf{U}' \rangle$.
(3) $\mathsf{U} * \mathsf{V} = \langle \mathsf{U}, \mathsf{V}' \rangle * \mathsf{V}$.
(4) $\langle \mathsf{U}, \mathsf{V}' \rangle \cap \mathsf{V} = \langle \mathsf{U} \cap \mathsf{V}, \mathsf{V}' \rangle$.

Lemma 7. *Let U, V and W be triangulated subcategory of T.*

(1) *Assume $\mathsf{U} * \mathsf{V} = \mathsf{V} * \mathsf{W}$ and is triangulated. Set $\mathsf{S} = \mathsf{U} * \mathsf{V}$ and let $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle$ be the canonical quotient functor. Then both $(Q(\mathsf{U}), Q(\mathsf{V}))$ and $(Q(\mathsf{V}), Q(\mathsf{W}))$ are stable t-structures of $\mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle$.*

(2) *Assume $\mathsf{U} * \mathsf{V} = \mathsf{V} * \mathsf{W} = \mathsf{W} * \mathsf{U}$ and is triangulated. Set $\mathsf{S} = \mathsf{U} * \mathsf{V}$ and let $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W}, \mathsf{W} \cap \mathsf{U} \rangle$ be the canonical quotient functor. Then $(Q(\mathsf{U}), Q(\mathsf{V}))$, $(Q(\mathsf{V}), Q(\mathsf{W}))$ and $(Q(\mathsf{W}), Q(\mathsf{U}))$are stable t-structures of $\mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W}, \mathsf{W} \cap \mathsf{U} \rangle$.*

Proof. (i). We have $\mathsf{S} = \langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle * \mathsf{V} = \mathsf{V} * \langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle$ and $\langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle \cap \mathsf{V} = \langle \mathsf{U} \cap \mathsf{V}, \mathsf{W} \cap \mathsf{V} \rangle = \mathsf{V} \cap \langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle$ from Lemma 6. Lemma 3 gives two stable t-structures $(Q(\langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle), Q(\mathsf{V}))$ and $(Q(\mathsf{V}), Q(\langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle))$ of $Q(\mathsf{S})$, but $(Q(\langle \mathsf{U}, \mathsf{W} \cap \mathsf{V} \rangle) = Q(\mathsf{U})$ and $Q(\langle \mathsf{W}, \mathsf{U} \cap \mathsf{V} \rangle) = Q(\mathsf{W})$ hence we are done.

(ii). From Lemma 6, we have $\mathsf{S} = \langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle * \mathsf{V} = \langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle * \langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle$ and $\langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle \cap \langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle = \langle \langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle, \mathsf{W} \cap \mathsf{U} \rangle$. Lemma 3 gives a stable t-structure $(Q(\langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle), Q(\langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle))$ but $(Q(\langle \mathsf{U}, \mathsf{V} \cap \mathsf{W} \rangle) = Q(\mathsf{U})$ and $Q(\langle \mathsf{V}, \mathsf{W} \cap \mathsf{U} \rangle) = Q(\mathsf{V})$. Analogously we obtain other stable t-structures. ☐

Theorem 8. *Let U, V and W be triangulated subcategories of T.*

(1) *Assume both $\mathsf{U} * \mathsf{V}$ and $\mathsf{V} * \mathsf{W}$ are triangulated. Set $\mathsf{S} = \mathsf{U} * \mathsf{V} \cap \mathsf{V} * \mathsf{W}$ and let $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W} \rangle$ be the canonical quotient functor. Then $(Q(\mathsf{U}_1), Q(\mathsf{V}))$, and $(Q(\mathsf{V}), Q(\mathsf{W}_1))$ are stable t-structures of $Q(\mathsf{S})$ where $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$ and $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$.*

(2) *Assume each of $\mathsf{U} * \mathsf{V}$, $\mathsf{V} * \mathsf{W}$ and $\mathsf{W} * \mathsf{U}$ is triangulated. Set $\mathsf{S} = \mathsf{U} * \mathsf{V} \cap \mathsf{V} * \mathsf{W} \cap \mathsf{W} * \mathsf{U}$ and let $Q : \mathsf{S} \to \mathsf{S}/\langle \mathsf{U} \cap \mathsf{V}, \mathsf{V} \cap \mathsf{W}, \mathsf{W} \cap \mathsf{U} \rangle$ be the canonical quotient functor. Then $(Q(\mathsf{U}_1), Q(\mathsf{V}_1))$, $(Q(\mathsf{V}_1), Q(\mathsf{W}_1))$ and $(Q(\mathsf{W}_1), Q(\mathsf{U}_1))$ are stable t-structures of $Q(\mathsf{S})$ where $\mathsf{U}_1 = \mathsf{U} \cap \mathsf{S}$, $\mathsf{V}_1 = \mathsf{V} \cap \mathsf{S}$ and $\mathsf{W}_1 = \mathsf{W} \cap \mathsf{S}$.*

Example 9 (The homotopy category of projective modules)**.** Let R be an Iwanaga-Gorenstein ring, that is, a noetherian ring which has finite injective dimension from either side as a module over itself. Let $\mathsf{T} = \mathsf{K}_{(b)}(\text{Prj } R)$ be the homotopy category of complexes of projective right-R-modules with bounded homology. Define subcategories of T by

$$\mathsf{X} = \mathsf{K}^{-}_{(b)}(\text{Prj } R) \ , \ \mathsf{Y} = \mathsf{K}_{ac}(\text{Prj } R) \ , \ \mathsf{Z} = \mathsf{K}^{+}_{(b)}(\text{Prj } R)$$

where $\mathsf{K}^{-}_{(b)}(\text{Prj } R)$ is the isomorphism closure of the class of complexes P with $P^i = 0$ for $i \gg 0$ and $\mathsf{K}^{+}_{(b)}(\text{Prj } R)$ is defined analogously, while $\mathsf{K}_{ac}(\text{Prj } R)$ is the subcategory of acyclic (that is, exact) complexes.

Note that Y is equal to $\mathsf{K}_{tac}(\text{Prj } R)$, the subcategory of totally acyclic complexes, that is, acyclic complexes which stay acyclic under the functor $\text{Hom}_R(-, Q)$ when Q is projective, see [3, cor. 5.5 and par. 5.12].

By [4, prop. 2.3(1), lem. 5.6(1), and rmk. 5.14] there are stable t-structures (X, Y), (Y, Z) in T.

If $P \in \mathsf{T}$ is given, then there is a distinguished triangle $P^{\geq 0} \to P \to P^{<0}$ where $P^{\geq 0}$ and $P^{<0}$ are hard truncations. Since $P^{\geq 0} \in \mathsf{Z}$ and $P^{<0} \in \mathsf{X}$, we have $\mathsf{T} = \mathsf{Z} * \mathsf{X}$.

We can hence apply Lemma 7. The intersection

$$\mathsf{X} \cap \mathsf{Z} \ = \ \mathsf{K}^{-}_{(b)}(\text{Prj } R) \cap \mathsf{K}^{+}_{(b)}(\text{Prj } R) \ = \ \mathsf{K}^{b}(\text{Prj } R)$$

is the isomorphism closure of the class of bounded complexes. If we use an obvious shorthand for quotient categories, Lemma 7 (ii) therefore provides a triangle of recollements

$$\big(\mathsf{K}^{-}_{(b)}/\mathsf{K}^{b}(\text{Prj } R) \ , \ \mathsf{K}_{ac}(\text{Prj } R) \ , \ \mathsf{K}^{+}_{(b)}/\mathsf{K}^{b}(\text{Prj } R)\big)$$

in $\mathsf{K}_{(b)}/\mathsf{K}^{b}(\text{Prj } R)$. Note that $\mathsf{K}_{ac}(\text{Prj } R)$ is equivalent to its projection to $\mathsf{K}_{(b)}/\mathsf{K}^{b}(\text{Prj } R)$ by [4, prop. 1.5], so we can write $\mathsf{K}_{ac}(\text{Prj } R)$ instead of the projection.

This example and its finite analogue were first obtained in [4, thms. 2.8 and 5.8] and motivated the definition of triangles of recollements.

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DEFINING RELATIONS OF 3-DIMENSIONAL QUADRATIC AS-REGULAR ALGEBRAS

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Abstract. Classification of AS-regular algebras is one of the main interests in noncommutative algebraic geometry. In this article, we focus on the 3-dimensional quadratic case. We find defining relations of 3-dimensional quadratic AS-regular algebras. Also, we classify these algebras up to isomorphism and up to graded Morita equivalence in terms of their defining relations.

1. Introduction

Classification of AS-regular algebras is one of the main interests in noncommutative algebraic geometry. In fact, the geometric classification of 3-dimensional AS-regular algebras due to Artin, Tate and Van den Bergh [2] is regarded as one of the starting points of the field. In this article, we focus on 3-dimensional quadratic AS-regular algebras. By restricting to these algebras, they are in one-to-one correspondence with geometric pairs (E, σ) classified by A-T-V [2]. In this article, we try to find defining relations of 3-dimensional quadratic AS-regular algebras and answer the question when algebras given by defining relations are isomorphic or graded Morita equivalent.

2. Preliminaries

Throughout this article, we fix an algebraically closed field k of characteristic zero. Let A be a graded k-algebra. We denote by $\mathrm{GrMod}A$ the category of graded right A-modules. We say that two graded algebras A and A' are graded Morita equivalent if the categories $\mathrm{GrMod}A$ and $\mathrm{GrMod}A'$ are equivalent.

The definition of an AS-regular algebra below is stronger than its original definition [**1**].

Definition 1. [1] A Noetherian connected graded algebra A is an AS-regular algebra of dimension d if

- $\mathrm{gldim}A = d < \infty$, and
- $\mathrm{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

Example 2. Let $A = k\langle x, y, z\rangle/(yx - \alpha z^2, zy - \beta x^2, xz - \gamma y^2)$, where $\alpha\beta\gamma \neq 0$. Then A is a 3-dimensional AS-regular algebra.

Artin, Tate and Van den Bergh classified 3-dimensional AS-regular algebras by using geometric techniques. In this article, we will focus on the quadratic case and classify 3-dimensional quadratic AS-regular algebras algebraically.

The detailed version of this paper will be submitted for publication elsewhere.

Let A be a graded algebra finitely generated in degree 1 over k. Note that A can be presented as $A = T(V)/I$ where V is a finite dimensional vector space over k, $T(V)$ is the tensor algebra on V, and I is a homogeneous two-sided ideal of $T(V)$. We say $A = T(V)/(R)$ is a quadratic algebra when $R \subset V \otimes_k V$ is a subspace and (R) is the two-sided ideal of $T(V)$ generated by R . By choosing a basis $\{x_1, \cdots, x_n\}$ for V over k , a quadratic algebra A is also presented as $A = k\langle x_1, \cdots, x_n\rangle/(f_1, \cdots, f_m)$ where $\deg x_i = 1$ for all i and f_j are homogeneous noncommutative polynomials of degree two for all j. For a quadratic algebra $A = T(V)/(R)$, we define

$$\mathcal{V}(R) := \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid \ f(p, q) = 0 \ \text{ for all } f \in R\}.$$

Definition 3. [3] A quadratic algebra $A = T(V)/(R)$ is called geometric if there exists a geometric pair (E, σ) where $E \subset \mathbb{P}(V^*)$ is a closed k-subscheme and σ is a k-automorphism of E such that

(G1) $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$,
(G2) $R = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0, \forall p \in E\}$.

If A satisfies the condition (G1), A determines a geometric pair (E, σ). Conversely, A is determined by a geometric pair (E, σ) if A satisfies the condition (G2) and we write $A = \mathcal{A}(E, \sigma)$ in this case.

By [2], every 3-dimensional quadratic AS-regular algebra A is geometric. Moreover, E is either $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$, that is, E is $\mathbb{P}^2$, a union of three lines which make a triangle, a union of three lines meeting at one point, a union of a line and a conic meeting at two point, a union of a line and a conic meeting at one point, a nodal curve, a cuspidal curve, a union of a line and a double line, a triple line , or an elliptic curve.

The types of (E, σ) of 3-dimensional quadratic AS-regular algebras are defined in [4] which are slightly modified from the original types defined in [1] and [2]. We follow the types defined in [4].

- Type $\mathbb{P}^2$: E is $\mathbb{P}^2$, and $\sigma \in \mathrm{Aut}_k\mathbb{P}^2 = \mathrm{PGL}_3(k)$.
- Type S_1 : E is a triangle, and σ stabilizes each component.
- Type S_2 : E is a triangle, and σ interchanges two of its components.
- Type S_3 : E is a triangle, and σ circulates three components.
- Type S_1' : E is a union of a line and a conic meeting at two points, and σ stabilizes each component and two intersection points.
- Type S_2' : E is a union of a line and a conic meeting at two points, and σ stabilizes each component and interchanges two intersection points.
- Type T_1 : E is a union of three lines meeting at one point, and σ stabilizes each component.
- Type T_2 : E is a union of three lines meeting at one point, and σ interchanges two of its components.
- Type T_3 : E is a union of three lines meeting at one point, and σ circulates three components.
- Type T_1' : E is a union of a line and a conic meeting at one point, and σ stabilizes each component.
- Type N : E is a nodal cubic curve.
- Type C : E is a cuspidal cubic curve.

We introduce some Lemmas which are used for classification.

Lemma 4. *Let A and A' be geometric algebras with $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$. Then $A \cong A'$ as graded algebras if and only if there exists $\tau \in \mathrm{Aut}_k\mathbb{P}(V^*)$ which restricts to an isomorphism $\tau : E \to E'$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E' \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow{\tau} & E' \end{array}$$

commutes.

Lemma 5. [3] *Let A and A' be geometric algebras with $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$. Then* $\mathrm{GrMod}A \cong \mathrm{GrMod}A'$ *if and only if there exists a sequence of automorphisms $\tau_n \in$* $\mathrm{Aut}_k\mathbb{P}(V^*)$ *which restricts to a sequence of isomorphisms $\tau_n : E \to E'$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\tau_n} & E' \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow{\tau_{n+1}} & E' \end{array}$$

commute for all $n \in \mathbb{Z}$.

In general, classifying quadratic algebras up to isomorphism is easier than classifying them up to graded Morita equivalence. Our method is to define a new graded algebra $\overline{A}$ and classify original algebra A up to graded Morita equivalence by classifying $\overline{A}$ up to isomorphism.

Remark 6. Let $A = T(V)/(R)$ be a 3-dimensional quadratic AS-regular algebra. Then A is Koszul and $A^! = T(V^*)/(R^{\perp})$ is Frobenius. It follows that we can take the Nakayama automorphism $\nu \in \mathrm{Aut}_k A^!$ of $A^!$. It was shown that ν naturally induces $\nu \in \mathrm{Aut}_k E$ by abuse of notation.

Using the automorphism $\nu \in \mathrm{Aut}_k E$, we define a new graded algebra $\overline{A}$ from A.

Definition 7. [4] Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quadratic AS-regular algebra and $\nu \in \mathrm{Aut}_k E$ the automorphism induced by the Nakayama automorphism of $A^!$. Define a new graded algebra by $\overline{A} := \mathcal{A}(E, \nu\sigma^3)$.

Theorem 8. [4] *Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be 3-dimensional quadratic AS-regular algebras. Suppose that (E, σ) and (E', σ') are the same Type $\mathbb{P}^2, S_i, S_i'$, T_i or T_i'. Then* $\mathrm{GrMod}A$ *is equivalent to* $\mathrm{GrMod}A'$ *if and only if $\overline{A}$ is isomorphic to $\overline{A'}$ as graded algebras.*

3. Main results

We completed classification of 3-dimensional quadratic AS-regular algebras in the cases of Type $\mathbb{P}^2$, S_i, S_i', T_i, T_i', N, C. We will explain our method using Thoerem 8 in some details for simplest case here. The remaining cases are also proved by using Lemma 4, Lemma 5, Theorem 8 and [5].

Example 9. Let (E,σ) be of Type S_1. Then we may assume that $E=\mathcal{V}(x)\cup\mathcal{V}(y)\cup\mathcal{V}(z)$ and $\sigma\in\mathrm{Aut}_k E$ is given by

$$\begin{aligned}\sigma|_{\mathcal{V}(x)}(0,b,c)&=(0,b,\alpha c),\\ \sigma|_{\mathcal{V}(y)}(a,0,c)&=(\beta a,0,c),\\ \sigma|_{\mathcal{V}(z)}(a,b,0)&=(a,\gamma b,0).\end{aligned}$$

where $\alpha\beta\gamma\neq 0,1$. We can determine $A=\mathcal{A}(E,\sigma)$ from the property (G2) of geometric algebra. In this case, $A=\mathcal{A}(E,\sigma)$ is given by

$$A=k\langle x,y,z\rangle/(yz-\alpha zy,zx-\beta xz,xy-\gamma yx)=:A_{\alpha,\beta,\gamma}.$$

By using Lemma 4 above, $A_{\alpha,\beta,\gamma}$ and $A_{\alpha',\beta',\gamma'}$ are isomorphic as graded algebras if and only if

$$(\alpha',\beta',\gamma')=\begin{cases}(\alpha,\beta,\gamma),(\beta,\gamma,\alpha),(\gamma,\alpha,\beta),\\ (\alpha^{-1},\gamma^{-1},\beta^{-1}),(\beta^{-1},\alpha^{-1},\gamma^{-1}),(\gamma^{-1},\beta^{-1},\alpha^{-1}).\end{cases}$$

Next, we define $\overline{A_{\alpha,\beta,\gamma}}$ to classify $A_{\alpha,\beta,\gamma}$ up to graded Morita equivalence. In this case, $\nu\in\mathrm{Aut}_k E$ is given by

$$\nu(a,b,c)=((\gamma/\beta)a,(\alpha/\gamma)b,(\beta/\alpha)c).$$

It follows that

$$\begin{aligned}\overline{A_{\alpha,\beta,\gamma}}&=\mathcal{A}(E,\nu\sigma^3)\\ &=k\langle x,y,z\rangle/(yz-\alpha\beta\gamma zy,zx-\alpha\beta\gamma xz,xy-\alpha\beta\gamma yx)\\ &=A_{\alpha\beta\gamma,\alpha\beta\gamma,\alpha\beta\gamma}\ .\end{aligned}$$

By Theorem 8,

$$\begin{aligned}\mathrm{GrMod}A_{\alpha,\beta,\gamma}\cong\mathrm{GrMod}A_{\alpha',\beta',\gamma'}&\Leftrightarrow\overline{A_{\alpha,\beta,\gamma}}\cong\overline{A_{\alpha',\beta',\gamma'}}\\ &\Leftrightarrow A_{\alpha\beta\gamma,\alpha\beta\gamma,\alpha\beta\gamma}\cong A_{\alpha'\beta'\gamma',\alpha'\beta'\gamma',\alpha'\beta'\gamma'}\\ &\Leftrightarrow\alpha'\beta'\gamma'=(\alpha\beta\gamma)^{\pm1}\ .\end{aligned}$$

We write down our results.

Theorem 10. *Let $A=\mathcal{A}(E,\sigma)$ be a 3-dimensional quadratic AS-regular algebra. In each type, we list the defining relations, when they are isomorphic and when they are graded Morita equivalent in terms of parameters in the defining relations as in Example 9.*

Type	*defining relations / isomorphism / graded Morita equivalent*
$\mathbb{P}^2$	*(case 1)*$xy-(\beta/\alpha)yx,yz-(\gamma/\beta)zy,zx-(\alpha/\gamma)xz$, *where* $\alpha\beta\gamma\neq 0$ *(case 2)*$xy-yx+y^2,xz-\alpha zx+\alpha zy,yz-\alpha zy$, *where* $\alpha\neq 0$ *(case 3)*$xy-yx+y^2-zx,xz+yz-zx,zy-yz-z^2$
	There are no isomorphic relations between different cases. *(case1)*$(\alpha',\beta',\gamma')=\begin{cases}(\alpha,\beta,\gamma),(\alpha,\gamma,\beta),(\beta,\alpha,\gamma),\\ (\beta,\gamma,\alpha),(\gamma,\alpha,\beta),(\gamma,\beta,\alpha)\end{cases}$ *in* $\mathbb{P}^2$ *(case2)*$A_\alpha\cong A_{\alpha'}$ *if and only if* $\alpha=\alpha'$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A,A' *of Type* $\mathbb{P}^2$

S_1	$yz-\alpha zy,\ zx-\beta xz,\ xy-\gamma yx,$ *where* $\alpha\beta\gamma\neq 0,1$
	$(\alpha',\beta',\gamma')=\begin{cases}(\alpha,\beta,\gamma),(\beta,\gamma,\alpha),(\gamma,\alpha,\beta),\\(\alpha^{-1},\gamma^{-1},\beta^{-1}),(\beta^{-1},\alpha^{-1},\gamma^{-1}),(\gamma^{-1},\beta^{-1},\alpha^{-1})\end{cases}$
	$\alpha'\beta'\gamma'=(\alpha\beta\gamma)^{\pm 1}$
S_2	$zx-\alpha yz, zy-\beta xz, x^2+\alpha\beta y^2,$ *where* $\alpha\beta\neq 0$
	$(\alpha',\beta')=(\alpha,\beta)$ *in* $\mathbb{P}^1$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A,A' *of Type* S_2
S_3	$yx-\alpha z^2, zy-\beta x^2, xz-\gamma y^2,$ *where* $\alpha\beta\gamma\neq 0,1$
	$\alpha'\beta'\gamma'=\alpha\beta\gamma$
	$\alpha'\beta'\gamma'=(\alpha\beta\gamma)^{\pm 1}$
S_1'	$xy-\beta yx, x^2+yz-\alpha zy, zx-\beta xz,$ *where* $\alpha\beta^2\neq 0,1$
	$(\alpha',\beta')=(\alpha,\beta),(\alpha^{-1},\beta^{-1})$
	$\alpha'\beta'^2=(\alpha\beta^2)^{\pm 1}$
S_2'	*Every algebra is isomorphic to* $k\langle x,y,z\rangle/(xy-zx, yx-xz, x^2+y^2+z^2)$
T_1	$\begin{cases}xy-yx,\\ xz-zx-\beta x^2+(\beta+\gamma)yx, \quad \textit{where } \alpha+\beta+\gamma\neq 0\\ yz-zy-\alpha y^2+(\alpha+\gamma)xy\end{cases}$
	$(\alpha',\beta',\gamma')=\begin{cases}(\alpha,\beta,\gamma),(\alpha,\gamma,\beta),(\beta,\alpha,\gamma),\\(\beta,\gamma,\alpha),(\gamma,\alpha,\beta),(\gamma,\beta,\alpha)\end{cases}$ *in* $\mathbb{P}^2$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A,A' *of Type* T_1
T_2	$\begin{cases}x^2-y^2,\\ xz-zy-\beta xy+(\beta+\gamma)y^2, \quad \textit{where } \alpha+\beta+\gamma\neq 0\\ yz-zx-\alpha yx+(\alpha+\gamma)x^2\end{cases}$
	$(\alpha'+\beta',\gamma')=(\alpha+\beta,\gamma)$ *in* $\mathbb{P}^1$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A,A' *of Type* T_2
T_3	$\begin{cases}x^2-xy+y^2,\\ xz+zy+\beta xy-(\beta+\gamma)y^2, \quad \textit{where } \alpha+\beta+\gamma\neq 0\\ \alpha yx+\gamma y^2-yz+zx-zy\end{cases}$
	$(\alpha',\beta',\gamma')=(\alpha,\beta,\gamma),(\beta,\gamma,\alpha),(\gamma,\alpha,\beta)$ *in* $\mathbb{P}^2$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A,A' *of Type* T_3
T_1'	$\begin{cases}\alpha x^2+\beta(\alpha+\beta)xy-xz+zx-(\alpha+\beta)zy,\\ xy-yx-\beta y^2, \quad \textit{where } \alpha+2\beta\neq 0\\ 2\beta xy-\beta^2y^2-yz+zy\end{cases}$
	$(\alpha',\beta')=(\alpha,\beta)$ *in* $\mathbb{P}^1$
	$\mathrm{GrMod}A\cong\mathrm{GrMod}A'$ *for any* A,A' *of Type* T_1'

N	$\alpha xy - yx, \alpha yz - zy + x^2, \alpha zx - xz + y^2$, *where* $\alpha(\alpha^3 - 1) \neq 0$
	$\alpha' = \alpha^{\pm 1}$
	$\alpha'^3 = \alpha^{\pm 3}$
C	*Every algebra is isomorphic to* $k\langle x, y, z\rangle / \begin{pmatrix} xy - yx - x^2, \\ zx - xz - x^2 - 3y^2, \\ zy - yz + 3y^2 + 2xz + 2xy \end{pmatrix}$

The classification of the cases when E is a union of a line and a double line, a triple line, or an elliptic curve is not finished yet and now in progress.

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A CLASSIFICATION OF CYCLOTOMIC KLR ALGEBRAS OF TYPE $A_n^{(1)}$

MASAHIDE KONISHI

Abstract. A Khovanov-Lauda-Rouquier algebra (KLR algebra for short) is defined by these two data, a quiver Γ and a weight ν on its vertices. Furthermore we obtain a cyclotomic KLR algebra by fixing another weight Λ on vertices. There exists idempotents called KLR idempotents in KLR algebras, but they are not primitive in general.

In past report, we fix a quiver Γ type $A_n^{(1)}$, ν and Λ some special case then we showed all the non-zero KLR idempotents are primitive in the cyclotomic KLR algebra.

In this report, we start from that and fix a quiver Γ type $A_n^{(1)}$, obtain ν and Λ such that non-zero KLR idempotents are all primitive in the cyclotomic KLR algebra.

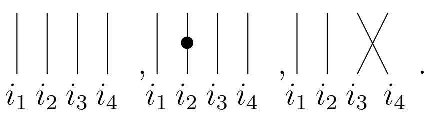
1. Definitions

At the beginning, we give definitions of KLR algebras and cyclotomic KLR algebras. Sometimes it is defined by using only generators and its relations, however the diagram interpretation of elements are quite useful, such as some statements are proved more simple. Because of that reason, we use diagrams in this report.

At first, we fix a quiver Γ without loops and multiple arrows. Each elements of its vertices set Γ_0 is used as colors of strands later, while the quiver are used for defining relations between diagrams.

Second, we fix a weight $\nu = \sum_{i \in \Gamma_0} a_i \nu_i (a_i \in \mathbb{Z}_{\geq 0})$ on vertices. This shows how many strands are there for each colors, furthermore the diagrams using $|\nu| = \sum_{i \in \Gamma_0} a_i$ strands are the generators of KLR algebras as a vector space.

We have not touched about what is the diagrams, roughly speaking, that is "colored braids with dots". There are some examples below;



We said "colored braid" just now, each i_k put below presents the color of the strand. Used colors are vertices of Γ (i.e. elements of Γ_0), furthermore the number of each colored strands is obtained from ν a weight on Γ_0. Those three diagrams are the main three kinds of diagrams, colored parallel strands, the dot and the crossing[1].

The detailed version of this paper will be submitted for publication elsewhere.

[1]The sum for colors is taken as dots and crossings.

Definition of the multiplication for two diagrams x and y is quite simple. We put the diadram y below the diagram x. If the colors of each strands then define the diagram xy as a concatenation, otherwise xy is 0. The leftmost diagram is an idempotent with this multiplication.

We put relations defined by quiver Γ to define a KLR algebra and take a quotient by an ideal defined from another weight Λ on Γ_0 to define a cyclotomic KLR algebra.

We use below notation for an information about colors. Set $m = |\nu|$,

$$\mathrm{Seq}(\nu) = \{(i_1, i_2, \cdots, i_m) \in (\Gamma_0)^m \mid \textit{each } i \in \Gamma_0 \textit{ appears } a_i \textit{ times}\}$$

For example, if $\Gamma_0 = \{0, 1\}$, $\nu = 2\nu_0 + \nu_1$, we get $\mathrm{Seq}(\nu) = \{(0,0,1), (0,1,0), (1,0,0)\}$.

We denote $\mathbf{e}(\mathbf{i})$ the diagram with parallel m strands colored $i_1, i_2, \cdots, i_m$ from left to right, $y_k\mathbf{e}(\mathbf{i})$ with a dot on kth strand, $\psi_l\mathbf{e}(\mathbf{i})$ with lth and $(l+1)$st strands crossed. We can obtain more complicated diagrams by fixing a shape with y_k and ψ_l and a color with $\mathbf{e}(\mathbf{i})$. y_k and ψ_l are the elements which took a sum about colors of strands, we can fix a color by multiplying $\mathbf{e}(\mathbf{i})$. In the definition below, there exists some cases we should divide relations by colors, so someimes put $\mathbf{e}(\mathbf{i})$.

Definition 1. KLR algebras $R_\Gamma(\nu)$ are defined by these generators and relations. Set $m = |\nu|$.

- Generators: $\{\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \mathrm{Seq}(\nu)\} \cup \{y_1, \cdots, y_m\} \cup \{\psi_1, \cdots, \psi_{m-1}\}$
- Relations:

$$\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}\mathbf{e}(\mathbf{i}),$$

$$\sum_{\mathbf{i}\in\mathrm{Seq}(\nu)} \mathbf{e}(\mathbf{i}) = 1,$$

$$y_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})y_k,$$

$$\psi_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{s_k}\cdot\mathbf{i})\psi_k \quad (\mathbf{s_k}\cdot\mathbf{i} = (i_1, \cdots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \cdots, i_m)),$$

$$y_k y_l = y_l y_k,$$

$$\psi_k y_l = y_l \psi_k \ (l \neq k, k+1),$$

$$\psi_k \psi_l = \psi_l \psi_k \ (|k-l| > 1),$$

$$\psi_k y_{k+1}\mathbf{e}(\mathbf{i}) = \begin{cases} (y_k\psi_k + 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+1}) \\ y_k\psi_k\mathbf{e}(\mathbf{i}) & (\textit{otherwise}) \end{cases},$$

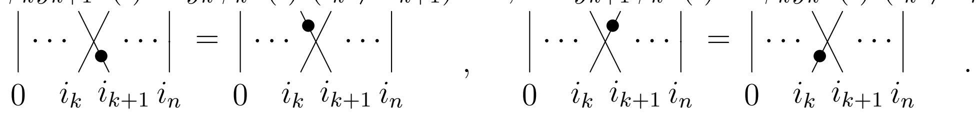
$$y_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \begin{cases} (\psi_k y_k + 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+1}) \\ \psi_k y_k\mathbf{e}(\mathbf{i}) & (\textit{otherwise}) \end{cases},$$

$$\psi_k^2\mathbf{e}(\mathbf{i}) = \begin{cases} 0 & (i_k = i_{k+1}) \\ \mathbf{e}(\mathbf{i}) & (\textit{no arrows between } i_k \textit{ and } i_{k+1}) \\ (y_{k+1} - y_k)\mathbf{e}(\mathbf{i}) & (i_k \to i_{k+1}) \\ (y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftarrow i_{k+1}) \\ (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftrightarrow i_{k+1}) \end{cases},$$

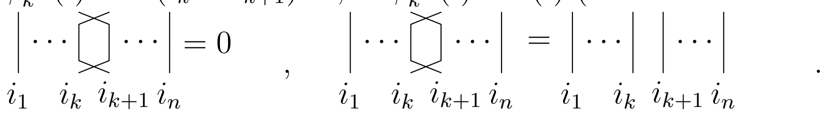
$$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \begin{cases} (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+2}\ ,\ i_k \to i_{k+1}) \\ (\psi_{k+1}\psi_k\psi_{k+1} - 1)\mathbf{e}(\mathbf{i}) & (i_k = i_{k+2}\ ,\ i_k \leftarrow i_{k+1}) \\ (\psi_{k+1}\psi_k\psi_{k+1} - 2y_{k+1} + y_k + y_{k+2})\mathbf{e}(\mathbf{i}) & (i_k = i_{k+2}\ ,\ i_k \leftrightarrow i_{k+1}) \\ \psi_{k+1}\psi_k\psi_{k+1}\mathbf{e}(\mathbf{i}) & (\textit{otherwise}) \end{cases},$$

We describe relations after 8th with diagrams from easier one.

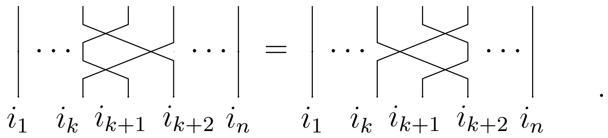
$\psi_k y_{k+1}\mathbf{e}(\mathbf{i}) = y_k\psi_k\mathbf{e}(\mathbf{i})$ $(i_k \neq i_{k+1})$, $y_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \psi_k y_k\mathbf{e}(\mathbf{i})$ $(i_k \neq i_{k+1})$.



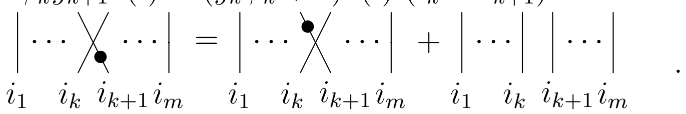
$\psi_k^2\mathbf{e}(\mathbf{i}) = 0$ $(i_k = i_{k+1})$, $\psi_k^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ (*no arrows between* i_k *and* i_{k+1}) .



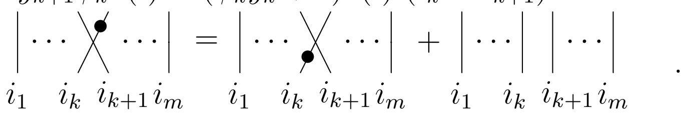
$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \psi_{k+1}\psi_k\psi_{k+1}\mathbf{e}(\mathbf{i})$ $(i_k \neq i_{k+2},$ *or no arrows between* i_k *and* $i_{k+1})$.



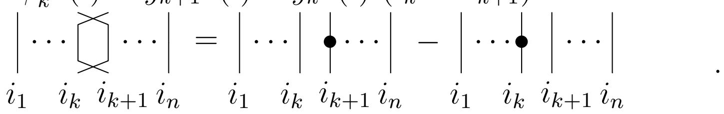
$\psi_k y_{k+1}\mathbf{e}(\mathbf{i}) = (y_k\psi_k + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+1})$



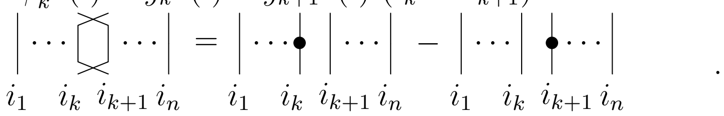
$y_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_k y_k + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+1})$



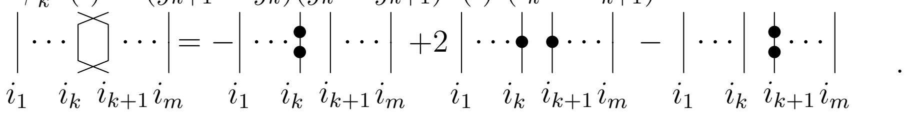
$\psi_k^2\mathbf{e}(\mathbf{i}) = y_{k+1}\mathbf{e}(\mathbf{i}) - y_k\mathbf{e}(\mathbf{i})$ $(i_k \to i_{k+1})$.



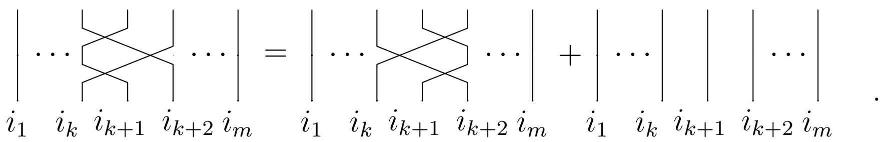
$\psi_k^2\mathbf{e}(\mathbf{i}) = y_k\mathbf{e}(\mathbf{i}) - y_{k+1}\mathbf{e}(\mathbf{i})$ $(i_k \leftarrow i_{k+1})$.



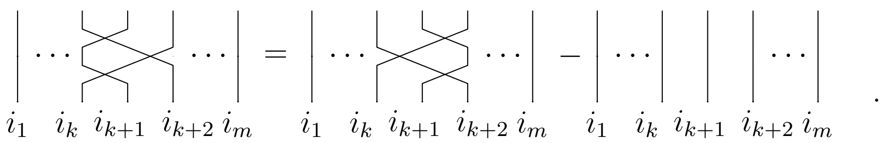
$\psi_k^2\mathbf{e}(\mathbf{i}) = (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i})$ $(i_k \leftrightarrow i_{k+1})$.



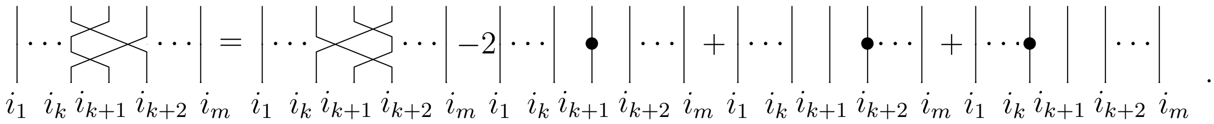
$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+2}\ ,\ i_k \to i_{k+1})$.



$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+2}\ ,\ i_k \leftarrow i_{k+1})$.



$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} - 2y_{k+1} + y_k + y_{k+2})\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+2}\ ,\ i_k \leftrightarrow i_{k+1})$.



We fix $\Lambda = \sum_{i\in\Gamma_0} b_i\Lambda_i (b_i \in \mathbb{Z}_{\geq 0})$ and let I^Λ be an ideal of $R_\Gamma(\nu)$ generated by $\{y_1^d\mathbf{e}(\mathbf{i}) \mid \mathbf{i} \in \mathrm{Seq}(\nu), d = b_{i_1}\}$. We call $R_\Gamma(\nu)^\Lambda = R_\Gamma(\nu)/I^\Lambda$ a cyclotomic KLR algebra.

Generators of the ideal are diagrams with b_{i_1} dots on the leftmost strand of each $\mathbf{e}(\mathbf{i})$. We see i_1 for $\mathbf{i}$ and then refer b_{i_1}, it is the place easy to confuse, be careful.

2. Problem

Those two relations break an idempotency of a KLR idenpotent $\mathbf{e}(\mathbf{i})$;

$\mathbf{e}(\mathbf{i}) = \psi_k y_{k+1}\mathbf{e}(\mathbf{i}) - y_k\psi_k\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+1})$,

$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = (\psi_{k+1}\psi_k\psi_{k+1} + 1)\mathbf{e}(\mathbf{i})$ $(i_k = i_{k+2}\ \ i_k \to i_{k+1})$.

Say inversely, only those two relations can break an idempotency. We note both relations can appear only if there exists a color used twice or more.

We can easily conclude that, on KLR algebras, the existence of a non-primitive KLR idempotent and a color used twice or more are equivalent.

However, on cyclotomic KLR algebras, sometimes there exists zero term in above relations and the above equivalence can be broken. There is one natural question, when all non-zero KLR idempotents are primitive on $R_\Gamma(\nu)$? (Characterize such ν and Γ!) To try to give an answer, we notice that depends on the shape of Γ [2].

We restrict the problem to the case "essentially type $A_n^{(1)}$" and obtain the answer.

Let a quiver $A_n^{(1)}$ $(n \geq 1)$ with vertices $\{0, \cdots, n\}$ and arrows from each k to $k+1$ $(0 \leq k \leq n-1)$ and from n to 0.

Moreover, assume $a_i > 0 (0 \leq i \leq n)$ in ν to reflect "essentially" the structure of KLR algebras.

We omit Γ from $R_\Gamma(\nu)^\Lambda$.

Theorem 2. *For a cyclotomic KLR algebraR_ν^Λ, all non-zero $\mathbf{e}(\mathbf{i})$ are primitive and ν and Λ satisfy one of followings are equivalent.*

(a) $R_\nu^\Lambda = 0$.

(b) $\nu = \sum_{0\leq i\leq n} \nu_i, \Lambda$ *is arbitrary.*

(c) $\nu = \sum_{0\leq i\leq n} \nu_i + \nu_k, \Lambda = \Lambda_k (0 \leq k \leq n)$.

[2] Roughly, the underlying graph of Γ is tree or not.

2.1. **Sketch of Proof.** Proof is done as following steps.
(i) Check for the case (b), (c).
(ii) Construct counterexample (non-zero non-primitive $\mathbf{e}(\mathbf{i})$) in "minimal case" about ν, Λ.
(iii) Check for induction on ν.
(iv) Check for induction on Λ.

We do (i) later and start with (ii) to (iv). Since the case $\Lambda = 0$ is included in case (a), we may assume $\Lambda \neq 0$. Since $A_n^{(1)}$ is rotation symmetry, we may assume $b_0 > 0$ in Λ.

In this situation we may take those two cases as (ii) minimal cases about ν,Λ:
(I) $\nu = \sum_{0 \leq i \leq n} \nu_i + \nu_k, \Lambda = \Lambda_1 \ (k \neq 1)$.
(II) $\nu = \sum_{0 \leq i \leq n} \nu_i + \nu_1, \Lambda = 2\Lambda_1$.

About (I), for example $k = n$, we set $\mathbf{i} = (0, 1, \cdots, n-1, n, n)$, then since $y_{n+1}\mathbf{e}(\mathbf{i}) \neq 0$ [3] and $y_{n+2}\mathbf{e}(\mathbf{i}) \neq 0$ [4], $\mathbf{e}(\mathbf{i})$ can be decomposed by the relation above.

About (II), we set $\mathbf{i} = (0, 0, 1, \cdots, n-1, n)$, then since $y_1\mathbf{e}(\mathbf{i}) \neq 0$ and $y_2\mathbf{e}(\mathbf{i}) \neq 0$, $\mathbf{e}(\mathbf{i})$ can be decomposed by the relation above.

(iii) Induction on ν. It's not in the case (b) hence there exists k satisfying $a_k \geq 2$. Moreover, it's not in the case (c) hence one of the followings is satisfied:
(O) There exists $l \neq k$ such that $b_l > 0$.
(T) $b_k \geq 2$.

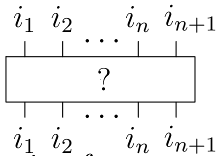
In both cases, since not in the case (a) there exists $\mathbf{i}$ with $\mathbf{e}(\mathbf{i}) \neq 0$. We try to construct $\mathbf{e}(\mathbf{i}') \neq 0$ like (I) or (II) by using $\mathbf{i}$. However to certify that we can't avoid using Graham-Lehrer conjecture[5] now, it is the most difficult part of the proof.

(iv) In the case (I), since I^Λ is maximal when $\Lambda = \Lambda_0$, non-zero non-primitive $\mathbf{e}(\mathbf{i})$ in $H_\nu^{\Lambda_0}$ is also in $H_\nu^{\Lambda'}$ where Λ' is another weight. For the case (II), we set $\Lambda = 2\Lambda_0$ and the same thing holds. [6]

We now back to (i). We can check it easily with following lemma.

Lemma 3. *Let A be associative algebra with unit, e be an idempotent in A. Then e is primitive and idempotents in eAe are only trivial two (0 and e) are equivalent.*

For (b), the elements in $\mathbf{e}(\mathbf{i})H_\nu^\Lambda\mathbf{e}(\mathbf{i})$ where $\mathbf{e}(\mathbf{i}) \neq 0$ are the linear combination of diagrams such as:



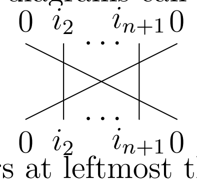
To fill up the "?" part, we use following fact:
"Every diagrams can be presented as linear combination of diagrams in which each strands cross at most once. "

[3]Refer [3].
[4]If the same color continues then "the number of dots we can put" is the same [1].
[5]solved.
[6]Comparing the case (I), we miss only the cases $\Lambda = c\Lambda_0$ ($c > 2$) in (II).

In this case, since each strands has different color we get diagrams in which no strands cross, in the other words, parallel strands and dots. We can only do "vanish the diagram with some dots" or "swap the dots".

Then $\mathbf{e}(\mathbf{i})H_\nu^\Lambda\mathbf{e}(\mathbf{i})$ is isomorphic to a quotient of polynomial algebra with m indeterminants by some homogeneous polynomials. Moreover, idempotents in that algebra is only 0 or 1.

For (c) we can apply the same method but be careful about $i_1 = 0$ and there are two strands colored 0. In this case, these diagrams can appear:



However in this case $i_2 \neq 0$ appears at leftmost then it is 0. Crossing can be appeared at leftmost, but in that case there is two 0 strands at leftmost then it is 0. Hence the same as case (b), there are only diagrams with parallel strands and dots.

That is the sketch of the proof.

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HEARTS OF TWIN COTORSION PAIRS ON EXACT CATEGORIES

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ABSTRACT. In the papers of Nakaoka, he introduced the notion of hearts of (twin) cotorsion pairs on triangulated categories and showed that they have structures of (semi-) abelian categories. We study in this article a twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ on an exact category $\mathcal{B}$ with enough projectives and injectives and introduce the notion of the heart. The heart of a twin cotorsion pair on $\mathcal{B}$ is semi-abelian. Moreover, the heart of a single cotorsion pair is abelian. These results are analog of Nakaoka's results in triangulated categories.

1. INTRODUCTION

The cotorsion pairs were first introduced by Salce, and it has been deeply studied in the representation theory during these years, especially in tilting theory and Cohen-Macaulay modules. Recently, the cotorsion pair are also studied in triangulated categories [3], in particular, Nakaoka introduced the notion of hearts of cotorsion pairs and showed that the hearts are abelian categories [6]. This is a generalization of the hearts of t-structure in triangulated categories [1] and the quotient of triangulated categories by cluster tilting subcategories [4]. Moreover, he generalized these results to a more general setting called twin cotorsion pair [7].

The aim of this paper is to give similar results for cotorsion pairs on Quillen's exact categories, which plays an important role in representation theory. An important class of exact categories is given by Frobenius categories, which gives most of important triangulated categories appearing in representation theory. We consider a *cotorsion pair* in an exact category, which is a pair $(\mathcal{U},\mathcal{V})$ of subcategories of an exact category $\mathcal{B}$ satisfying $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V})=0$ (*i.e.* $\mathrm{Ext}^1_{\mathcal{B}}(U,V)=0$, $\forall U\in\mathcal{U}$ and $\forall V\in\mathcal{V}$) and any $B\in\mathcal{B}$ admits two short exact sequences $V_B\rightarrowtail U_B\twoheadrightarrow B$ and $B\rightarrowtail V^B\twoheadrightarrow U^B$ where $V_B,V^B\in\mathcal{V}$ and $U_B,U^B\in\mathcal{U}$. Let

$$\mathcal{B}^+:=\{B\in\mathcal{B}\mid \mathcal{U}_B\in\mathcal{V}\},\quad \mathcal{B}^-:=\{B\in\mathcal{B}\mid \mathcal{V}^B\in\mathcal{U}\}.$$

We define the *heart* of $(\mathcal{U},\mathcal{V})$ as the quotient category

$$\underline{\mathcal{H}}:=(\mathcal{B}^+\cap\mathcal{B}^-)/(\mathcal{U}\cap\mathcal{V}).$$

Now we state our first main result, which is an analogue of [6, Theorem 6.4].

Theorem 1. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on an exact category $\mathcal{B}$ with enough projectives and injectives. Then $\underline{\mathcal{H}}$ is abelian.*

The detailed version [5] of this paper has been submitted for publication.

Moreover, following Nakaoka, we consider pairs of cotorsion pairs $(\mathcal{S},\mathcal{T})$ and $(\mathcal{U},\mathcal{V})$ in $\mathcal{B}$ such that $\mathcal{S}\subseteq\mathcal{U}$, we also call such a pair a *twin cotorsion pair*. The notion of hearts is generalized to such pairs, and our second main result is the following, which is an analogue of [7, Theorem 5.4].

Theorem 2. *Let $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair on $\mathcal{B}$. Then the heart of this cotorsion pair is semi-abelian.*

2. Notations

For briefly review of the important properties of exact categories, we refer to [5, §2]. For more details, we refer to [2].

Throughout this paper, let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the full subcategory of projectives (resp. injectives) of $\mathcal{B}$.

Definition 3. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{B}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\operatorname{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V})=0$.

(b) For any object $B\in\mathcal{B}$, there exits two short exact sequences

$$V_B\rightarrowtail U_B\twoheadrightarrow B,\quad B\rightarrowtail V^B\twoheadrightarrow U^B$$

satisfying $U_B,U^B\in\mathcal{U}$ and $V_B,V^B\in\mathcal{V}$.

By definition of a cotorsion pair, we can immediately conclude:

Proposition 4. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair of $\mathcal{B}$, then*

(a) *B belongs to $\mathcal{U}$ if and only if $\operatorname{Ext}^1_{\mathcal{B}}(B,\mathcal{V})=0$.*

(b) *B belongs to $\mathcal{V}$ if and only if $\operatorname{Ext}^1_{\mathcal{B}}(\mathcal{U},B)=0$.*

(c) *$\mathcal{U}$ and $\mathcal{V}$ are closed under extension.*

(d) *$\mathcal{P}\subseteq\mathcal{U}$ and $\mathcal{I}\subseteq\mathcal{V}$.*

Definition 5. A pair of cotorsion pairs $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ on $\mathcal{B}$ is called a *twin cotorsion pair* if it satisfies:

$$\mathcal{S}\subseteq\mathcal{U}.$$

By the definition of the cotorsion pair and Proposition 4 this condition is equivalent to $\operatorname{Ext}^1_{\mathcal{B}}(\mathcal{S},\mathcal{V})=0$, and also to $\mathcal{V}\subseteq\mathcal{T}$.

Definition 6. For any twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, put

$$\mathcal{W}:=\mathcal{T}\cap\mathcal{U}.$$

(a) $\mathcal{B}^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$V_B\rightarrowtail U_B\twoheadrightarrow B$$

where $U_B\in\mathcal{W}$ and $V_B\in\mathcal{V}$.

(b) $\mathcal{B}^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$B \rightarrowtail T^B \twoheadrightarrow S^B$$

where $T^B \in \mathcal{W}$ and $S^B \in \mathcal{S}$.

Definition 7. Let $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair of $\mathcal{B}$, we denote the quotient of $\mathcal{B}$ by $\mathcal{W}$ as $\underline{\mathcal{B}} := \mathcal{B}/\mathcal{W}$. For any subcategory $\mathcal{C}$ of $\mathcal{B}$, we denote by $\underline{\mathcal{C}}$ the subcategory of $\underline{\mathcal{B}}$ consisting of the same objects as $\mathcal{C}$. Put

$$\mathcal{H} := \mathcal{B}^+ \cap \mathcal{B}^-.$$

Since $\mathcal{H} \supseteq \mathcal{W}$, we have an additive full quotient subcategory

$$\underline{\mathcal{H}} := \mathcal{H}/\mathcal{W}$$

which we call the *heart* of twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$.
The heart of a cotorsion pair $(\mathcal{U},\mathcal{V})$ is defined to be the heart of twin cotorsion pair $(\mathcal{U},\mathcal{V})$, $(\mathcal{U},\mathcal{V})$.

3. Main results

A additive category is called preabelian if every morphism has both kernel and cokernel.

Proposition 8. *For any twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, its heart $\underline{\mathcal{H}}$ is preabelian.*

A preabelian category is abelian if every monomorphism is a kernel and every epimorphism is a cokernel. For a single cotorsion pair, we have the following result:

Theorem 9. *For any cotorsion pair $(\mathcal{U},\mathcal{V})$ on $\mathcal{B}$, its heart $\underline{\mathcal{H}}$ is an abelian category.*

For the hearts of twin cotorsion pairs, we can not get the same result.

Definition 10. A preabelian category $\mathcal{A}$ is called *left semi-abelian* if in any pull-back diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"'] & D \end{tikzcd}$$

in $\mathcal{A}$, α is an epimorphism whenever δ is a cokernel. *Right semi-abelian* is defined dually. $\mathcal{A}$ is called *semi-abelian* if it is both left and right semi-abelian.

We can prove the following theorem.

Theorem 11. *For any twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, its heart $\underline{\mathcal{H}}$ is semi-abelian.*

4. examples

In this section we give several examples of twin cotorsion pair, and we also give some view of the relation between the heart of a cotorsion pair and the hearts of its two components.

Recall that $\mathcal{M}$ is *cluster tilting* if it satisfies the following conditions:

(a) $\mathcal{M}$ is contravariantly finite and covariantly finite in $\mathcal{B}$.

(b) $\mathcal{M}^{\perp_1} = \mathcal{M}$.
(c) ${}^{\perp_1}\mathcal{M} = \mathcal{M}$.

Proposition 12. *A subcategory $\mathcal{M}$ in $\mathcal{B}$ is cluster tilting if and only if $(\mathcal{M}, \mathcal{M})$ is a cotorsion pair on $\mathcal{B}$.*

Proposition 13. *If $\mathcal{M}$ is a cluster tilting subcategory, then the heart of $(\mathcal{M}, \mathcal{M})$ is $\mathcal{B}/\mathcal{M}$.*

In the following examples, we denote by "$\circ$" in a quiver the objects belong to a subcategory and by "$\cdot$" the objects do not.

Example 14. Let Λ be the path algebra of the following quiver

$$\begin{tikzcd} 1 & 2 \arrow[l] & 3 \arrow[l] & 4 \arrow[l] \arrow[lll, dotted, dash, bend right] \end{tikzcd}$$

then we obtain the AR-quiver of $\operatorname{mod}\Lambda$.

$$\begin{tikzcd}[column sep=small] 1 \arrow[dr] \arrow[rr, dotted, dash] & & 2 \arrow[dr] \arrow[rr, dotted, dash] & & 3 \arrow[dr] \arrow[rr, dotted, dash] & & 4 \\ & {\begin{smallmatrix}2\\1\end{smallmatrix}} \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & {\begin{smallmatrix}3\\2\end{smallmatrix}} \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & {\begin{smallmatrix}4\\3\end{smallmatrix}} \arrow[ur] & \\ & & {\begin{smallmatrix}3\\2\\1\end{smallmatrix}} \arrow[ur] & & {\begin{smallmatrix}4\\3\\2\end{smallmatrix}} \arrow[ur] & & \end{tikzcd}$$

Let $\mathcal{M} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_{\mathcal{B}}(X, \Lambda) = 0\}$, then $(\mathcal{M}, \mathcal{M}^{\perp_1})$ is a cotorsion pair on $\operatorname{mod}\Lambda$. But

$$\mathcal{M} = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \circ \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

which consisting of all the direct sums of indecomposable projectives and indecomposable injectives. We observe that in fact $\mathcal{M} = \mathcal{M}^{\perp_1}$ and hence it is a cluster tilting subcategory. And the quiver of the quotient category $(\operatorname{mod}\Lambda)/\mathcal{M}$ is

$$\begin{tikzcd} 2 \arrow[dr] \arrow[rr, dotted, dash] & & 3 \\ & {\begin{smallmatrix}3\\2\end{smallmatrix}} \arrow[ur] & \end{tikzcd}$$

which is equivalent to the AR-quiver of A_2.

Example 15. Take the notion of the former example, Let

$$\mathcal{M}' = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

then $(\mathcal{M}', \mathcal{M}'^{\perp_1})$ is a cotorsion pair and

$$\mathcal{M}'^{\perp_1} = \begin{array}{ccccccc} \circ & & \cdot & & \circ & & \circ \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

hence it contains Λ. Obviously it is closed under extension and contravariantly finite, then $(\mathcal{M}'^{\perp_1}, (\mathcal{M}'^{\perp_1})^{\perp_1})$ is also a cotorsion pair on $\operatorname{mod}\Lambda$ and

$$(\mathcal{M}'^{\perp_1})^{\perp_1} = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \circ \\ & \cdot & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

Thus we get a twin cotorsion pair

$$(\mathcal{M}', \mathcal{M}'^{\perp_1}), (\mathcal{M}'^{\perp_1}, (\mathcal{M}'^{\perp_1})^{\perp_1})$$

The heart of this twin cotorsion pair is $(\operatorname{mod}\Lambda)/\mathcal{M}'^{\perp_1}$, and the AR-quiver of it is ${\scriptstyle 2} \to {\scriptstyle \substack{3\\2}}$. Thus it is not abelian.

Example 16. Let Λ be the k-algebra given by the quiver

$$\begin{tikzcd} & 1 \arrow[dl, "\beta"'] & \\ 3 \arrow[rr, "\gamma"'] & & 2 \arrow[ul, "\alpha"'] \end{tikzcd}$$

and bound by $\alpha\beta = 0$ and $\beta\gamma\alpha = 0$. Then the AR-quiver of $\operatorname{mod}\Lambda$ is given by

$$\begin{tikzcd}[column sep=small] 1 \arrow[dr] \arrow[rr, dotted, no head] & & 2 \arrow[dr] \arrow[rr, dotted, no head] & & 3 \arrow[dr] \arrow[rr, dotted, no head] & & 1 \\ & \substack{2\\1} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \substack{3\\2} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \substack{1\\3} \arrow[ur] & \\ & & \substack{3\\2\\1} \arrow[ur] & & \substack{1\\3\\2} \arrow[ur] & & \end{tikzcd}$$

Here, the first and the last columns are identified. Let

$$\mathcal{S} = \begin{array}{ccccccc} \cdot & & \circ & & \cdot & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array} \quad \mathcal{T} = \begin{array}{ccccccc} \cdot & & \cdot & & \circ & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

and

$$\mathcal{U} = \begin{array}{ccccccc} \circ & & \circ & & \cdot & & \circ \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array} \quad \mathcal{V} = \begin{array}{ccccccc} \cdot & & \cdot & & \cdot & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

The heart of cotorsion pair $(\mathcal{S}, \mathcal{T})$ is $\operatorname{add}({\scriptstyle 1})$ and the heart of cotorsion pair $(\mathcal{U}, \mathcal{V})$ is $\operatorname{add}({\scriptstyle 3})$. But when we consider the twin cotorsion pair $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$, we get $\mathcal{W} = \mathcal{V}$ and

$$(\operatorname{mod}\Lambda)^-/\mathcal{W} = \operatorname{add}({\scriptstyle 1} \oplus {\scriptstyle 2}) \text{ and } (\operatorname{mod}\Lambda)^+/\mathcal{W} = \operatorname{add}({\scriptstyle 3})$$

hence its heart is zero.

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STABLE SET POLYTOPES OF TRIVIALLY PERFECT GRAPHS

KAZUNORI MATSUDA

Abstract. We give necessary and sufficient conditions for strong Koszulness of toric rings associated with stable set polytopes of graphs.

1. Introduction

Let G be a simple graph on the vertex set $V(G) = [n]$ with the edge set $E(G)$. $S \subset V(G)$ is said to be *stable* if $\{i, j\} \notin E(G)$ for all $i, j \in S$. Note that $\emptyset$ is stable. For each stable set S of G, we define $\rho(S) = \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^n$, where $\mathbf{e}_i$ is the i-th unit coordinate vector in $\mathbb{R}^n$.

The convex hull of $\{\rho(S) \mid S$ is a stable set of $G\}$ is called the *stable set polytope* of G (see [2]) , denoted by $\mathcal{Q}_G$. $\mathcal{Q}_G$ is a kind of $(0, 1)$-polytope. For this polytope, we define the subring of $k[T, X_1, \ldots, X_n]$ as follows:

$$k[\mathcal{Q}_G] := k[T \cdot X_1^{a_1} \cdots X_n^{a_n} \mid (a_1, \ldots, a_n) \text{ is a vertex of } \mathcal{Q}_G],$$

where k is a field. $k[\mathcal{Q}_G]$ is called the *toric ring associated with the stable set polytope of* G. We can regard $k[\mathcal{Q}_G]$ as a graded k-algebra by setting $\deg T \cdot X_1^{a_1} \cdots X_n^{a_n} = 1$.

In the theory of graded algebras, the notion of Koszulness (introduced by Priddy [15]) plays an important role and is closely related to the Gröbner basis theory.

Let $\mathcal{P}$ be an integral convex polytope (i.e., a convex polytope each of whose vertices has integer coordinates) and $k[\mathcal{P}] := k[T \cdot X_1^{a_1} \cdots X_n^{a_n} \mid (a_1, \ldots, a_n)$ is a vertex of $\mathcal{P}]$ be the toric ring associated with $\mathcal{P}$. In general, it is known that

$$\begin{gathered}\text{The defining ideal of } k[\mathcal{P}] \text{ possesses a quadratic Gröbner basis} \\ \Downarrow \\ k[\mathcal{P}] \text{ is Koszul} \\ \Downarrow \\ \text{The defining ideal of } k[\mathcal{P}] \text{ is generated by quadratic binomials}\end{gathered}$$

follows from general theory (for example, see [1]).

In this note, we study the notion of a *strongly Koszul* algebra. In [7], Herzog, Hibi, and Restuccia introduced this concept and discussed the basic properties of strongly Koszul algebras. Moreover, they proposed the conjecture that the strong Koszulness of R is at the top of the above hierarchy, that is,

Conjecture 1 (see [7]). *The defining ideal of a strongly Koszul algebra $k[\mathcal{P}]$ possesses a quadratic Gröbner basis.*

The final version of this paper has been submitted for publication elsewhere.

A ring R is *trivial* if R can be constructed by starting from polynomial rings and repeatedly applying tensor and Segre products. In this note, we propose the following conjecture.

Conjecture 2. *Let $\mathcal{P}$ be a $(0,1)$-polytope and $k[\mathcal{P}]$ be the toric ring generated by $\mathcal{P}$. If $k[\mathcal{P}]$ is strongly Koszul, then $k[\mathcal{P}]$ is trivial.*

In the case of a $(0,1)$-polytope, Conjecture 2 implies Conjecture 1. If $\mathcal{P}$ is an order polytope or an edge polytope of bipartite graphs, then Conjecture 2 holds true [7].

In this note, we prove Conjecture 2 for stable set polytopes. The main theorem of this note is the following:

Theorem 3 ([13])**.** *Let G be a graph. Then the following assertions are equivalent:*

(1) *$k[\mathcal{Q}_G]$ is strongly Koszul.*
(2) *G is a trivially perfect graph.*

In particular, if $k[\mathcal{Q}_G]$ is strongly Koszul, then $k[\mathcal{Q}_G]$ is trivial.

Throughout this note, we will use the standard terminologies of graph theory in [4].

2. Strongly Koszul algebra

Let k be a field, R be a graded k-algebra, and $\mathfrak{m} = R_+$ be the homogeneous maximal ideal of R.

Definition 4 ([7])**.** A graded k-algebra R is said to be *strongly Koszul* if $\mathfrak{m}$ admits a minimal system of generators $\{u_1, \dots, u_t\}$ which satisfies the following condition:

> For all subsequences $u_{i_1}, \dots, u_{i_r}$ of $\{u_1, \dots, u_t\}$ $(i_1 \leq \cdots \leq i_r)$ and for all $j = 1, \dots, r-1$, $(u_{i_1}, \dots, u_{i_{j-1}}) : u_{i_j}$ is generated by a subset of elements of $\{u_1, \dots, u_t\}$.

A graded k-algebra R is called Koszul if $k = R/\mathfrak{m}$ has a linear resolution. By the following theorem, we can see that a strongly Koszul algebra is Koszul.

Proposition 5 ([7, Theorem 1.2])**.** *If R is strongly Koszul with respect to the minimal homogeneous generators $\{u_1, \dots, u_t\}$ of $\mathfrak{m} = R_+$, then for all subsequences $\{u_{i_1}, \dots, u_{i_r}\}$ of $\{u_1, \dots, u_t\}$, $R/(u_{i_1}, \dots, u_{i_r})$ has a linear resolution.*

The following proposition plays an important role in the proof of the main theorem.

Theorem 6 ([7, Proposition 2.1])**.** *Let S be a semigroup and $R = k[S]$ be the semigroup ring generated by S. Let $\{u_1, \dots, u_t\}$ be the generators of $\mathfrak{m} = R_+$ which correspond to the generators of S. Then, if R is strongly Koszul, then for all subsequences $\{u_{i_1}, \dots, u_{i_r}\}$ of $\{u_1, \dots, u_t\}$, $R/(u_{i_1}, \dots, u_{i_r})$ is also strongly Koszul.*

By this theorem, we have

Corollary 7 (see [14])**.** *If $k[\mathcal{Q}_G]$ is strongly Koszul, then $k[\mathcal{Q}_{G_W}]$ is strongly Koszul for all induced subgraphs G_W of G.*

3. Hibi ring and comparability graph

In this section, we introduce the concepts of a Hibi ring and a comparability graph. Both are defined with respect to a partially ordered set.

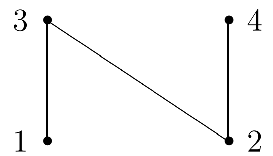
Let $P = \{p_1, \ldots, p_n\}$ be a finite partially ordered set consisting of n elements, which is referred to as a *poset*. Let $J(P)$ be the set of all poset ideals of P, where a poset ideal of P is a subset I of P such that if $x \in I$, $y \in P$, and $y \leq x$, then $y \in I$. Note that $\emptyset \in J(P)$.

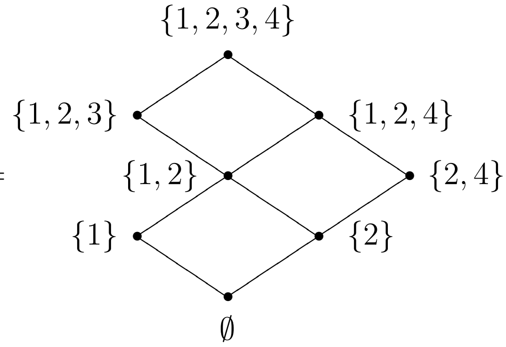
First, we give the definition of the Hibi ring introduced by Hibi.

Definition 8 ([8])**.** For a poset $P = \{p_1, \ldots, p_n\}$, the *Hibi ring* $\mathcal{R}_k[P]$ is defined as follows:

$$\mathcal{R}_k[P] := k[T \cdot \prod_{i \in I} X_i \mid I \in J(P)] \subset k[T, X_1, \ldots, X_n]$$

Example 9. Consider the following poset $P = (1 \leq 3, 2 \leq 3 \text{ and } 2 \leq 4)$.

$P =$ 

$J(P) =$ 

Then we have

$$\mathcal{R}_k[P] = k[T, TX_1, TX_2, TX_1X_2, TX_2X_4, TX_1X_2X_3, TX_1X_2X_4, TX_1X_2X_3X_4].$$

Hibi showed that a Hibi ring is always normal. Moreover, a Hibi ring can be represented as a factor ring of a polynomial ring: if we let

$$I_P := (X_I X_J - X_{I \cap J} X_{I \cup J} \mid I, J \in J(P), I \not\subseteq J \text{ and } J \not\subseteq I)$$

be the binomial ideal in the polynomial ring $k[X_I \mid I \in J(P)]$ defined by a poset P, then $\mathcal{R}_k[P] \cong k[X_I \mid I \in J(P)]/I_P$. Hibi also showed that I_P has a quadratic Gröbner basis for any term order which satisfies the following condition: the initial term of $X_I X_J - X_{I \cap J} X_{I \cup J}$ is $X_I X_J$. Hence a Hibi ring is always Koszul from general theory.

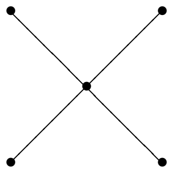
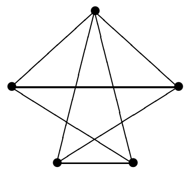
Next, we introduce the concept of a comparability graph.

Definition 10. A graph G is called a *comparability graph* if there exists a poset P which satisfies the following condition:

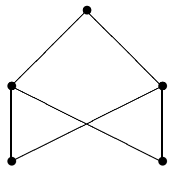
$$\{i, j\} \in E(G) \iff i \geq j \quad \text{or} \quad i \leq j \quad \text{in} \quad P.$$

We denote the comparability graph of P by $G(P)$.

Example 11. The lower-left poset P defines the comparability graph $G(P)$.

$P =$  $G(P) =$ 

Remark 12. It is possible that $P \neq P'$ but $G(P) = G(P')$. Indeed, for the following poset P', $G(P')$ is identical to $G(P)$ in the above example.

$P' =$ 

Complete graphs are comparability graphs of totally ordered sets. Bipartite graphs and trivially perfect graphs (see the next section) are also comparability graphs. Moreover, if G is a comparability graph, then the suspension (e.g., see [11, p.4]) of G is also a comparability graph.

Recall the following definitions of two types of polytope which are defined by a poset.

Definition 13 (see [16]). Let $P = \{p_1, \ldots, p_n\}$ be a finite poset.

(1) The *order polytope* $\mathcal{O}(P)$ of P is the convex polytope which consists of $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $0 \leq a_i \leq 1$ with $a_i \geq a_j$ if $p_i \leq p_j$ in P.

(2) The *chain polytope* $\mathcal{C}(P)$ of P is the convex polytope which consists of $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $0 \leq a_i \leq 1$ with $a_{i_1} + \cdots + a_{i_k} \leq 1$ for all maximal chain $p_{i_1} < \cdots < p_{i_k}$ of P.

Let $\mathcal{C}(P)$ and $\mathcal{O}(P)$ be the chain polytope and order polytope of a finite poset P, respectively. In [16], Stanley proved that

$$\{\text{The vertices of } \mathcal{O}(P)\} = \{\rho(I) \mid I \text{ is a poset ideal of } P\},$$
$$\{\text{The vertices of } \mathcal{C}(P)\} = \{\rho(A) \mid A \text{ is an anti-chain of } P\},$$

where $A = \{p_{i_1}, \ldots, p_{i_k}\}$ is an anti-chain of P if $p_{i_s} \not\leq p_{i_t}$ and $p_{i_s} \not\geq p_{i_t}$ for all $s \neq t$. Hence we have $\mathcal{Q}_{G(P)} = \mathcal{C}(P)$.

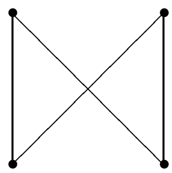
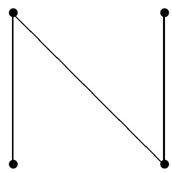
In [9], Hibi and Li answered the question of when $\mathcal{C}(P)$ and $\mathcal{O}(P)$ are unimodularly equivalent. From their study, we have the following theorem.

Theorem 14 ([9, Theorem 2.1]). *Let P be a poset and $G(P)$ be the comparability graph of P. Then the following are equivalent:*

(1) *The X-poset in Example 3.4 does not appear as a subposet* (*refer to* [17, *Chapter* 3]) *of P.*

(2) $\mathcal{R}_k[P] \cong k[\mathcal{Q}_{G(P)}]$.

Example 15. The cycle of length 4 C_4 and the path of length 3 P_4 are comparability graphs of Q_1 and Q_2, respectively.

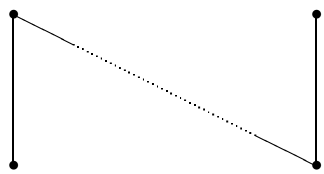
$Q_1 =$  $Q_2 =$ 

Hence $k[\mathcal{Q}_{C_4}] \cong \mathcal{R}_k[Q_1]$ and $k[\mathcal{Q}_{P_4}] \cong \mathcal{R}_k[Q_2]$.

A ring R is *trivial* if R can be constructed by starting from polynomial rings and repeatedly applying tensor and Segre products. Herzog, Hibi and Restuccia gave an answer for the question of when is a Hibi ring strongly Koszul.

Theorem 16 (see [7, Theorem 3.2]). *Let P be a poset and $R = \mathcal{R}_k[P]$ be the Hibi ring constructed from P. Then the following assertions are equivalent:*

(1) *R is strongly Koszul.*
(2) *R is trivial.*
(3) *The N-poset as described below does not appear as a subposet of P.*



By this theorem, Corollary 7, and Example 15, we have

Corollary 17. *If G contains C_4 or P_4 as an induced subgraph, then $k[\mathcal{Q}_G]$ is not strongly Koszul.*

4. TRIVIALLY PERFECT GRAPH

In this section, we introduce the concept of a trivially perfect graph. As its name suggests, a trivially perfect graph is a kind of perfect graph; it is also a kind of comparability graph, as described below.

Definition 18. For a graph G, we set

$$\alpha(G) := \max\{\#S \mid S \text{ is a stable set of } G\},$$

$$m(G) := \#\{\text{the set of maximal cliques of } G\}.$$

We call $\alpha(G)$ the *stability number* (or *independence number*) of G.

In general, $\alpha(G) \leq m(G)$. Moreover, if G is chordal, then $m(G) \leq n$ by Dirac's theorem [5]. In [6], Golumbic introduced the concept of a trivially perfect graph.

Definition 19 ([6])**.** We say that a graph G is *trivially perfect* if $\alpha(G_W) = m(G_W)$ for any induced subgraph G_W of G.

For example, complete graphs and star graphs (i.e., the complete bipartite graph $K_{1,r}$) are trivially perfect.

We define some additional concepts related to perfect graphs. Let C_G be the set of all cliques of G. Then we define

$$\omega(G) := \max\{\#C \mid C \in C_G\},$$
$$\theta(G) := \min\{s \mid C_1 \coprod \cdots \coprod C_s = V(G), C_i \in C_G\},$$
$$\chi(G) := \theta(\overline{G}),$$

where $\overline{G}$ is the complement of G. These invariants are called the *clique number*, *clique covering number*, and *chromatic number* of G, respectively.

In general, $\alpha(G) = \omega(\overline{G})$, $\theta(G) \le m(G)$ and $\omega(G) \le \chi(G)$. The definition of a perfect graph is as follows.

Definition 20. We say that a graph G is *perfect* if $\omega(G_W) = \chi(G_W)$ for any induced subgraph G_W of G.

Lovász proved that G is perfect if and only if $\overline{G}$ is perfect [12]. The theorem is now called the weak perfect graph theorem. With it, it is easy to show that a trivially perfect graph is perfect.

Proposition 21. *Trivially perfect graphs are perfect.*

Proof. Assume that G is trivially perfect. By [12], it is enough to show that $\overline{G}$ is perfect. For all induced subgraphs $\overline{G}_W$ of $\overline{G}$, we have

$$m(G_W) = \alpha(G_W) = \omega(\overline{G_W}) \le \chi(\overline{G_W}) = \theta(G_W) \le m(G_W)$$

by general theory (note that $\overline{G}_W = \overline{G_W}$). □

Golumbic gave a characterization of trivially perfect graphs.

Theorem 22 ([6, Theorem 2])**.** *The following assertions are equivalent:*

(1) *G is trivially perfect.*

(2) *G is C_4, P_4-*free*, that is, G contains neither C_4 nor P_4 as an induced subgraph.*

Proof. (1) $\Rightarrow$ (2): It is clear since $\alpha(C_4) = 2$, $m(C_4) = 4$, and $\alpha(P_4) = 2$, $m(P_4) = 3$.

(2) $\Rightarrow$ (1): Assume that G contains neither C_4 nor P_4 as an induced subgraph. If G is not trivially perfect, then there exists an induced subgraph G_W of G such that $\alpha(G_W) < m(G_W)$. For this G_W, there exists a maximal stable set S_W of G_W which satisfies the following:

There exists $s \in S_W$ such that $s \in C_1 \cap C_2$ for some distinct pair of cliques $C_1, C_2 \in C_{G_W}$.

Note that $\#S_W > 1$ since G_W is not complete. Then there exist $x \in C_1$ and $y \in C_2$ such that $\{x, s\}$, $\{y, s\} \in E(G_W)$ and $\{x, y\} \notin E(G_W)$.

Let $u \in S_W \setminus \{s\}$. If $\{x, u\} \in E(G_W)$ or $\{y, u\} \in E(G_W)$, then the induced graph $G_{\{x,y,s,u\}}$ is C_4 or P_4, a contradiction. Hence $\{x, u\} \notin E(G_W)$ and $\{y, u\} \notin E(G_W)$. Then

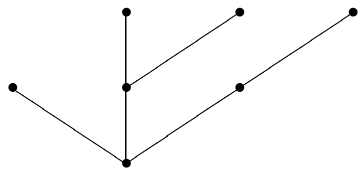
$\{x, y\} \cup \{S \setminus \{s\}\}$ is a stable set of G_W, which contradicts that S is maximal. Therefore, G is trivially perfect. □

Next, we show that a trivially perfect graph is a kind of comparability graph. First, we define the notion of a tree poset.

Definition 23 (see [18])**.** A poset P is a *tree* if it satisfies the following conditions:

(1) Each of the connected components of P has a minimal element.
(2) For all p, $p^{'} \in P$, the following assertion holds: if there exists $q \in P$ such that $p, p^{'} \leq q$, then $p \leq p^{'}$ or $p \geq p^{'}$.

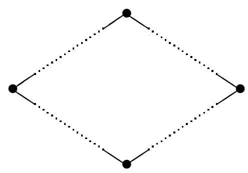
Example 24. The following poset is a tree:



Tree posets can be characterized as follows.

Proposition 25. *Let P be a poset. Then the following assertions are equivalent:*

(1) *P is a tree.*
(2) *Neither the X-poset in Example 11, the N-poset in Theorem 16, nor the diamond poset as described below appears as a subposet of P.*



In [18], Wolk discussed the properties of the comparability graphs of a tree poset and showed that such graphs are exactly the graphs that satisfy the "diagonal condition". This condition is equivalent to being C_4, P_4-free, and hence we have

Corollary 26. *Let G be a graph. Then the following assertions are equivalent:*

(1) *G is trivially perfect.*
(2) *G is a comparability graph of a tree poset.*
(3) *G is C_4, P_4-free.*

Remark 27. A graph G is a *threshold graph* if it can be constructed from a one-vertex graph by repeated applications of the following two operations:

(1) Add a single isolated vertex to the graph.
(2) Take a suspension of the graph.

The concept of a threshold graph was introduced by Chvátal and Hammer [3]. They proved that G is a threshold graph if and only if G is C_4, P_4, $2K_2$-free. Hence a trivially perfect graph is also called a *quasi-threshold graph.*

5. Proof of Main theorem

In this section, we prove the main theorem.

Theorem 28 ([13])**.** *Let G be a graph. Then the following assertions are equivalent:*

(1) *$k[\mathcal{Q}_G]$ is strongly Koszul.*
(2) *G is trivially perfect.*

Proof. We assume that G is trivially perfect. Then there exists a tree poset P such that $G = G(P)$ from Corollary 26. This implies that neither the X-poset in Example 11 nor the N-poset in Theorem 16 appears as a subposet of P by Proposition 25, and hence $k[\mathcal{Q}_{G(P)}] \cong \mathcal{R}_k[P]$ is strongly Koszul by Theorems 14 and 16.

Conversely, if G is not trivially perfect, G contains C_4 or P_4 as an induced subgraph by Corollary 26. Therefore, we have that $k[\mathcal{Q}_G]$ is not strongly Koszul by Corollary 17. $\square$

Remark 29. On the recent work with Hibi and Ohsugi, we have that the Conjecture 2 is false [10]. We proved that there exist infinite many non-trivial strongly Koszul edge rings.

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QF RINGS AND DIRECT SUMMAND CONDITIONS

MANABU MATSUOKA

Abstract. In this paper we investigate the rings with the direct summand condition, and we give the applications to coding theory. We study the linear codes over the finite ring with this condition. In particular, we consider dual codes and cyclic codes.

Key Words: direct summand conditions, dual codes, cyclic codes.

2010 *Mathematics Subject Classification*: Primary 94B05; Secondary 94B15, 16D40.

1. Introduction

For a ring R, we consider the condition that every finitely generated free submodule N of a finitely generated free R-module M is a direct summand of M. For example QF rings satisfy this condition. In [4], Y. Hirano proved that a commutative artinian ring satisfies this condition. He also found some class of noncommutative rings with this condition.

In [10], T. Sumiyama studied maximal Galois subrings of finite local rings. Y. Hirano characterized finite frobenius rings in [3]. By the way, since several years, codes over finite Frobenius rings draw considerable attension in coding theory. In [2], M. Greferath investigated splitting codes over finite rings. In [1], A. A. Andrade and Palazzo Jr. studied linear codes over finite rings. J. A. Wood established the extension theorem and MacWilliams identities over finite Frobenius rings in [11]. K. Shiromoto and L. Storme gave a Griesner type bound for linear codes over finite QF rings in [9].

Throughout this paper, R denotes a ring with $1 \neq 0$, n denotes a natural number with $n \geq 2$, unless otherwise stated.

2. Rings with the direct summand condition

For a ring R, we consider the following direct summand condition for free left modules:

$(\mathrm{DS})_{\mathrm{l}}$ Every finitely generated free submodule N of a finitely generated free left R-module M is a direct summand of M.

Similarly, we consider the direct summand condition for free right modules:

$(\mathrm{DS})_{\mathrm{r}}$ Every finitely generated free submodule N of a finitely generated free right R-module M is a direct summand of M.

If R satisfies the both conditions, it is said that it has the condition (DS).

For a semisimple ring, every module is semisimple, and every submodule of a semisimple module is a direct summand. Thus, a semisimple ring satisfies the condition (DS).

Definition 1. For a ring R, R is called a QF (quasi-Frobenius) ring if R is left artinian and left self-injective.

The detailed version of this paper will be submitted for publication elsewhere.

It is well-known that the definition of a QF ring is left-right symmetric.

Proposition 2. *Let R be a QF ring. Then R satisfies the condition* (DS)*.*

For any left R-module ${}_RM$, $M^* = \mathrm{Hom}_R({}_RM, {}_RR)$ is a right R-module. In fact the right R-actions of M^* is defined by

$$(f\cdot r)(m) = f(m)\cdot r$$

where $r \in R$, $m \in M$ and $f \in M^*$.

The natural homomorphism $\xi : M \to M^{**}$ is defined by

$$\xi(m)(f) = f(m)$$

where $m \in M$ and $f \in M^*$. A module M is called torsionless if the natural homomorphism $\xi : M \to M^{**}$ is injective. A torsionless module M is said to be reflexive if the natural injection $\xi : M \to M^{**}$ is an isomorphism.

Proposition 3. *Let R be a ring, and let ${}_RN$ be a left R-submodule of R^n. If ${}_RN$ is a direct summand of R^n, then ${}_RN$ is reflexive.*

For any submodule $A \subseteq M$, let $A^\circ = \{f \in M^* \mid f(A) = 0\}$, which is a submodule of M^*. And, for any submodule $I \subseteq M^*$, let $I^\diamond = \cap_{f\in I}\ker(f)$, which is a submodule of M.

Lemma 4. *Let R be a ring, and let ${}_RM$ be a reflexive left R-module. If I is a right R-submodule of M^*, then $I^\diamond \cong I^\circ$ as left R-modules.*

Lemma 5. *Let R be a ring, and let ${}_RM$ be a free left R-module. If A is a direct summand of M, then $A^{\circ\diamond} = A$.*

By Lemma 4 and Lemma 5, we get the following theorem.

Theorem 6. *Let R be a ring, and let ${}_RM$ be a reflexive free left R-module. If A is a direct summand of M, then $A^{\circ\circ} \cong A$ as left R-modules.*

Corollary 7. *Let R be a ring with the condition* (DS)*, and let ${}_RN$ be a finitely generated free left R-submodule of R^n. Then $N^{\circ\circ} \cong N$ as left R-modules.*

3. Codes over finite rings with the condition (DS)$_l$

Let R be a finite ring. A linear left(right) code C of length n over R is a left(right) R-submodule of the left(right) R-module $R^n = \{(a_0, \cdots, a_{n-1}) \mid a_i \in R\}$. If C is a free R-module, C is said to be a free code.

On R^n define the standard inner product by

$$< x, y >= \sum_{i=0}^{n-1} x_i y_i$$

for $x = (x_0, x_1, \cdots, x_{n-1})$, $y = (y_0, y_1, \cdots, y_{n-1}) \in R^n$.

The dual code $C^\perp$ of a linear left code C is defined by

$$C^\perp = \{a \in R^n \mid \ < c, a >= 0 \text{ for any } c \in C\}.$$

Clearly, $C^\perp$ is a linear right code over R.

Similarly, for a linear right code D, we can define the dual code

$$D^\perp = \{b \in R^n \mid \ < b, d >= 0 \text{ for any } d \in D\},$$

Then $D^{\perp}$ is a linear left code over R.

For a left(right) code $C \subseteq R^n$, C is called a self-dual code if $C = C^{\perp}$. In this case, C is a bi-module.

For any left R-submodule $C \subseteq R^n$, C° is defined by

$$C^{\circ} = \{\lambda \in \mathrm{Hom}_R({}_RR^n, {}_RR) \mid \lambda(C) = 0\}.$$

Then C° is a right R-submodule of a right R-module $Hom_R({}_RR^n, {}_RR)$.

For every $x \in R^n$, we define a right R-module homomorphism $\delta_x : R^n \to R$ as $\delta_x(y) =< y, x >$.

Let $e_1, \cdots, e_n$ be fundamental vectors. We define a natural right R-module homomorphism $\epsilon : ({}_RR^n)^* \to R^n_R$ as $\epsilon(f) = (f(e_1), \cdots, (e_n))$. Then ϵ is an isomorphism. In fact $\delta : R^n_R \to ({}_RR^n)^*$ with $\delta(x) = \delta_x$ is an inverse map.

Proposition 8. *Let R be a finite ring, and let $C \subseteq R^n$ be a linear left code. Then $C^{\perp} \cong C^{\circ}$ as right R-modules.*

Theorem 9. *Let R be a finite ring with the condition* (DS). *For a free left code $C \subseteq R^n$, $(C^{\perp})^{\perp} = C$.*

Given any subset $T \subseteq R$, a left annihilator of T is a set

$$\mathrm{l.ann}_R(T) = \{r \in R \mid rt = 0 \text{ for all } t \in T\}$$

which is a left ideal of R. A right annihilator $\mathrm{r.ann}_R(T)$ is defined, similarly.

Then we can get the following corollary.

Corollary 10. *Let R be a finite ring with the condition* (DS). *For a free left submodule C of ${}_RR$, we have*

$$\mathrm{l.ann}_R(\mathrm{r.ann}_R C) = C.$$

Similarly, if R satisfies the direct summand condition for free modules, then we have $\mathrm{r.ann}_R(\mathrm{l.ann}_R D) = D$ for any free right submodule D of R_R.

Theorem 11. *Let R be a finite ring with the condition* $(\mathrm{DS})_1$. *If $C \subseteq R^n$ is a free left code of finite rank, then $C^{\perp}$ is a free right code of finite rank and* $\mathrm{rank}C^{\perp} = n - \mathrm{rank}C$.

4. Cyclic codes

Let R be a finite ring. A linear left(right) code $C \subseteq R^n$ is called cyclic if

$$(a_0, a_1, \cdots, a_{n-1}) \in C \text{ implies } (a_{n-1}, a_0, a_1, \cdots, a_{n-2}) \in C.$$

Let E be the following square matrix;

$$E = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

It follows that a left code $C \subseteq R^n$ is cyclic if and only if it is invariant under right multiplication by E.

Proposition 12. *Let R be a finite ring, and let $C \subseteq R^n$ be a linear left code. If C is a cyclic left code, $C^{\perp}$ is a cyclic right code.*

By Theorem 9 and Proposition 12, we get the following corollary.

Corollary 13. *Let R be a finite ring with the condition* (DS)*, and let $C \subseteq R^n$ be a free left code. Then C is a cyclic left code if and only if $C^{\perp}$ is a cyclic right code.*

In what follows, we shall use the following conventions:

$(g)_l$ is the left ideal generated by $g \in R[X]$.

$(g)_r$ is the right ideal generated by $g \in R[X]$.

(g) is the two-sided ideal generated by $g \in R[X]$.

Cyclic codes are understood in terms of left ideals in quotient rings of polynomial rings. The left R-module isomorphism $\rho : R^n \to R[X]/(X^n - 1)$ sending the vector $a = (a_0, a_1, \cdots, a_{n-1})$ to the equivalence class of polynomial $a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$, allows us to identify the cyclic left code with the left ideal of $R[X]/(X^n - 1)$.

Notice that $X^n - 1$ is the central element of $R[X]$.

Theorem 14. *Let R be a finite ring. There is a one to one correspondence between cyclic left codes in R^n and left ideals of $R[X]/(X^n - 1)$.*

Definition 15. Let R be a finite ring, and let C be a cyclic left code in $R[X]/(X^n - 1)$. If there exist monic polynomials g and h such that $\rho(C) = (g)_l/(X^n - 1)$ and $X^n - 1 = hg$, then C is called the principal cyclic left code. In this case, $g(X)$ is called the generator polynomial and $h(X)$ is called the parity check polynomial of C. Similarly, for a cyclic right code C, C is called the principal cyclic right code if $\rho(C) = (g)_r/(X^n - 1)$ and $X^n - 1 = gh$.

Proposition 16. *Let R be a finite ring, and let $C \subseteq R^n$ be a principal cyclic left code with the generator polynomial $g(X)$ of degree $n - k$. Then C is a free left code of rank k.*

Let $C \subseteq R^n$ be a free left(right) code. If a basis of C is used as rows of a matrix G, the matrix G is called a generator matrix of C. If G is a $k \times n$ generator matrix of a free left code C, then, for any $c \in C$, we have $c = aG$ for some $a \in R^k$. If G is a $k \times n$ generator matrix of a free right code D, then, for any $d \in D$, we have ${}^t d = {}^t G {}^t b$ for some $b \in R^k$. A generator matrix of $C^{\perp}$ is called a parity check matrix of C.

Proposition 17. *Let R be a finite ring, and let $C \subseteq R^n$ be a principal cyclic left code with the generator polynomial*

$$g(X) = g_{n-k}X^{n-k} + \cdots + g_1 X + g_0$$

with $g_{n-k} = 1$. Then C has the $k \times n$ generator matrix G of the form

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-k} & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & \cdots & g_{n-k} & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & g_0 & g_1 & \cdots & g_{n-k} \end{pmatrix}.$$

The generator matrix of the principal cyclic right code $\rho(C) = (g)_r/(X^n - 1)$ with $X^n - 1 = gh$ is the same form.

Next, we determine the parity check matrix of a principal cyclic left code.

Proposition 18. *Let R be a finite ring with the condition* $(\mathrm{DS})_1$*, and let $C \subsetneq R^n$ be a principal cyclic left code with the generator polynomial $g(X)$ of degree $n-k$ and the parity check polynomial*

$$h(X) = h_k X^k + \cdots + h_1 X + h_0$$

with $h_k = 1$. Suppose $X^n - 1 = hg = gh \in R[X]$. Then C has the $(n-k) \times n$ parity check matrix H of the form

$$H = \begin{pmatrix} h_k & \cdots & h_1 & h_0 & 0 & \cdots & 0 \\ 0 & h_k & \cdots & h_1 & h_0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & h_k & \cdots & h_1 & h_0 \end{pmatrix}.$$

Corollary 19. *Let R be a finite ring with the condition* $(\mathrm{DS})_1$*, and let $C \subsetneq R^n$ be a principal cyclic left code with the generator polynomial $g(X)$ of degree $n-k$ and the parity check polynomial*

$$h(X) = h_k X^k + \cdots + h_1 X + h_0$$

with $h_k = 1$. Suppose $X^n - 1 = hg = gh \in R[X]$. Then $C^\perp$ is the principal cyclic right code, and we have

$$\rho(C^\perp) = (h^\perp)_r/(X^n - 1),$$

where $h^\perp(X) = (h_0 X^k + \cdots + h_{k-1}X + h_k)h_0^{-1}$.

Proposition 20. *Let R be a finite ring, and let $C \subsetneq R^n$ be a principal cyclic left code with the generator polynomial $g(X)$ and the parity check polynomial $h(X)$. Suppose $X^n - 1 = hg = gh \in R[X]$. Then $a \in C$ if and only if $\rho(a)\overline{h} = 0$ in $R[X]/(X^n - 1)$.*

Then we get the following corollary.

Corollary 21. *Let R be a finite ring, and let $C \subsetneq R^n$ be a principal cyclic left code with the generator polynomial $g(X)$ and the parity check polynomial $h(X)$. Suppose $X^n - 1 = hg = gh \in R[X]$. Set $\overline{R} = R[X]/(X^n - 1)$. Then we have*

$$\rho(C) = (g)_l/(X^n - 1) = \mathrm{l.ann}_{\overline{R}}\left(\overline{h}\right).$$

By Corollary 19 and Proposition 20, we get the following corollary.

Corollary 22. *Let R be a finite ring with the condition* $(\mathrm{DS})_1$*, and let $C \subsetneq R^n$ be a principal cyclic left code with the generator polynomial*

$$g(X) = g_{n-k}X^{n-k} + \cdots + g_1 X + g_0$$

with $g_{n-k} = 1$ and the parity check polynomial $h(X)$. Suppose $X^n - 1 = hg = gh \in R[X]$. Set $\overline{R} = R[X]/(X^n - 1)$. Then we have

$$\rho(C^\perp) = (h^\perp)_r/(X^n - 1) = \mathrm{r.ann}_{\overline{R}}\left(\overline{g^\perp}\right),$$

where $g^\perp(X) = g_0^{-1}(g_0 X^{n-k} + \cdots + g_{n-k-1}X + g_{n-k})$.

Now we give a basic example.

Example 23. Let $\mathbf{Z}_2$ be a finite field of two elements, and $M_2(\mathbf{Z}_2)$ be a set of 2×2 matrices over $\mathbf{Z}_2$. Let $R = D_2(\mathbf{Z}_2)$, where

$$D_2(\mathbf{Z}_2) = \left\{ \left(\begin{array}{cc} a & b \\ 0 & a \end{array} \right) \in M_2(\mathbf{Z}_2) \;\middle|\; a, b \in \mathbf{Z}_2 \right\}.$$

R is a finite commutative local ring with the unique maximal ideal

$$M = \left\{ \left(\begin{array}{cc} 0 & b \\ 0 & 0 \end{array} \right) \in M_2(\mathbf{Z}_2) \;\middle|\; b \in \mathbf{Z}_2 \right\}.$$

Then R satisfies the condition (DS).

Now set $i = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$. Then we have

$$R = \{\ 0,\ 1,\ i,\ 1+i\ \}$$

with $i^2 = 1$. Thus we get $D_2(\mathbf{Z}_2) = \mathbf{Z}_2[\ i\ ]$.

Now we get the following factorizations:

$$X^4 - 1 = (X^2 + (1+i)X + i)(X^2 + (1+i)X + i).$$

Set $\rho(C) = (X^2 + (1+i)X + i)/(X^4 - 1)$. Then C is a principal cyclic code of rank 2. And we get $\rho(C^{\perp}) = (X^2 + (1+i)X + i)/(X^4 - 1)$. Hence this is a self-dual code.

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IDEALS AND TORSION THEORIES

HIROYUKI MINAMOTO

Abstract. We introduce ideal theoretic conditions on an ideal $\mathcal{I}$ of an abelian category $\mathcal{A}$, which are show to be equivalent to the condition that the ideal is associated to a torsion class (resp. pre-torsion class, Serre subcategory) of $\mathcal{A}$. We also discuss an ideal which is associated to a radical (a sub functor of the identity functor $\mathrm{id}_{\mathcal{A}}$ which has a special property) of $\mathcal{A}$.

This work came out from an attempt to obtain a formalism of the argument which is given in the proof of the following theorem on 2-representation infinite (2-RI) algebras ([1]): Let Λ be a 2-RI algebra and τ_2, τ_2^- be 2-Auslander-Reiten translations. A Λ-module M is called θ-minimal if the canonical morphism $M \to \tau_2\tau_2^- M$ is injective.

1. Motivation: Auslander-Reiten component of a 2-representation infinite algebra.

This work came out from an attempt to obtain a formalism of the argument which is given in the proof of the following theorem on 2-representation infinite algebra.

First we recall the definition of n-representation infinite algebra introduced by Herschend-Iyama-Oppermann [1]. Let A be a finite dimensional algebra over a field k of finite global dimension. Recall that the Nakayama functor ν is the derived tensor product $- \otimes_A^{A} D(A)$ of the k-dual $D(A) := \mathrm{Hom}_k(A, k)$, which gives a triangle autoequivalence of the bounded derived category $\mathcal{D}_A := \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$. For a integer $n \in \mathbb{Z}$ we set $\nu_n := \nu \circ [-n]$.

Definition 1. A finite dimensional algebra A is called *n-representation infinite (n-RI) algebra* if it is of global dimension n and the complex $\nu_n^{-p}(A)$ belongs to the standard heart mod-A for $p \in \mathbb{N}$.

Let A be n-RI algebra. The Hom-$\otimes$ adjunction induced by the A-A bi-module $\theta := \mathrm{Ext}^n_A(D(A), A)$ is called n-Auslander-Reiten translations. More precisely, we set $\tau_n(-) := \mathrm{Hom}_A(\theta, -)$ and $\tau_n^- := - \otimes_A \theta$.

$$\tau_n^- := - \otimes_A \theta : \text{mod-}A \rightleftarrows \text{mod-}A : \tau_n(-) := \mathrm{Hom}_A(\theta, -).$$

We remark that τ_n and τ_n^- dose not give equivalences and, in particular, are not inverse to each other. Hence, the adjoint unit morphism $M \to \tau_n\tau_n^- M$ is possibly not an isomorphism.

To state our motivating theorem, we introduce one terminology. An A-module M is called *θ-minimal* if the adjoint unit morphism $M \to \tau_n\tau_n^- M$ is injective.

Theorem 2. *Let A be a 2-RI algebra and $M \neq 0$ be a θ-minimal indecomposable A-module. Assume that* $\mathrm{Hom}_A(M, A) = 0$. *(e.g., M is a non-projective 2-preprojective module or 2-regular module.) Then the Auslander-Reiten component Γ_M containing M is*

The detailed version of this paper will be submitted for publication elsewhere.

of type $\mathbb{Z}\mathrm{A}_\infty$ unless it contains projective module or injective modules and M is placed in the bottom of Γ_M.

The proof goes as follows: from the property $\mathrm{Hom}_A(M, A) = 0$, we can deduce that $(\tau_1 M) \otimes_A \theta = 0$ where τ_1 is the usual Auslander-Reiten translation. Together with the θ-minimality of M, the latter property implies that the middle term M_1 of the Auslander-Reiten sequence is indecomposable.

$$0 \to \tau_1 M \to M_1 \to M \to 0.$$

This part of argument is quite formal and leads us to the theory of quasi torsion ideals, which will be discussed in the main body of this note. Using similar argument, we conclude the results.

2. quasi torsion ideals

Let $\mathcal{A}$ be an ableian category. Recall that a sub functor of the Hom-functor $\mathcal{A}(-,+) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$ is called an ideal of $\mathcal{A}$. Here we view $\mathcal{A}$ as a ring with several objects.

Definition 3. Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ be an ideal of $\mathcal{A}$.

(1) An object $M \in \mathcal{A}$ is called *$\mathcal{I}$-minimal* if $\mathcal{I}(-, M) = 0$;

(2) An exact sequence
$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$
is called a *quasi-torsion sequence*(qt sequence) with respect to $\mathcal{I}$, if N is $\mathcal{I}$-minimal and f belongs to the ideal $\mathcal{I}$.

(3) An ideal $\mathcal{I}$ is called *quasi-torsion ideal*(qt ideal), if there exists a qt sequence
$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$
for all objects M of $\mathcal{A}$.

Lemma 4. *For an object M, a qt sequence the middle term of which is M, is unique up to isomorphism (if it exists).*

If an ideal $\mathcal{I}$ is quasi-torsion. Then for an object $M \in \mathcal{A}$, an object $L \in \mathcal{A}$ which appearers in the left term of a qt-sequence $0 \to L \to M \to N \to 0$ whose middle term is M is uniquely determined up to isomorphisms. We denote such an object by $\mathsf{com}M$ and call *cominimal pert of M*.

Corollary 5. *Let $\mathcal{I}$ be a qt ideal. Let $0 \to L \to M \to N \to 0$ be a non-split qt sequence. If L and N are indecomposable, then the middle term M is also indecomposable.*

Example 6. Let A be a Noetherian ring and ρ be an A-A-bi-module. For A-modules M and N we set
$$\mathcal{I}(M, N) := \{f : M \to N \mid f \otimes_A \rho = 0\}.$$
It is easy to see that $\mathcal{I}$ is an ideal of the category mod-A of finite A-modules. Moreover, we can prove that $\mathcal{I}$ is a qt-ideal and $\mathcal{I}$-minimal objects are precisely modules M such that the unite morphism $M \to \mathrm{Hom}_A(\rho, M \otimes \rho)$ is injective.

Let $0 \to L \to M \to N \to 0$ be an exact sequence of A-modules. Assume that N is $\mathcal{I}$-minimal and $L \otimes_A \rho = 0$, then the above sequence is quasi-torsion. Assume moreover that

L and N are indecomposable. The by the above corollary, we conclude that the middle term M is also indecomposable. This is the way that we prove the indecomposability of the middle term M_1 in the proof of Theorem 2.

Recall that a *preradical* $\mathbf{r}$ of an abelian category $\mathcal{A}$ is a sub functor of the identity functor $\mathsf{id}_{\mathcal{A}}$. (Note that we can easily see that a preradical $\mathbf{r}$ is an additive functor.) A preradical $\mathbf{r}$ is called *radical* if it satisfies $\mathbf{r}(M/\mathbf{r}(M)) = 0$ for all $M \in \mathcal{A}$. (For details, see, e.g. [2]). For a radical $\mathbf{r}$, we define an ideal $\mathcal{I}_{\mathbf{r}}$ to be

$$\mathcal{I}_{\mathbf{r}}(M,N) := \{f \in \mathcal{A}(M,N) \mid f \text{ factors through } \mathbf{r}(M).\}.$$

Lemma 7. *A preradical* $\mathbf{r}$ *is a radical if and only if the ideal* $\mathcal{I}_{\mathbf{r}}$ *is quasi-torsion. If this is the case we have* $\mathbf{r}M = \mathsf{com}M$.

Theorem 8. *The following gives a one to one correspondence between quasi-torsion ideals and radicals on* $\mathcal{A}$.

$$\mathcal{I} \mapsto \mathsf{com}_{\mathcal{I}}, \ \mathbf{r} \mapsto \mathcal{I}_{\mathbf{r}}.$$

Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ be a quasi-torsion ideals.

Lemma 9. *The factor category* $\mathcal{A}/\mathcal{I}$ *has pseudo-kernels. Hence the category* $\mathrm{mod}\mathcal{A}/\mathcal{I}$ *of coherent functors is abelian.*

We discuss the global dimension of the category $\mathrm{mod}\mathcal{A}/\mathcal{I}$. When we view an abelian category $\mathcal{A}$ as a ring with several objects, this category is the category of finitely presented modules. We show that the global dimension has a relationship with nilpotensy of the qt-ideal $\mathcal{I}$.

Theorem 10. *Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{I}$ *be a quasi-torsion ideals.*

(1) *If* $\mathcal{A}$ *is Artinian, then every coherent* $\mathcal{A}/\mathcal{I}$*-module has finite projective dimension.*
(2) *If* $\mathcal{I}^n = \mathcal{I}^{n+1}$*, then* $\mathrm{gldim}(\mathrm{mod}\mathcal{A}/\mathcal{I}) \leq 2n$.
(3) *If* $\mathrm{gldim}(\mathrm{mod}\mathcal{A}/\mathcal{I}^2) \leq n-1$*, then* $\mathcal{I}^n = \mathcal{I}^{n+1}$ *and* $\mathrm{gldim}(\mathrm{mod}\mathcal{A}/\mathcal{I}) \leq 2n$.

As far as the author knows, such a phenomena rarely happen for a ring with single object, i.e., a ring in the usual sense. Therefore we can expect that, even from ring theoretical view point, a ring with several objects has special features which are not possessed by a ring with single objects.

We end this note by proposing a question: Can we get more strong result, like

$$\mathrm{gldimmod}\mathcal{A}/\mathcal{I} \leq 2n \quad \Longleftrightarrow \quad \mathcal{I}^n = \mathcal{I}^{n+1}????$$

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SUPPORT τ-TILTING MODULES AND PREPROJECTIVE ALGEBRAS

YUYA MIZUNO

Abstract. We study support τ-tilting modules over preprojective algebras of Dynkin type. We classify basic support τ-tilting modules by giving a bijection with elements in the corresponding Weyl groups. We also study g-matrices of support τ-tilting modules, which show terms of minimal projective presentations of indecomposable direct summands. We give an explicit description of g-matrices and prove that cones given by g-matrices coincide with chambers of the associated root systems.

1. Introduction

The preprojective algebra associated to a quiver was introduced by Gelfand-Ponomarev [GP] to study the preprojective representations of a quiver. Since then, they have been studied extensively not only from the viewpoint of representation theory of algebras (for example [BGL, DR1, DR2, Ri1]) but also in several contexts such as (algebraic, differential, symplectic) geometry and quantum groups.

In [BIRS] (also in [IR1]), the authors studied preprojective algebras via tilting theory for non-Dynkin quivers. By making heavy use of tilting theory, they succeed to give several important results such as a method for constructing a large class of 2-Calabi-Yau categories which have close connections with cluster algebras. On the other hand, in Dynkin cases (i.e. the underlying graph of a quiver is A_n $(n \geq 1)$, D_n $(n \geq 4)$ and E_n $(n = 6, 7, 8)$), the preprojective algebras are selfinjective, so that all tilting modules are trivial (i.e. projective). In this note, we will show that, instead of tilting modules, *support τ-tilting modules* play an important role in this case.

The notion of support τ-tilting modules was introduced in [AIR], which gives a generalization of tilting modules. They have several nice properties. For example, it is shown that there are deep connections between τ-tilting theory, torsion theory, silting theory, cluster theory and so on (refer to an introductory article [IR2]). Moreover, support τ-tilting modules have nicer mutation theory than tilting modules. Namely, any basic almost-complete support τ-tilting module is the direct summand of exactly two basic support τ-tilting modules. It implies that mutation of support τ-tilting modules is always possible and this property admits interesting combinatorial descriptions for support τ-tilting graphs. Furthermore, certain support τ-tilting modules over selfinjective algebras provide tilting complexes [M1]. It is therefore fruitful to investigate these remarkable modules for given algebras.

Conventions. Let K be an algebraically closed field and we denote by $D := \mathrm{Hom}_K(-, K)$. By a finite dimensional algebra Λ, we mean a basic finite dimensional algebra over K. By a module, we mean a right module unless stated otherwise. We denote by $\mathsf{mod}\Lambda$ the

The detailed version of this paper will be submitted for publication elsewhere.

category of finitely generated Λ-modules and by $\mathsf{proj}\Lambda$ the category of finitely generated projective Λ-modules. For $X \in \mathsf{mod}\Lambda$, we denote by $\mathsf{Sub}X$ (respectively, $\mathsf{Fac}X$) the subcategory of $\mathsf{mod}\Lambda$ whose objects are submodules (respectively, factor modules) of finite direct sums of copies of X. We denote by $\mathsf{add}M$ the subcategory of $\mathsf{mod}\Lambda$ consisting of direct summands of finite direct sums of copies of $M \in \mathsf{mod}\Lambda$.

2. Preliminaries

2.1. **Preprojective algebras.** In this subsection, we recall the definition of preprojective algebras and some properties of them.

Definition 1. Let Q be a finite connected acyclic quiver with vertices $Q_0 = \{1, \dots, n\}$. The preprojective algebra associated to Q is the algebra

$$\Lambda_Q = \Lambda := K\overline{Q}/\langle \sum_{a \in Q_1} (aa^* - a^*a)\rangle$$

where $\overline{Q}$ is the double quiver of Q, which is obtained from Q by adding for each arrow $a : i \to j$ in Q_1 an arrow $a^* : i \leftarrow j$ pointing in the opposite direction.

Note that Λ_Q does not depend on the orientation of Q.

We collect some basic properties of preprojective algebras.

Proposition 2. *Let Q be an acyclic quiver and Λ the preprojective algebra of Q. Then Q is a Dynkin quiver if and only if Λ is a finite dimensional algebra. Further, if these equivalent conditions hold, then Λ is selfinejctive.*

Note that, even if Q is a Dynkin quiver, Λ is infinite representation type (i.e. there exists infinitely many indecomposable Λ-modules) in general [DR2].

We define the *Coxeter group* W_Q associated to Q, which is defined by the generators $s_1, \dots, s_n$ and relations

- $s_i^2 = 1$,
- $s_is_j = s_js_i$ if there is no arrow between i and j in Q,
- $s_is_js_i = s_js_is_j$ if there is precisely one arrow between i and j in Q.

Each element $w \in W_Q$ can be written in the form $w = s_{i_1} \cdots s_{i_k}$. If k is minimal among all such expressions for w, then k is called the *length* of w and we denote by $l(w) = k$. In this case, we call $s_{i_1} \cdots s_{i_k}$ a *reduced expression* of w.

2.2. **Support τ-tilting modules.** In this subsection, we give definitions of support τ-tilting modules.

Definition 3. Let Λ be a finite dimensional algebra.

(a) We call X in $\mathsf{mod}\Lambda$ *τ-rigid* if $\mathrm{Hom}_\Lambda(X, \tau X) = 0$.

(b) We call X in $\mathsf{mod}\Lambda$ *τ-tilting* (respectively, *almost complete τ-tilting*) if X is τ-rigid and $|X| = |\Lambda|$ (respectively, $|X| = |\Lambda| - 1$), where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X.

(c) We call X in $\mathsf{mod}\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that X is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

We also give the following definitions.

(d) We call a pair (X, P) of $X \in \mathsf{mod}\Lambda$ and $P \in \mathsf{proj}\Lambda$ *τ-rigid* if X is τ-rigid and $\mathrm{Hom}_\Lambda(P, X) = 0$.

(e) We call a τ-rigid pair (X, P) a *support τ-tilting* (respectively, *almost support τ-tilting*) pair if $|X| + |P| = |\Lambda|$ (respectively, $|X| + |P| = |\Lambda| - 1$).

We call (X, P) *basic* if X and P are basic, and we say that (X, P) is a *direct summand* of (X', P') if X is a direct summand of X' and P is a direct summand of P'. We denote by sτ-tiltΛ the set of isomorphism classes of basic support τ-tilting Λ-modules.

Note that (X, P) is a τ-rigid (respectively, support τ-tilting) pair for Λ if and only if X is a τ-rigid (respectively, τ-tilting) $(\Lambda/\langle e\rangle)$-module, where e is an idempotent of Λ such that $\mathsf{add}P = \mathsf{add}e\Lambda$ [AIR, Proposition 2.3]. Moreover, if (X, P) and (X, P') are support τ-tilting pairs for Λ, then we get $\mathsf{add}P = \mathsf{add}P'$. Thus, a basic support τ-tilting module X determines basic support τ-tilting pair (X, P) uniquely and we can identify basic support τ-tilting modules with basic support τ-tilting pairs.

Example 4. Let $\Lambda := KQ$ be the path algebra given by the following quiver

$$Q := \ (1 \longleftarrow 2)\ .$$

Then one can check that there exist support τ-tilting modules as follows

$$e_1\Lambda \oplus e_2\Lambda,\ S_2 \oplus e_2\Lambda,\ e_1\Lambda,\ S_2 \text{ and } 0.$$

They can be identified with support τ-tilting pairs

$$(e_1\Lambda \oplus e_2\Lambda, 0),\ (S_2 \oplus e_2\Lambda, 0),\ (e_1\Lambda, e_2\Lambda),\ (S_2, e_1\Lambda) \text{ and } (0, e_1\Lambda \oplus e_2\Lambda),$$

respectively.

One of the important properties of support τ-tilting modules is a partial order.

Definition 5. Let Λ be a finite dimensional algebra. For $T, T' \in$ sτ-tiltΛ, we write

$$T' \geq T$$

if $\mathsf{Fac}T' \supset \mathsf{Fac}T$, where $\mathsf{Fac}X$ the subcategory of $\mathsf{mod}\Lambda$ whose objects are factor modules of finite direct sums of copies of X. Then $\geq$ gives a partial order on sτ-tiltΛ [AIR, Theorem 2.18]. Clearly, Λ is the unique maximal element and 0 is the unique minimal element.

Example 6. Let $\Lambda := KQ$ be the path algebra given by the following quiver

$$Q := \ (1 \longleftarrow 2)\ .$$

Let $T_1 := e_1\Lambda \oplus e_2\Lambda,\ T_2 := S_2 \oplus e_2\Lambda,\ T_3 := e_1\Lambda,\ T_4 := S_2$ and $T_5 := 0$.
Then we have

$$\mathsf{Fac}T_1 = \mathsf{add}\{e_1\Lambda \oplus e_2\Lambda \oplus S_2\},\ \mathsf{Fac}T_2 = \mathsf{add}\{S_2 \oplus e_2\Lambda\},\ \mathsf{Fac}T_3 = \mathsf{add}\{e_1\Lambda\},\ \mathsf{Fac}T_4 = \mathsf{add}\{S_2\},$$

where $\mathsf{add}X$ denote the subcategory of $\mathsf{mod}\Lambda$ consisting of direct summands of finite direct sums of $X \in \mathsf{mod}\Lambda$. Then, from Definition 5, one can obtain the following Hasse quiver.

$$\begin{tikzcd}
 & T_2 \arrow[r] & T_4 \arrow[dr] & \\
T_1 \arrow[ur] \arrow[r] & T_3 \arrow[rr] & & T_5.
\end{tikzcd}$$

3. Support τ-tilting modules and the Weyl group

Our main aim is to give a complete description of all support τ-tilting modules over preprojective algebras of Dynkin type. For this purpose, we give the following bijection.

Theorem 7. *Let Q be a Dynkin quiver and Λ the preprojective algebra of Q. There exist bijections between the isomorphism classes of basic support τ-tilting Λ-modules and the elements of W_Q.*

In the following two subsections, we explain Theorem 7.

3.1. **Support τ-tilting ideals.** Let Q be a Dynkin quiver with $Q_0 = \{1, \ldots, n\}$ and Λ the preprojective algebra of Q. We denote by the two-sided ideal I_i of Λ generated by $1 - e_i$, where e_i is a primitive idempotent of Λ for $i \in Q_0$, that is, $I_i := \Lambda(1 - e_i)\Lambda$ for $i \in Q_0$. We see that the ideal has the following property.

Lemma 8. *Let $X \in \mathsf{mod}\Lambda$. Then XI_i is maximal amongst submodules Y of X such that any composition factor of X/Y is isomorphic to a simple module Λ/I_i.*

We denote by $\langle I_1, \ldots, I_n\rangle$ the set of ideals of Λ which can be written as

$$I_{i_1} I_{i_2} \cdots I_{i_k}$$

for some $k \geq 0$ and $i_1, \ldots, i_k \in Q_0$.

Our aim in this subsection is to show the following.

Theorem 9. *Any $T \in \langle I_1, \ldots, I_n\rangle$ is a basic support τ-tilting modules of Λ.*

For a proof, we use the following important property.

Definition 10. [AIR, Theorem 2.18 and 2.28] Let Λ be a finite dimensional algebra. Then

($*$) any basic almost support τ-tilting pair (U, Q) is a direct summand of exactly two basic support τ-tilting pairs (T, P) and (T', P'). Moreover we have $T > T'$ or $T < T'$.

Under the above setting, let X be an indecomposable direct summand of T or P. We write $(T', P') = \mu_X(T, P)$ or simply $T' = \mu_X(T)$ and say that T' is a *mutation* of T. In particular, we write $T' = \mu_X^-(T)$ if $T > T'$ (respectively, $T' = \mu_X^+(T)$ if $T < T'$) and we say that T' is a *left mutation* (respectively, *right mutation*) of T. By ($*$), exactly one of the left mutation or right mutation occurs.

Using mutations, we give the following key proposition.

Proposition 11. *Let $T \in \langle I_1, \ldots, I_n\rangle$ and assume that T is a basic support τ-tilting Λ-module. If $I_iT \neq T$, then there is a left mutation of T associated to e_iT and $\mu_{e_iT}^-(T) \cong I_iT$. In particular, I_iT is a basic support τ-tilting Λ-module.*

Namely, a multiplication by I_i gives a left mutation of T if $I_iT \neq T$. Using this property inductively, we can obtain Theorem 9.

There exists a close relationship between mutations and partial orders.

Definition 12. [AIR, Corollary 2.34] Let Λ be a finite dimensional algebra. We define the *support τ-tilting quiver* $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ as follows.
- The set of vertices is $\mathrm{s}\tau\text{-tilt}\,\Lambda$.
- Draw an arrow from T to T' if T' is a left mutation of T.

Then $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ coincides with the Hasse quiver of the partially ordered set $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

Hence, the Hasse quiver of Example 6 gives behavior of mutations of support τ-tilting modules.

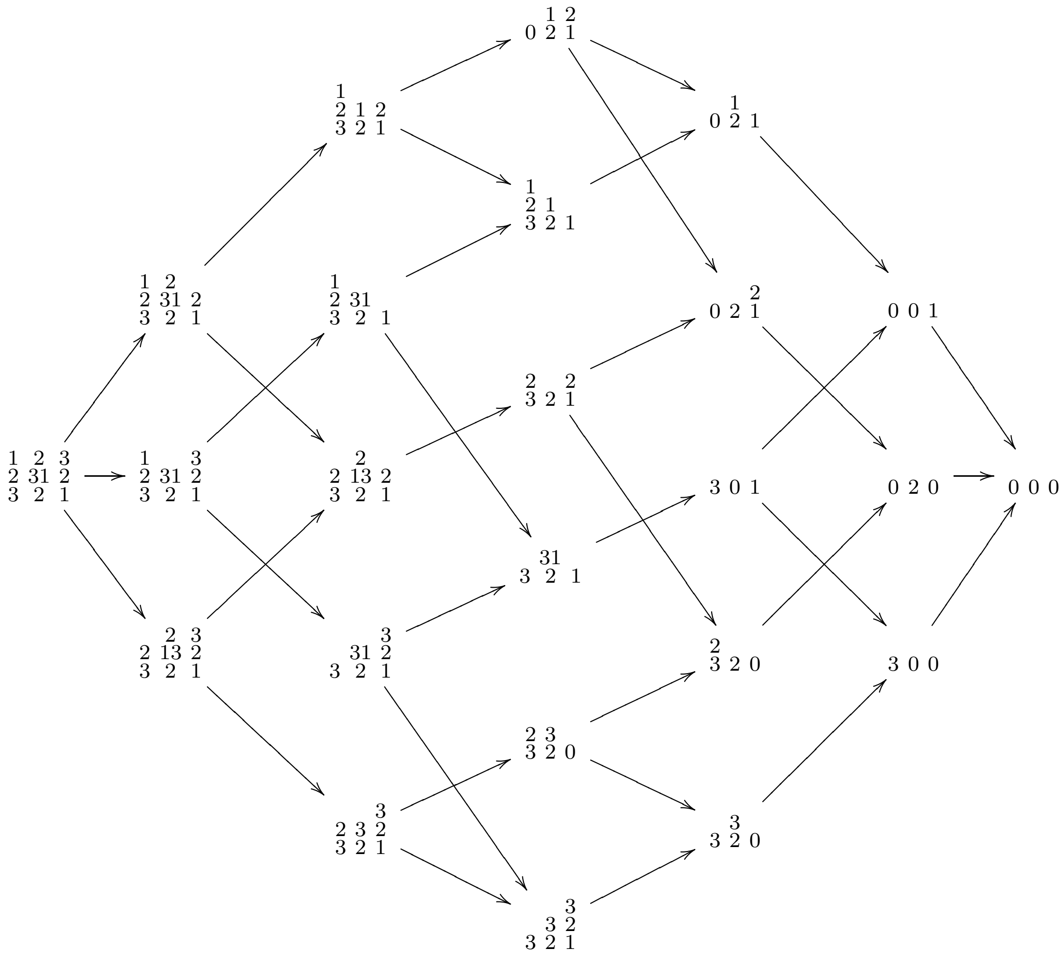
Now using support τ-tilting quiver, we describe support τ-tilting modules of preprojective algebras.

Example 13. (a) Let Λ be the preprojective algebra of type A_2. In this case, $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is given as follows.

$$\begin{tikzcd}
 & \substack{1\\2}\ 1 \arrow[r, "I_1"] & 0\ 1 \arrow[dr, "I_2"] & \\
\substack{1\\2}\ \substack{2\\1} \arrow[ur, "I_2"] \arrow[dr, "I_1"'] & & & 0\ 0 \\
 & 2\ \substack{2\\1} \arrow[r, "I_2"'] & 2\ 0 \arrow[ur, "I_1"'] &
\end{tikzcd}$$

Here we represent modules by their radical filtrations and we write a direct sum $X \oplus Y$ by XY.

(b) Let Λ be the preprojective algebra of type A_3. In this case, $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is given as follows



Remark 14. In these examples, $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ consists of a finite connected component. We will show that this is always true for preprojective algebras of Dynkin type in the next subsection. Thus, all support τ-tilting modules can be obtained by mutations from Λ.

3.2. **A connection with the Weyl group.** Let Q be a Dynkin quiver with $Q_0 = \{1, \ldots, n\}$ and Λ the preprojective algebra of Q. To give an explicit description of support τ-tilting Λ-modules, we provide a connection with the Weyl group.

We use the following result (see [BIRS, M2]).

Theorem 15. *There exists a bijection* $W_Q \to \langle I_1, \ldots, I_n \rangle$. *It is given by* $w \mapsto I_w = I_{i_1} I_{i_2} \cdots I_{i_k}$ *for any reduced expression* $w = s_{i_1} \cdots s_{i_k}$.

From this theorem and Theorem 9, we obtain one finite connected component in $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$. Then we can apply the following result.

Proposition 16. [AIR, Corollary 2.38] *If* $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ *has a finite connected component* C, *then* $C = \mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$.

As a conclusion, we can obtain the following statement.

Theorem 17. *Any basic support τ-tilting Λ-module is isomorphic to an element of $\langle I_1, \dots, I_n \rangle$.*

We also use the following lemma.

Lemma 18. *If right ideals T and U are isomorphic as Λ-modules, then $T = U$.*

Then, combining the above results, we get the desired statement.

Proof of Theorem 7. We will give a bijection between sτ-tiltΛ and W_Q. A bijection – and W_Q can be given similarly.

By Theorem 9 and 15, we have a map $W_Q \ni w \mapsto I_w \in$ sτ-tiltΛ. This map is surjective since any support τ-tilting Λ-module is isomorphic to I_w for some $w \in W_Q$ by Theorem 17. Moreover it is injective by Theorem 15 and Lemma 18. Thus we get the conclusion. $\square$

At the end of this subsection, we briefly give a relationship between a partial order of support τ-tilting modules and that of W_Q.

Definition 19. Let $u, w \in W_Q$. We write $u \leq_L w$ if there exist $s_{i_k}, \dots, s_{i_1}$ such that $w = s_{i_k} \dots s_{i_1} u$ and $l(s_{i_j} \dots s_{i_1} u) = l(u) + j$ for $0 \leq j \leq k$. Clearly $\leq_L$ gives a partial order on W_Q, and we call $\leq_L$ the *left order* (it is also called *weak order*). We denote by $\mathcal{H}(W_Q, \leq_L)$ the Hasse quiver of left order on W_Q.

Then we have the following results.

Theorem 20. *The bijection in $W_Q \to$ sτ-tiltΛ in Theorem 7 gives an isomorphism of partially ordered sets $(W_Q, \leq_L)$ and $(\text{s}\tau\text{-tilt}\Lambda, \leq)^{op}$.*

We remark that the *Bruhat order* on W_Q coincides with the reverse inclusion relation on $\langle I_1, \cdots I_n \rangle$ [ORT, Lemma 6.5].

4. g-matrices and cones

In this last section, we introduce the notion of *g-vectors* [DK] (which is also called *index* [AR, P]) and *g-matrices* of support τ-tilting modules. We refer to [AIR, section 5] for a background of this notion.

Definition 21. Let Λ be a finite dimensional algebra and $K_0(\mathsf{proj}\Lambda)$ the Grothendieck group of the additive category $\mathsf{proj}\Lambda$. Then the isomorphism classes $e_1\Lambda, \dots, e_n\Lambda$ of indecomposable projective Λ-modules form a basis of $K_0(\mathsf{proj}\Lambda)$. Consider X in $\mathsf{mod}\Lambda$ and let

$$P_1^X \longrightarrow P_0^X \longrightarrow X \longrightarrow 0$$

be its minimal projective presentation in $\mathsf{mod}\Lambda$. Then we define the *g-vector* of X as the element of the Grothendieck group given by

$$g(X) := [P_0^X] - [P_1^X] = \sum_{i=1}^{n} g_i(X) e_i \Lambda.$$

Let (X,P) be a support τ-tilting pair for Λ with $X=\bigoplus_{i=1}^{\ell}X_i$ and $P=\bigoplus_{i=\ell+1}^{n}P_i$, where X_i and P_i are indecomposable. Then define $g(X_i)$ as above and $g(P_i):=-[P_i]$. We define the *g-matrix* of (X,P) by

$$g(X,P):=(g(X_1),\cdots,g(X_\ell),g(P_{\ell+1}),\cdots,g(P_n))\in \mathrm{GL}(\mathbb{Z}^n).$$

Note that it forms a basis of $K_0(\mathsf{proj}\Lambda)$ [AIR, Theorem 5.1].

Moreover, define its *cone* by

$$C(X,P):=\{a_1g(X_1)+\cdots+a_\ell g(X_\ell)+a_{\ell+1}g(P_{\ell+1})+\cdots+a_ng(P_n)\mid a_i\in\mathbb{R}_{>0}\}.$$

Now we give an example.

Example 22. Let $\Lambda:=KQ$ be the path algebra given by the following quiver.

$$Q:=\ (1\longleftarrow 2)\ .$$

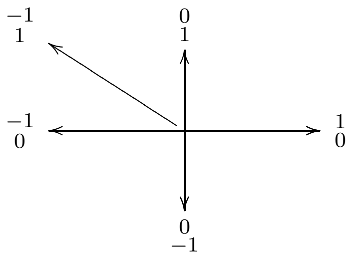
As we have seen before, we have support τ-tilting pairs as follows

$$(e_1\Lambda\oplus e_2\Lambda,0),\ (S_2\oplus e_2\Lambda,0),\ (e_1\Lambda,e_2\Lambda),\ (S_2,e_1\Lambda)\text{ and }(0,e_1\Lambda\oplus e_2\Lambda).$$

Then we have their g-matrices as follows

$$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1&0\\1&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1&-1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$$

and their cones can be described as follows.



It is quite interesting to investigate behavior of cones of support τ-tilting modules for given algebras (cf. cones of tilting modules [H]).

At the end of this note, we give a description of cones of preprojective algebras. Let Q be a Dynkin quiver with vertices $Q_0=\{1,\dots,n\}$ and Λ the preprojective algebra of Q. Then we have the following result.

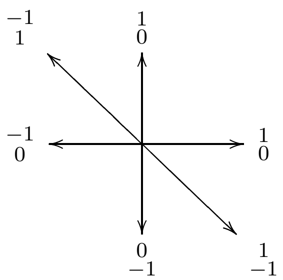
Theorem 23. *The set of g-matrices of support τ-tilting Λ-modules coincides with the subgroup $\langle\sigma_1,\dots,\sigma_n\rangle$ of $GL(\mathbb{Z}^n)$ generated by σ_i for all $i\in Q_0$, where σ_i is the contragradient of the geometric representation* [BB]. *In particular, cones of basic support τ-tilting Λ-modules give chambers of the associated root system of Q.*

Thus, cones of preprojective algebras are completely determined by simple calculations.

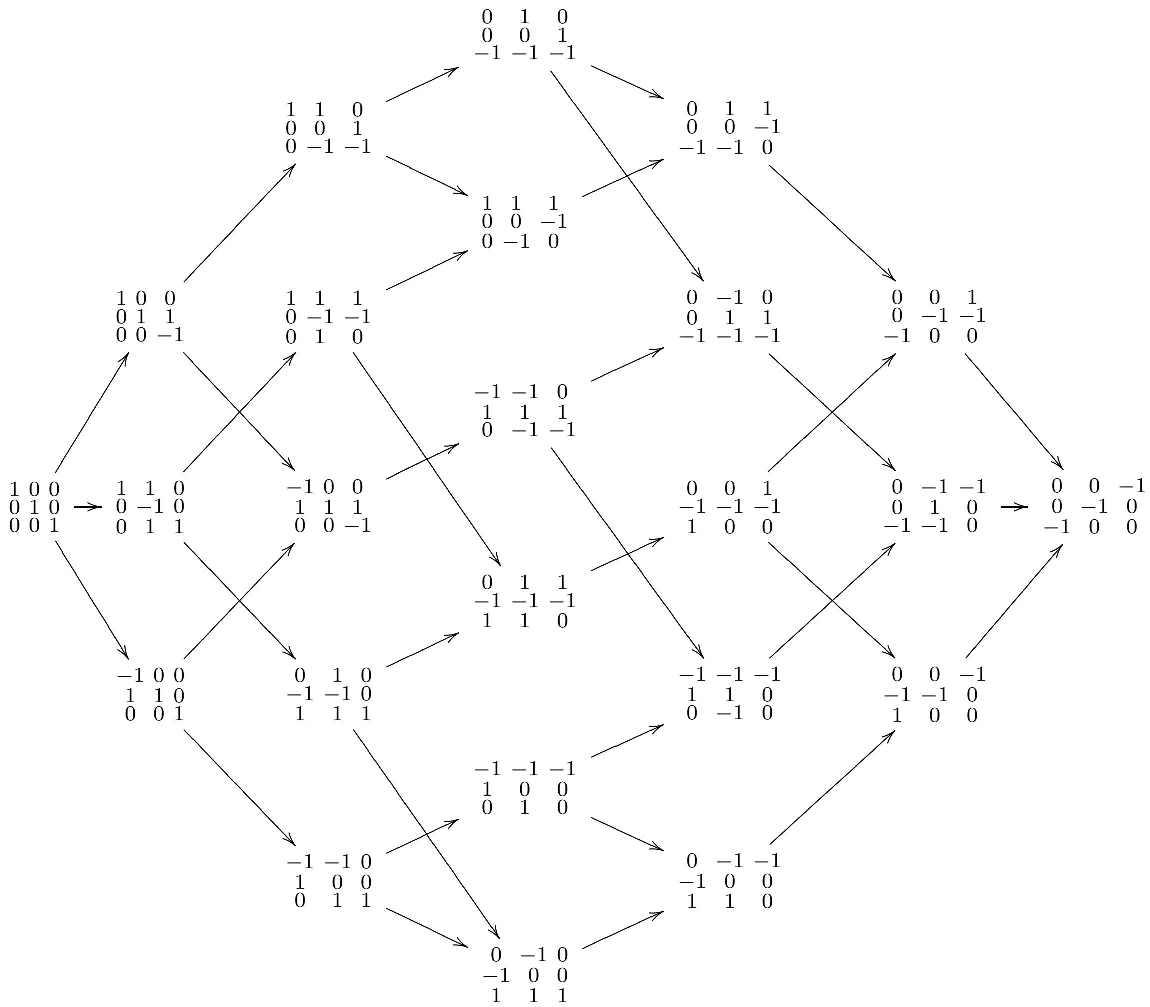
Example 24. (a) Let Λ be the preprojective algebra of type A_2. In this case, the g-matrices of Example 13 (a) are given as follows, where $\sigma_1=\left(\begin{smallmatrix}-1&0\\1&1\end{smallmatrix}\right)$ and $\sigma_2=\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right)$.

$$
\begin{tikzcd}
 & \begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix} \arrow[r, "\sigma_1"] & \begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix} \arrow[dr, "\sigma_2"] & \\
\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \arrow[ur, "\sigma_2"] \arrow[dr, "\sigma_1"'] & & & \begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix} \\
 & \begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix} \arrow[r, "\sigma_2"'] & \begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix} \arrow[ur, "\sigma_1"'] &
\end{tikzcd}
$$

Hence, their cones are given as follows.



(b) Let Λ be the preprojective algebra of type A_3. In this case, the g-matrices of Example 13 (b) are given as follows.



5. Further connections

In this section, we explain connections with other works. Here, let $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ be the bounded derived category of $\mathsf{mod}\Lambda$ and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ the bounded homotopy category of $\mathsf{proj}\Lambda$. Then we have the following bijections

Theorem 25. *Let Q be a Dynkin quiver with vertices $Q_0 = \{1, \dots, n\}$ and Λ the preprojective algebra of Q. There are bijections between the following objects.*

(a) *The elements of the Weyl group W_Q.*
(b) *The set $\langle I_1, \dots, I_n \rangle$.*
(c) *The set of isomorphism classes of basic support τ-tilting Λ-modules.*
(d) *The set of isomorphism classes of basic support τ-tilting Λ^{op}-modules.*
(e) *The set of torsion classes in $\mathsf{mod}\Lambda$.*
(f) *The set of torsion-free classes in $\mathsf{mod}\Lambda$.*
(g) *The set of isomorphism classes of basic two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.*
(h) *The set of intermediate bounded co-t-structures in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ with respect to the standard co-t-structure.*

(i) *The set of intermediate bounded t-structures in* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ *with length heart with respect to the standard t-structure.*
(j) *The set of isomorphism classes of two-term simple-minded collections in* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$*.*
(k) *The set of quotient closed subcategories in* $\mathsf{mod}KQ$*.*
(l) *The set of subclosed subcategories in* $\mathsf{mod}KQ$*.*

We have given bijections between (a), (b), (c) and (d) in the previous section. Bijections between (g), (h), (i) and (j) are the restriction of [KY] and it is given in [BY, Corollary 4.3] (it is stated for Jacobian algebras, but the statement holds for any finite dimensional algebra). Bijections between (a), (k) and (l) are given by [ORT] (note that a bijection (a) and (k) holds for any acyclic quiver with a slight modification, see [ORT]).

A bijection between (c) and (g) is shown by [AIR, Theorem 3.2] for any finite dimensional algebra.

Bijections between (a), (e) and (f) follow from the next statement, which provides complete descriptions of torsion classes and torsion-free classes in $\mathsf{mod}\Lambda$.

Proposition 26. (i) *For any torsion class $\mathcal{T}$ in* $\mathsf{mod}\Lambda$*, there exists $w \in W_Q$ such that* $\mathcal{T} = \mathsf{Fac}I_w$*.*
(ii) *For any torsion-free class $\mathcal{F}$ in* $\mathsf{mod}\Lambda$*, there exists $w \in W_Q$ such that* $\mathcal{F} = \mathsf{Sub}\Lambda/I_w$*.*

Remark 27. It is shown that objects $\mathsf{Fac}I_w$ and $\mathsf{Sub}\Lambda/I_w$ have several nice properties. For example, $\mathsf{Fac}I_w$ and $\mathsf{Sub}\Lambda/I_w$ are *Frobenius* categories and, moreover, *stable 2-CY* categories which have cluster-tilting objects. They also play important roles in the study of cluster algebra structures for a coordinate ring of the unipotent cell associated with w (see [BIRS, GLS]).

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CLASSICAL DERIVED FUNCTORS AS FULLY FAITHFUL EMBEDDINGS

PEDRO NICOLÁS AND MANUEL SAORÍN

Abstract. Given associative unital algebras A and B and a complex $T^\bullet$ of $B-A-$bimodules, we give necessary and sufficient conditions for the derived functors $\mathbf{R}\mathrm{Hom}_A(T^\cdot, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ and $? \otimes^{\mathbf{L}}_B T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ to be fully faithful. We also give criteria for these functors to be one of the fully faithful functors appearing in a recollement of derived categories. In the case when $T^\bullet$ is just a $B-A-$bimodule, we connect the results with (infinite dimensional) tilting theory and show that some open question on the fully faithfulness of $\mathbf{R}\mathrm{Hom}_A(T, ?)$ is related to the classical Wakamatsu tilting problem.

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1. Introduction

In October 2013, the second named author was invited to the *46th Japan Symposium on Ring and Representation Theory* and the title of one of his talks was exactly the title of this paper, which tries to be a much expanded version of that talk. Several results that we will present here are particular cases of results given in [32] in the language of dg categories and will be published elsewhere. For different reasons, the language of dg categories tends to be difficult to understand by people working both in Ring Theory and Representation Theory, and it is specially so for beginners in the field. The main motivation of the present work is to isolate the material of [32] which applies to ordinary (always associative unital) algebras and rings, and use it to go further in its applications in terms of recollement situations that were only indirectly considered in [32]. We hope in this way that the results that we present interest ring and representation theorists. Only minor references to dg algebras will be needed, but the bulk of the contents stays within the scope of ordinary algebras and rings.

Apart from the extraordinary hospitality of the organizers, the most captivating thing for the mentioned author was the very active, lively and enthusiastic Japanese youth community in the field, who presented their recent work, sometimes impressive. This paper is written thinking mainly on them. Aimed at beginners, in the initial sections of the paper we have tried to be as self-contained as possible, referring to the written literature only for technical definitions, some proofs and specific details.

All throughout the paper the term 'algebra' will denote an associative unital algebra over a ground commutative ring k, fixed in the sequel. Unless otherwise stated, 'module' will mean 'right module' and the corresponding category of modules over an algebra A will be denoted by $\mathrm{Mod} - A$. Left A-modules will be looked at as right modules over

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the opposite algebra A^{op}. Then $\mathcal{D}(A)$ and $\mathcal{D}(A^{op})$ will denote the derived categories of the categories of right and left A-modules, respectively. On what concerns set-theoretical matters, unlike [32], in this paper we will avoid the universe axiom and, instead, we will distinguish between 'sets' and '(proper) classes'. All families will be set-indexed families and an expression of the sort 'it has (co)products' will always mean 'it has set-indexed (co)products'.

By now, the following is a classical result due to successive contributions by Happel , Rickard and Keller (see [15], [34], [35] and [19]). We refer the reader to sections 2 and 3 for the pertinent definitions.

Theorem 1.1. *Let A and B be ordinary algebras and let $T^\bullet$ be a complex of $B-A-$bimodules. The following assertions are equivalent:*

1. *The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is an equivalence of categories;*
2. *The functor $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is an equivalence of categories;*
3. *$T_A^\bullet$ is a classical tilting object of $\mathcal{D}(A)$ such that the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism.*

It is natural to ask what should be the substitute of assertion (3) in this theorem when, in assertion (1) (resp. assertion (2)), we only require that $? \otimes_B^{\mathbf{L}} T^\bullet$ (resp. $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$) be fully faithful. That is the first goal of this paper. Namely, given a complex $T^\bullet$ of $B-A-$bimodules, we want to give necessary and sufficient conditions for $? \otimes_B^{\mathbf{L}} T^\bullet$ and $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ to be fully faithful functors.

On the other hand, a weaker condition than the one in the theorem appears when the algebras A and B admit a recollement situation

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "i_*=i_!" description] & \mathcal{D} \arrow[l, shift right=2, "i^*"'] \arrow[l, shift left=2, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{D}'' \arrow[l, shift right=2, "j_!"'] \arrow[l, shift left=2, "j_*"] \end{tikzcd}$$

as defined by Beilinson, Bernstein and Deligne ([7]), where either $\{\mathcal{D}, \mathcal{D}'\} = \{\mathcal{D}(A), \mathcal{D}(B)\}$ or $\{\mathcal{D}, \mathcal{D}''\} = \{\mathcal{D}(A), \mathcal{D}(B)\}$. In these cases, the functors $i_* = i_!$, $j_!$ and j_* are also fully faithful. This motivates the second goal of the paper. We want to give necessary and sufficient conditions for those recollements to exist, but imposing the condition that some of the functors in the picture be either $? \otimes_B^{\mathbf{L}} T^\bullet$ or $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$.

Finally, the following are natural questions for which we want to have an answer.

Questions 1.2. *Let $T^\bullet$ be a complex of $B-A-$bimodules.*

1. *Suppose that $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful.*
   (a) *Is there a recollement*

$$\begin{tikzcd} \mathcal{D}(A) \arrow[r, "i_*=i_!" description] & \mathcal{D}(B) \arrow[l, shift right=2, "i^*"'] \arrow[l, shift left=2, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{D}'' \arrow[l, shift right=2, "j_!"'] \arrow[l, shift left=2, "j_*"] \end{tikzcd} ,$$

   with $i_* = \mathbf{R}Hom_A(T^\bullet, ?)$, for some triangulated category $\mathcal{D}''$?
   (b) *Is there a recollement*

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "i_*=i_!" description] & \mathcal{D}(B) \arrow[l, shift right=2, "i^*"'] \arrow[l, shift left=2, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{D}(A) \arrow[l, shift right=2, "j_!"'] \arrow[l, shift left=2, "j_*"] \end{tikzcd} ,$$

whith $j_* = \mathbf{R}Hom_A(T^\bullet, ?)$, *for some triangulated category* $\mathcal{D}'$*?*

(2) *Suppose that* $? \otimes^{\mathbf{L}}_B T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ *is fully faithful.*

(a) *Is there a recollement*

$$\mathcal{D}(B) \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i_*=i_!}}}}} \mathcal{D}(A) \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{j^!=j^*}}}}} \mathcal{D}'' ,$$

with $i_* = ? \otimes^{\mathbf{L}}_B T^\bullet$, *for some triangulated category* $\mathcal{D}''$*?*

(b) *Is there a recollement*

$$\mathcal{D}' \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i_*=i_!}}}}} \mathcal{D}(A) \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{j^!=j^*}}}}} \mathcal{D}(B) ,$$

whith $j_! = ? \otimes^{\mathbf{L}}_B T^\bullet$, *for some triangulated category* $\mathcal{D}'$*?*

The organization of the paper goes as follows. In section 2 we give the preliminary results on triangulated categories and the corresponding terminology used in the paper. This part has been prepared as an introductory material for beginners and, hence, tends to be as self-contained as possible.

Section 3 is specifically dedicated to the derived functors of Hom and the tensor product, but, due to the requirements of some later proofs in the paper, the development is made for derived categories of bimodules. In this context the material seems to be unavailable in the literature. Special care is put on describing the behavior of these derived functors when passing from the derived categoy of bimodules to derived category of modules on one side. In the final part of the section, we give a brief introduction to dg algebras and give a generalization of Rickard theorem, in the case of the derived category of a k-flat dg algebra (Theorem 3.15).

Section 4 contains the main results in the paper. We first show that the compact objects in the derived category of an algebra are precisely those for which the associated derived tensor product preserves products (proposition 4.2). We then go towards the mentioned goals of the paper. Proposition 4.4 gives a criterion for the fully faithfulness of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$, while proposition 4.10 gives criteria for the fully faithfulness of $? \otimes^{\mathbf{L}}_B T^\bullet$. In a parallel way, in corollary 4.5, theorem 4.6, theorem 4.13 and corollary 4.14 we give criteria for the existence of the recollements mentioned in questions 1.2 (1.a, 2.a, 2.b and 1.b, respectively). As a confluent point, when $T^\bullet_A$ is exceptional in $\mathcal{D}(A)$ and the algebra morphism $B \longrightarrow \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism, we show in theorem 4.18 that $T^\bullet$ defines a recollement as in question 1.2(1.b) if, only if, it defines a recollement as in question 1.2(2.b) on the derived categories of left modules, and this is turn equivalent to saying that A is in the thick subcategory of $\mathcal{D}(A)$ generated by $T^\bullet_A$. We end the section by giving counterexamples to all questions 1.2 and by proposing some other questions which remain open.

In the final section 5, we explicitly re-state some of the results of section 4 in the particular case that $T^\bullet = T$ is just a $B-A-$bimodule. One of the questions asked in section 4 asks whether $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ preserves compact objects, when it is fully faithful, $T^\bullet_A$ is exceptional in $\mathcal{D}(A)$ and B is isomorphic to $\mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$. We end the paper by showing that, when $T^\bullet = T$ is a bimodule, this question is related to the classical Wakamatsu tilting problem.

Some of our results are connected to recent results of Bazzoni-Mantese-Tonolo [5], Bazzoni-Pavarin [6], Chen-Xi ([10], [11]), Han [14] and Yang [41]. All throughout sections 4 and 5, we give remarks showing these connections.

2. Preliminaries on triangulated categories and derived functors

The results of this section are well-known, but they sometimes appear scattered in the literature and with different notation. We give them here for the convenience of the reader and, also, as a way of unifying the terminology that we shall use throughout the paper. Most of the material is an adaptation of Verdier's work (see [38]), but we will refer also to several texts like [29], [40], [16], [18], [25]..., for specific results and proofs. As mentioned before, we will work over a fixed ground commutative ring k. Then the term 'category' will mean always 'k-category'. If $\mathcal{C}$ is such a category, then the set of morphisms $\mathcal{C}(C, D)$ has a structure of k-module, for all $C, D \in \mathcal{D}$, and compositions of morphisms are k-bilinear. Unless otherwise stated, subcategories will be always full and closed under taking isomorphic objects.

The reader is referred to [40, Chapter 10] for the explicit definition *triangulated category*, although some of its notation is changed. If $\mathcal{D}$ is such a category, then the *shifting*, also called *suspension (or translation) functor* $\mathcal{D} \longrightarrow \mathcal{D}$ will be denoted here by $?[1]$ and a triangle in $\mathcal{D}$ will be denoted by $X \longrightarrow Y \longrightarrow Z \stackrel{+}{\longrightarrow}$, although we will also write $X \longrightarrow Y \longrightarrow Z \stackrel{h}{\longrightarrow} X[1]$ when the connecting morphism h need be emphasized. Recall that Z is determined by the morphism $f : X \longrightarrow Y$ up to non-unique isomorphism. We will call Z the *cone of* f. A functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ between triangulated categories will be called a *triangulated or triangle-preserving functor* when it takes triangles to triangules.

2.1. **The triangulated structure of the stable category of a Frobenius exact category.** An exact category (in the sense of Quillen) is an additive category $\mathcal{C}$, together with a class of short exact sequences, called *conflations* or *admissible short exact sequences.* If $0 \rightarrow C \stackrel{f}{\longrightarrow} D \stackrel{g}{\longrightarrow} E \rightarrow 0$ is a conflation, then we say that f is an *inflation* or *admissible monomorphism* and that g is a *deflation* or *admissible epimorphism.* The class of conflations must satisfy the following axioms and their duals (see [18] and [8]):

Ex0 The identity morphism 1_X is a deflation, for each $X \in \mathrm{Ob}(C)$;
Ex1 The composition of two deflations is a deflation;
Ex2 Pullbacks of deflations along any morphism exist and are deflations.

Obviously the dual of an exact category is an exact category with the 'same' class of conflations. An object P of an exact category is called *projective* when the functor $\mathcal{C}(P, ?) : \mathcal{C} \longrightarrow k-\mathrm{Mod}$ takes deflations to epimorphisms. The dual notion is that of *injective* object. We say that $\mathcal{C}$ *has enough projectives (resp. injectives)* when, for each object $C \in \mathcal{C}$, there is a deflation $P \longrightarrow C$ (resp. inflation $C \longrightarrow I$), where P (resp. I) is a projective (resp. injective) object. The exact category $\mathcal{C}$ is said to be *Frobenius exact* when it has enough projectives and enough injectives and the classes of projective and injective objects coincide.

Given any exact Frobenius category $\mathcal{C}$, we can form its *stable category*, denoted $\underline{\mathcal{C}}$. Its objects are those of $\mathcal{C}$, but we have $\underline{\mathcal{C}}(C,D) = \frac{\mathcal{C}(C,D)}{\mathcal{P}(C,D)}$, where $\mathcal{P}(C,D)$ is the k-submodule of $\mathcal{C}(C,D)$ consisting of those morphisms which factor through some projective(=injective) object. The new category comes with a *projection functor* $p_{\mathcal{C}} : \mathcal{C} \longrightarrow \underline{\mathcal{C}}$, which has the property that each functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which vanishes on the projective (=injective) objects factors through $p_{\mathcal{C}}$ in a unique way.

The so-called *(first) syzygy functor* $\Omega : \underline{C} \longrightarrow \underline{\mathcal{C}}$ assigns to each object C the kernel of any deflation (=admissible epimorphism) $\epsilon : P \twoheadrightarrow C$ in $\mathcal{C}$, where P is a projective object. Up to isomorphism in $\underline{\mathcal{C}}$, the object $\Omega(C)$ does not depend on the projective object P or the deflation ϵ. Moreover, Ω is an equivalence of categories and its quasi-inverse is called the *(first) cosyzygy functor* $\Omega^{-1} : \underline{C} \longrightarrow \underline{C}$.

If $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ a conflation in the Frobenius exact category $\mathcal{C}$, then we have the following commutative diagram, where the rows are conflations and I is a projective(=injective) object:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "f"] \arrow[d, equal] & D \arrow[r, "g"] \arrow[d, dotted] & E \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & C \arrow[r] & I \arrow[r] & \Omega^{-1}C \arrow[r] & 0
\end{tikzcd}$$

The following result is fundamental (see [16, Section I.2]):

Proposition 2.1 (Happel). *If $\mathcal{C}$ is a Frobenius exact category, then its stable category admits a structure of triangulated category, where:*

(1) *the suspension functor is the first cosyzygy functor;*
(2) *the distinguished triangles are those isomorphic in $\underline{\mathcal{C}}$ to a sequence of morphisms*

$$C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} \Omega^{-1}(C)$$

coming from a commutative diagram in $\mathcal{C}$ as above.

2.2. **The Frobenius exact structure on the category of chain complexes. The homotopy category.** In the rest of this section $\mathcal{A}$ will be an abelian category. The *graded category* associated to $\mathcal{A}$, denoted $\mathcal{A}^{\mathbb{Z}}$ has as objects the $\mathbb{Z}$-indexed families $X^\bullet := (X^n)_{n\in\mathbb{Z}}$, where $X_n \in \mathcal{A}$ for each $n \in \mathbb{Z}$. If $X^\bullet, Y^\bullet \in \mathcal{A}^{\mathbb{Z}}$ then $\mathcal{A}^{\mathbb{Z}}$ consists of the families $f = (f^n)_{n\in\mathbb{Z}}$, where $f^n \in \mathcal{A}(X^n, Y^n)$ for each $n \in \mathbb{Z}$. This category is abelian and comes with a canonical self-equivalence $?[1] : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$, called the *suspension* or *(1-)shift*, given by the rule $X^\bullet[1]^n = X^{n+1}$, for all $n \in \mathbb{Z}$. Keeping the same class of objects, we can increase the class of morphisms and form a new category $\mathrm{GR}\mathcal{A}$, where $\mathrm{GR}\mathcal{A}(X^\bullet, Y^\bullet) = \oplus_{n\in\mathbb{Z}}\mathcal{A}^{\mathbb{Z}}(X^\bullet, Y^\bullet[n])$. This is obviously a graded category where a *morphisms of degree* n is just a morphism $X^\bullet \longrightarrow Y^\bullet[n]$ in $\mathcal{A}^{\mathbb{Z}}$, and where $g \circ f$ is the composition $X^\bullet \xrightarrow{f} Y^\bullet[m] \xrightarrow{g[m]} Z[m+n]$ in $\mathcal{A}^{\mathbb{Z}}$, whenever f and g are morphisms in $GR\mathcal{A}$ of degrees m and n.

A *chain complex* of objects of $\mathcal{A}$ is a pair $(X^\bullet, d)$ consisting of an object $X^\bullet$ of $\mathrm{GR}\mathcal{A}$ together with a morphism $d : X^\bullet \longrightarrow X^\bullet$ in $GR\mathcal{A}$ of degree $+1$ such that $d \circ d = 0$. The *category of chain complexes* of objects of $\mathcal{A}$ will be denoted by $\mathcal{C}(\mathcal{A})$. It has as objects the chain complexes of objects of $\mathcal{A}$ and a morphism $f : X^\bullet \longrightarrow Y^\bullet$, usually called a *chain map*, will be just a morphism of degree 0 in GR$\mathcal{A}$ such that $f \circ d = d \circ f$. The category $\mathcal{C}(\mathcal{A})$ is abelian, with pointwise calculation of limits and colimits, and we have an obvious forgetful functor $\mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}^{\mathbb{Z}}$ which is faithful and dense, but not full. What is even more important for us is that $\mathcal{C}(\mathcal{A})$ admits a structure of Quillen exact category, usually called the *semi-split exact structure*. Recall that a short exact sequence $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ in $\mathcal{C}(\mathcal{A})$ is called *semi-split* when its image by the forgetful functor $\mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is a split exact sequence of $\mathcal{A}^{\mathbb{Z}}$. That is, when the sequence $0 \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0$ splits in $\mathcal{A}$, for each $n \in \mathbf{Z}$. For the semi-split exact structure in $\mathcal{C}(\mathcal{A})$ the conflations are precisely the semi-split exact sequences. The suspension functor of $\mathcal{A}^{\mathbb{Z}}$ induces a corresponding suspension functor $?[1] : \mathcal{C}(\mathcal{A}) \xrightarrow{\cong} \mathcal{C}(\mathcal{A})$, which is a self-equivalence and take $(X^\bullet, d_X) \rightsquigarrow (X^\bullet[1], d_{X[1]}$, where $d^n_{X^\bullet[1]} = -d^{n+1}_X$ for each $n \in \mathbb{Z}$.

It is well-known (see, e.g., [16, Section I.3.2]) that $\mathcal{C}(\mathcal{A})$ is a *Frobenius exact category* when considered with this exact structure. Its projective (=injective) objects with respect to the conflations are precisely the *contractible complexes*. These are those complexes isomorphic in $\mathcal{C}(\mathcal{A})$ to a coproduct of complexes of the form $\mathrm{C}_n(X) : ...0 \longrightarrow X \xrightarrow{1_X} X \longrightarrow 0....$, where $C_n(X)$ is concentrated in degrees $n-1, n$ for all $n \in \mathbb{Z}$ and all $X \in \mathcal{A}$. For each $X^\bullet \in \mathcal{C}(\mathcal{A})$, we have a conflation

$$0 \to X^\bullet[-1] \longrightarrow \textstyle\coprod_{n \in \mathbb{Z}} C_n(X_{n-1}) \longrightarrow X^\bullet \to 0.$$

Then the syzygy functor of $\mathcal{C}(\mathcal{A})$ with respect to the semi-split exact structure is identified with the inverse $?[-1] : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{A})$ of the suspension functor. The stable category of $\mathcal{C}(\mathcal{A})$ has a very familiar description due to the following result (see [16, p. 28]):

Lemma 2.2. *Let $(X^\bullet, d_X)$ and $(Y^\bullet, d_Y)$ be chain complexes of objects of $\mathcal{A}$ and let $f : X^\bullet \longrightarrow Y^\bullet$ be a chain map. The following assertions are equivalent:*

(1) *f factors through a contractible complex;*
(2) *f is* null-homotopic, *i.e., there exists a morphism $\sigma : X^\bullet \longrightarrow Y^\bullet$ of degree -1 in GR$\mathcal{A}$ such that $f = \sigma \circ d_X + d_Y \circ \sigma$.*

As a consequence of the previous result, the morphisms in $\mathcal{C}(\mathcal{A})$ which factor through a projective (=injective) object, with respect to the semi-split exact structure, are precisely the null-homotopic chain maps. As a consequence the associated stable category $\underline{\mathcal{C}(\mathcal{A})}$ is the *homotopy category* of $\mathcal{A}$, denoted $\mathcal{H}(\mathcal{A})$ in the sequel. We then get:

Corollary 2.3. *The homotopy category $\mathcal{H}(\mathcal{A})$ has a structure of triangulated category such that*

(1) *the suspension functor $?[1] : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ is induced from the suspension functor of $\mathcal{C}(\mathcal{A})$;*
(2) *each distinguished triangle in $\mathcal{H}(\mathcal{A})$ comes from a semi-split exact sequence $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ in $\mathcal{C}(\mathcal{A})$ in the form described in proposition 2.1.*

Recall that if $(\mathcal{D}, ?[1])$ is a triangulated category and $\mathcal{A}$ is an abelian category, then a *cohomological functor* $H : \mathcal{D} \longrightarrow \mathcal{A}$ is an additive functor such that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is any triangle in $\mathcal{D}$ and one puts $H^k = H \circ (?[k])$, for each $k \in \mathbb{Z}$, then we get a long exact sequence

$$...H^{k-1}(Z) \overset{H^{k-1}(h)}{\longrightarrow} H^k(X) \overset{H^k(f)}{\longrightarrow} H^k(Y) \overset{H^k(g)}{\longrightarrow} H^k(Z) \overset{H^k(h)}{\longrightarrow} H^{k+1}(X) \overset{H^{k+1}(f)}{\longrightarrow} H^{k+1}(Y)...$$

Recall that if $X^\bullet$ is a chain complex and $k \in \mathbb{Z}$, then the *k-th object of homology of* X is $H^k(X^\bullet) = \frac{\mathrm{Ker}(d^k)}{\mathrm{Im}(d^{k-1})}$, where $d : X^\bullet \longrightarrow X^\bullet$ is the differential. The assignment $X^\bullet \rightsquigarrow H^k(X^\bullet)$ is the definition on objects of a functor $H^k : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}$. The following is well-known:

Corollary 2.4. *Let $\mathcal{A}$ be any abelian category, $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ the projection functor and $k \in \mathbb{Z}$ be an integer. The functor $H^k : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ vanishes on null-homotopic chain maps and there is a unique k-linear functor $\bar{H}^k : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{A}$ such that $\bar{H}^k \circ p = H^k$. Moreover, $\bar{H}^k$ is a cohomological functor.*

Remark 2.5. *We will forget the overlining of H and will still denote by H^k the functor $\bar{H}^k : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{A}$.*

2.3. **Localization of triangulated categories.** As in the previous sections, the symbol $\mathcal{A}$ will denote a fixed abelian category. We will use the term *big category* to denote a concept defined as an usual category, but where we do not require that the morphsims between two objects form a set. The following is well-known (see [13, Chapter 1]):

Proposition 2.6. *Given a category $\mathcal{C}$ and a class $\mathcal{S}$ of morphisms in $\mathcal{C}$, there is a big category $\mathcal{C}[\mathcal{S}^{-1}]$, together with a dense functor $q : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ satisfying the following properties:*

1. *$q(s)$ is an isomorphism, for each $s \in \mathcal{S}$;*
2. *if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is any functor between categories such that $F(s)$ is an isomorphism for each $s \in \mathcal{S}$, then there is a unique functor $\bar{F} : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{D}$ such that $\bar{F} \circ q = F$.*

Definition 1. $\mathcal{C}[\mathcal{S}^{-1}]$ *is called the* localization of $\mathcal{C}$ with respect to $\mathcal{S}$.

Remarks 2.7. (1) *The pair $(\mathcal{C}[\mathcal{S}^{-1}], q)$ is uniquely determined up to equivalence.*
(2) *A sufficient condition to guarantee that $\mathcal{C}[\mathcal{S}^{-1}]$ is an usual category is that the functor $q : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ has a left or a right adjoint. If, say, $R : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}$ is right adjoint to q, then we have a bijection $\mathcal{C}[\mathcal{S}^{-1}](q(X), q(Y)) \cong \mathcal{C}(X, Rq(Y))$, for all $X, Y \in \mathcal{C}$. This proves that the morphisms between two objects of $\mathcal{C}[\mathcal{S}^{-1}]$ form a set since q is dense. A dual argument works in case q has a left adjoint.*

The explicit definition of $\mathcal{C}[\mathcal{S}^{-1}]$ was given in [13, Chapter 1], but the morphisms in this category are intractable in general. Then Gabriel and Zisman introduced some condition on $\mathcal{S}$ which makes much more tractable the morphisms in $\mathcal{C}[\mathcal{S}^{-1}]$.

Definition 2. *We shall say that $\mathcal{S}$* admits a calculus of left fractions *when it satisfies the following conditions:*

1. *$1_X \in \mathcal{S}$, for all $X \in \mathcal{C}$;*
2. *for each diagram $X' \overset{f'}{\longrightarrow} Y' \overset{s'}{\longleftarrow} Y$ in $\mathcal{C}$, with $s' \in \mathcal{S}$, there exists a diagram $X' \overset{s}{\longleftarrow} X \overset{f}{\longrightarrow} Y$ such that $s \in \mathcal{S}$ and $f' \circ s = s' \circ f$.*

(3) *if $f, g : X \longrightarrow Y$ are morphisms in $\mathcal{C}$ and there exists $t \in \mathcal{S}$ such that $t \circ f = t \circ g$, then there exists $s \in \mathcal{S}$ such that $f \circ s = g \circ s$.*

We say that $\mathcal{S}$ admits a calculus of right fractions *when it satisfies the duals of properties 1-3 above. Finally, we will say that $\mathcal{S}$* admits a calculus of fractions*, or that $\mathcal{S}$ is a* multiplicative system of morphisms*, when it admits both a calculus of left fractions and a calculus of right fractions.*

When $\mathcal{S}$ admits a calculus of left fractions, the morphisms in $\mathcal{C}[\mathcal{S}^{-1}]$ have a more tractable form. Indeed, if $X, Y \in \mathrm{Ob}(C) = \mathrm{Ob}(\mathcal{C}[\mathcal{S}^{-1}])$, then $\mathcal{C}[\mathcal{S}^{-1}](X, Y)$ consists of the formal left fractions $s^{-1}f$. Such a formal left fraction is the equivalence class of the pair (s, f) with respect to some equivalence relation defined in the class of diagrams $X \xleftarrow{s} X' \xrightarrow{f} Y$, with $s \in \mathcal{S}$. We refer the reader to [13, Chapter 1] for the precise definition of the equivalence relation and the composition of morphisms in $\mathcal{C}[\mathcal{S}^{-1}]$.

The process of localizing a category with respect to a class of morphism was developed in the context of triangulated categories by Verdier (see [38, Section II.2]).

Definition 3. *Let $\mathcal{D}$ be a triangulated category. A multiplicative system $\mathcal{S}$ in $\mathcal{D}$ is said to be* compatible with the triangulation *when the following properties hold:*

(1) *if $s : X \longrightarrow Y$ is a morphism in $\mathcal{S}$ and $n \in \mathbb{Z}$, then $s[n] : X[n] \longrightarrow Y[n]$ is in $\mathcal{S}$;*
(2) *each commutative diagram in $\mathcal{D}$*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[r, "g"] \arrow[d, "s'"] & Z \arrow[r, "h"] \arrow[d, "s''"] & X[1] \arrow[d, "s[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

where the rows are triangles and where $s, s' \in \mathcal{S}$, can be completed commutatively by an arrow s'' which is in $\mathcal{S}$.

As we can expect, one gets:

Proposition 2.8 (Verdier, Théorème 2.2.6)**.** *Let $\mathcal{D}$ be a triangulated category and $\mathcal{S}$ be a multiplicative system compatible with the triangulation in $\mathcal{D}$. There exists a unique structure of triangulated category in $\mathcal{D}[\mathcal{S}^{-1}]$ such that the canonical functor $q : \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{S}^{-1}]$ is triangulated. Moreover, if $\mathcal{A}$ is an abelian category and $H : \mathcal{D} \longrightarrow \mathcal{A}$ is a cohomological functor such that $H(s)$ is an isomorphism, for each $s \in \mathcal{S}$, then there is a unique cohomological funtor $\tilde{H} : \mathcal{D}[\mathcal{S}^{-1}] \longrightarrow \mathcal{A}$ such that $\tilde{H} \circ q = H$.*

2.4. **Semi-orthogonal decompositions. Brown representability theorem.** Let $\mathcal{D}$ be a triangulated category. A *t-structure* in $\mathcal{D}$ (see [7]) is a pair $(\mathcal{U}, \mathcal{W})$ of full subcategories which satisfy the following properties:

i) $\mathcal{D}(U, W[-1]) = 0$, for all $U \in \mathcal{U}$ and $W \in \mathcal{W}$;
ii) $\mathcal{U}[1] \subseteq \mathcal{U}$;
iii) For each $X \in Ob(\mathcal{D})$, there is a triangle $U \longrightarrow X \longrightarrow V \xrightarrow{+}$ in $\mathcal{D}$, where $U \in \mathcal{U}$ and $V \in \mathcal{W}[-1]$.

It is easy to see that in such case $\mathcal{W} = \mathcal{U}^{\perp}[1]$ and $\mathcal{U} = {}^{\perp}(\mathcal{W}[-1]) = {}^{\perp}(\mathcal{U}^{\perp})$. For this reason, we will write a t-structure as $(\mathcal{U}, \mathcal{U}^{\perp}[1])$. We will call $\mathcal{U}$ and $\mathcal{U}^{\perp}$ the *aisle* and the *co-aisle* of the t-structure, respectively. The objects U and V in the above triangle are uniquely determined by X, up to isomorphism, and define functors $\tau_{\mathcal{U}} : \mathcal{D} \longrightarrow \mathcal{U}$ and $\tau^{\mathcal{U}^{\perp}} : \mathcal{D} \longrightarrow \mathcal{U}^{\perp}$ which are right and left adjoints to the respective inclusion functors. We call them the *left and right truncation functors* with respect to the given t-structure.

Recall that a subcategory $\mathcal{X}$ of a triangulated category $\mathcal{D}$ is *closed under extensions* when, given any triangle $X' \longrightarrow Y \longrightarrow X \stackrel{+}{\longrightarrow}$ in $\mathcal{D}$, with $X, X' \in \mathcal{X}$, the object Y is also in $\mathcal{X}$.

Definition 4. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{U} \subseteq \mathcal{D}$ a full subcategory. We say that $\mathcal{U}$ is*

(1) suspended *when it is closed under extensions in $\mathcal{D}$ and $\mathcal{U}[1] \subseteq \mathcal{U}$;*
(2) triangulated *when it is closed under extensions and $\mathcal{U}[1] = \mathcal{U}$*
(3) thick *when it is triangulated and closed under taking direct summands.*

Notation and terminology.- Given a class $\mathcal{S}$ of objects of $\mathcal{D}$, we shall denote by $\mathrm{susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $\mathrm{tria}_{\mathcal{D}}(\mathcal{S})$, resp. $\mathrm{thick}_{\mathcal{D}}(\mathcal{S})$) the smallest suspended (resp. triangulated, resp. thick) subcategory of $\mathcal{D}$ containing $\mathcal{S}$. If $\mathcal{U}$ is any suspended (resp. triangulated, resp. thick) subcategory of $\mathcal{D}$ and $\mathcal{U} = \mathrm{susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $\mathcal{U} = \mathrm{tria}_{\mathcal{D}}(\mathcal{S})$, resp. $\mathrm{thick}_{\mathcal{D}}(\mathcal{S})$), we will say that $\mathcal{U}$ is the *suspended (resp. triangulated, resp. thick) subcategory of $\mathcal{D}$ generated by $\mathcal{S}$*. When $\mathcal{D}$ has coproducts, we will denote by $\mathrm{Susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $\mathrm{Tria}_{\mathcal{D}}(\mathcal{S})$) the smallest suspended (resp. triangulated) subcategory of D containing $\mathcal{S}$ which is closed under taking coproducts in $\mathcal{D}$.

Remark 2.9. *If $\mathcal{D}$ has coproducts and $\mathcal{T}$ is a full subcategory closed under taking coproducts, then it is a triangulated subcategory if, and only if, it is thick (see* [29, Prop.1.6.8 and its proof]*).*

It is an easy exercise to prove now the following useful result.

Lemma 2.10. *Let $F : \mathcal{D} \longrightarrow \mathcal{D}'$ be a triangulated functor between triangulated categories. If $\mathcal{S} \subseteq \mathcal{D}$ is any class of objects, then $F(\mathrm{susp}_{\mathcal{D}}(\mathcal{S})) \subseteq \mathrm{susp}_{\mathcal{D}'}(F(\mathcal{S}))$ (resp. $F(\mathrm{tria}_{\mathcal{D}}(\mathcal{S})) \subseteq \mathrm{tria}_{\mathcal{D}'}(F(\mathcal{S}))$, resp. $F(\mathrm{thick}_{\mathcal{D}}(\mathcal{S})) \subseteq \mathrm{thick}_{\mathcal{D}'}(F(\mathcal{S}))$). When $\mathcal{D}$ and $\mathcal{D}'$ have coproducts and F preserves coproducts, we also have that $F(\mathrm{Susp}_{\mathcal{D}}(\mathcal{S})) \subseteq \mathrm{Susp}_{\mathcal{D}'}(F(\mathcal{S}))$ (resp. $F(\mathrm{Tria}_{\mathcal{D}}(\mathcal{S})) \subseteq \mathrm{Tria}_{\mathcal{D}'}(F(\mathcal{S}))$).*

The following result of Keller and Vossieck [23, Proposition 1.1] is fundamental to deal with t-structures and semi-orthogonal decompositions.

Proposition 2.11. *A full subcategory $\mathcal{U}$ of $\mathcal{D}$ is the aisle of a t-structure if, and only if, it is a suspended subcategory such that the inclusion functor $\mathcal{U} \hookrightarrow \mathcal{D}$ has a right adjoint.*

The type of t-structure which is most useful to us in this paper is the following.

Definition 5. *A* semi-orthogonal decomposition *or* Bousfield localization pair *in $\mathcal{D}$ is a t-structure $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ such that $\mathcal{U}[1] = \mathcal{U}$ (equivalently, such that $\mathcal{U}^{\perp} = \mathcal{U}^{\perp}[1]$). That is, a semi-orthogonal decomposition is a t-structure such that $\mathcal{U}$ (resp. $\mathcal{U}^{\perp}$) is a triangulated* *subcategory of $\mathcal{D}$. In such case we will use $(\mathcal{U},\mathcal{U}^{\perp})$ instead of $(\mathcal{U},\mathcal{U}^{\perp}[1])$ to denote the semi-orthogonal decomposition.*

Certain adjunctions of triangulated functors provide semi-orthogonal decompositions.

Proposition 2.12. *Let $F:\mathcal{D}\longrightarrow\mathcal{D}'$ and $G:\mathcal{D}'\longrightarrow\mathcal{D}$ be triangulated functors between triangulated categories such that (F,G) is an adjoint pair. The following assertions hold:*

1. *If F is fully faithful, then $(Im(F), Ker(G))$ is a semi-orthogonal decomposition of $\mathcal{D}'$;*
2. *If G is fully faitfhful, then $(Ker(F), Im(G))$ is a semi-orthogonal decomposition of $\mathcal{D}$.*

Proof. Assertion (2) follows from assertion (1) by duality. To prove (1), note that the unit $\lambda:1_{\mathcal{D}}\longrightarrow G\circ F$ is an isomorphism (see [17, Proposition II.7.5]) and, by the adjunction equations, we then get that $G(\delta)$ is also an isomorphism, where $\delta:F\circ G\longrightarrow 1_{\mathcal{D}'}$ is the counit. This implies that if $M\in\mathcal{D}'$ is any object and we complete δ_M to a triangle

$$(F\circ G)(M)\xrightarrow{\delta_M}M\longrightarrow Y_M\xrightarrow{+},$$

then $Y_M\in\mathrm{Ker}(G)$.

But, by the adjunction, we have that $\mathcal{D}'(F(D),Y)=\mathcal{D}(D,G(Y))=0$, for each $Y\in\mathrm{Ker}(G)$. It follows that $(\mathrm{Im}(F),\mathrm{Ker}(G))$ is a semi-orthogonal decomposition of $\mathcal{D}'$. □

Given a triangulated subcategory $\mathcal{T}$ of $\mathcal{D}$, we shall denote by $\Sigma_{\mathcal{T}}$ the class of morphisms $s:X\longrightarrow Y$ in $\mathcal{D}$ whose cone is an object of $\mathcal{T}$. The following is a fundamental result of Verdier:

Proposition 2.13. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{T}$ a thick subcategory. The following assertions hold:*

1. *$\Sigma_{\mathcal{T}}$ is a multiplicative system of $\mathcal{D}$ compatible with the triangulation. The category $\mathcal{D}[\Sigma_{\mathcal{T}}^{-1}]$ is denoted by $\mathcal{D}/\mathcal{T}$ and called the* quotient category of $\mathcal{D}$ by $\mathcal{T}$.
2. *The canonical functor $q:\mathcal{D}\longrightarrow\mathcal{D}/\mathcal{T}$ satisfies the following universal property:*
   (*) For each triangulated category $\mathcal{D}'$ and each triangulated functor $F:\mathcal{D}\longrightarrow\mathcal{D}'$ such that $F(\mathcal{T})=0$, there is a triangulated functor $\bar{F}:\mathcal{D}/\mathcal{T}\longrightarrow\mathcal{D}'$, unique up to natural isomorphism, such that $\bar{F}\circ q\cong F$.
3. *The functor $q:\mathcal{D}\longrightarrow\mathcal{D}/\mathcal{T}$ has a right adjoint if, and only if, $(\mathcal{T},\mathcal{T}^{\perp})$ is a semi-orthogonal decomposition in $\mathcal{D}$. In this case, the functor $\tau^{\mathcal{T}^{\perp}}:D\longrightarrow\mathcal{T}^{\perp}$ induces an equivalence of triangulated categories $\mathcal{D}/\mathcal{T}\xrightarrow{\cong}\mathcal{T}^{\perp}$.*
4. *The functor $q:\mathcal{D}\longrightarrow\mathcal{D}/\mathcal{T}$ has a left adjoint if, and only if, $({}^{\perp}\mathcal{T},\mathcal{T})$ is a semi-orthogonal decomposition in $\mathcal{D}$. In this case, the functor $\tau_{{}^{\perp}\mathcal{T}}:D\longrightarrow{}^{\perp}\mathcal{T}$ induces an equivalence of triangulated categories $\mathcal{D}/\mathcal{T}\xrightarrow{\cong}{}^{\perp}\mathcal{T}$.*

Proof. Assertions (1) is [38, Proposition II.2.1.8] while assertion (2) is included in [38, Corollaire II.2.2.11]. Assertions (3) and (4) are dual to each other. Assertion (3) is implicit in [38, Proposition II.2.3.3]. For an explicit proof, see [25, Proposition 4.9.1] and use proposition 2.11. □

In many situations, we will need criteria for a given triangulated functor to have adjoints. The main tool is the following.

Definition 6. *Let $\mathcal{D}$ be a triangulated category with coproducts. We shall say that $\mathcal{D}$* satisfies Brown representability theorem *when any cohomological contravariant functor $H : \mathcal{D} \longrightarrow Mod - k$ which takes coproducts to products is representable. That is, there exists an object Y of $\mathcal{D}$ such that H is naturally isomorphic to $\mathcal{D}(?, Y)$.*

The key point is the following.

Proposition 2.14. *Let $\mathcal{D}$ be a triangulated subcategory which satisfies Brown representability theorem and let $\mathcal{D}'$ be any triangulated category. Each triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ which preserves coproducts has a right adjoint.*

Proof. See [29, Theorem 8.8.4]. □

Definition 7. *Let $\mathcal{D}$ have coproducts. An object X of $\mathcal{D}$ is called* compact *when the functor $\mathcal{D}(X, ?) : \mathcal{D} \longrightarrow Mod{-}k$ preserves coproducts. The category $\mathcal{D}$ is called* compactly generated *when there is a <u>set</u> $\mathcal{S}$ of compact objects such that $Tria_{\mathcal{D}}(\mathcal{S}) = \mathcal{D}$. We then say that $\mathcal{S}$ is a* set of compact generators *of $\mathcal{D}$.*

Corollary 2.15. *The following assertions hold, for any compactly generated triangulated category $\mathcal{D}$ and any covariant triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$:*

1. *F preserves coproducts if, and only if, it has a right adjoint;*
2. *F preserves products if, and only if, it has a left adjoint.*

Proof. See [25, Proposition 5.3.1]. □

The following lemma, whose proof can be found in [32], is rather useful.

Lemma 2.16. *Let $F : \mathcal{D} \longrightarrow \mathcal{D}'$ be a triangulated functor between triangulated categories and suppose that it has a left adjoint L and a right adjoint R. Then L is fully faithful if, and only if, so is R.*

2.5. **Recollements and TTF triples.** The following concept, introduced in [7], is fundamental in the theory of triangulated categories.

Definition 8. *A* recollement *of triangulated categories consists of a triple $(\mathcal{D}', \mathcal{D}, \mathcal{D}'')$ of triangulated categories and of six triangulated functors between them, assembled as follows*

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "i_*=i_!" description] & \mathcal{D} \arrow[l, shift right=2, "i^*"'] \arrow[l, shift left=2, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{D}'' \arrow[l, shift right=2, "j_!"'] \arrow[l, shift left=2, "j_*"] \end{tikzcd} ,$$

which satisfy the following conditons:

1. *Each functor in the picture is left adjoint to the one immediately below, when it exists;*
2. *The composition $i^! \circ j_*$ (and hence also $j^! \circ i_! = j^* \circ i_*$ and $i^* \circ j_!$) is the zero functor;*
3. *The functors i_*, j_* and $j_!$ are fully faithful (and hence the unit maps $1_{\mathcal{D}'} \longrightarrow i^! \circ i_* = i^! \circ i_!$, $1_{\mathcal{D}''} \longrightarrow j^* \circ j_! = j^! \circ j_!$ and the counit maps $i^* \circ i_! = i^* \circ i_* \longrightarrow 1_{\mathcal{D}'}$, $j^! \circ j_* = j^* \circ j_* \longrightarrow 1_{\mathcal{D}''}$ are all isomorphisms);*
4. *The remaining unit and counit maps of the different adjunctions give rise, for each object $X \in \mathcal{D}$, to triangules*

$$(i_! \circ i^!)(X) \longrightarrow X \longrightarrow (j_* \circ j^*)(X) \xrightarrow{+}$$
and
$$(j_! \circ j^!)(X) \longrightarrow X \longrightarrow (i_* \circ i^*)(X) \xrightarrow{+}.$$

In such situation, we will say that $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$.

A less familiar concept, coming from torsion theory in module categories, is the following (see [31]):

Definition 9. *Given a triangulated category $\mathcal{D}$, a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories is called a* TTF triple *in $\mathcal{D}$ when the pairs $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are both semi-orthogonal decompositions of $\mathcal{D}$.*

As shown in [31, Section 2.1], it turns out that recollements and TTF triples are equivalent concepts in the following sense:

Proposition 2.17. *Let $\mathcal{D}$ be a triangulated category. The following assertions hold:*

(1) *If*

$$\mathcal{D}' \underset{i^!}{\overset{i^*}{\leftleftarrows}}\!\!\xrightarrow{i_*=i_!} \mathcal{D} \underset{j_*}{\overset{j_!}{\leftleftarrows}}\!\!\xrightarrow{j^!=j^*} \mathcal{D}''$$

is a recollement of the triangulated category $\mathcal{D}$, then $(Im(j_!), Im(i_*), Im(j_*))$ is a TTF triple in $\mathcal{D}$;

(2) *If $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}$, then*

$$\mathcal{Y} \underset{i^!}{\overset{i^*}{\leftleftarrows}}\!\!\xrightarrow{i_*=i_!} \mathcal{D} \underset{j_*}{\overset{j_!}{\leftleftarrows}}\!\!\xrightarrow{j^!=j^*} \mathcal{X}$$

is a recollement, where:

(a) $i_* = i_! : \mathcal{Y} \longrightarrow \mathcal{D}$ *and* $j_! : \mathcal{X} \longrightarrow \mathcal{D}$ *are the inclusion functors;*

(b) $i^* = \tau^{\mathcal{Y}} : \mathcal{D} \longrightarrow \mathcal{Y}$ *is the right truncation with respect to the semi-orthogonal decomposition $(\mathcal{X}, \mathcal{Y})$ and $i^! = \tau_{\mathcal{Y}} : \mathcal{D} \longrightarrow \mathcal{Y}$ is the left truncation with respect to the semi-orthogonal decomposition $(\mathcal{Y}, \mathcal{Z})$;*

(c) $j^! = j^* = \tau_{\mathcal{X}} : \mathcal{D} \longrightarrow \mathcal{X}$ *is the left truncation with respect to the semi-orthogonal decomposition $(\mathcal{X}, \mathcal{Y})$;*

(d) j_* *is the composition $\mathcal{X} \hookrightarrow \mathcal{D} \xrightarrow{\tau^{\mathcal{Z}}} \mathcal{Z} \hookrightarrow \mathcal{D}$, where the hooked arrows are the inclusions and $\tau^{\mathcal{Z}}$ is the right truncation with respect to the semi-orthogonal decomposition $(\mathcal{Y}, \mathcal{Z})$.*

Remark 2.18. *In the rest of the paper, whenever*

$$\mathcal{D}' \underset{i^!}{\overset{i^*}{\leftleftarrows}}\!\!\xrightarrow{i_*=i_!} \mathcal{D} \underset{j_*}{\overset{j_!}{\leftleftarrows}}\!\!\xrightarrow{j^!=j^*} \mathcal{D}''$$

is a recollement of triangulated categories, we will simply write $\mathcal{D}' \equiv \mathcal{D} \equiv \mathcal{D}''$ and the six functors of the recollement will be understood.

We now give a criterion for a triangulated functor to be one of the two central arrows of a recollement.

Proposition 2.19. *The following assertions hold:*

1. *Let $F : \mathcal{D}' \longrightarrow \mathcal{D}$ be a triangulated functor between triangulated categories. There is a recollement $\mathcal{D}' \equiv \mathcal{D} \equiv \mathcal{D}''$, with $F = i_*$, for some triangulated category $\mathcal{D}''$, if and only if F is fully faithful and has both a left and a right adjoint;*
2. *Let $G : \mathcal{D} \longrightarrow \mathcal{D}''$ be a triangulated functor between triangulated categories. There is a recollement $\mathcal{D}' \equiv \mathcal{D} \equiv \mathcal{D}''$, with $G = j^*$, for some triangulated category $\mathcal{D}'$, if and only if G has both a left and a right adjoint, and one of these adjoints is fully faithful.*

Proof. (1) We just need to prove the 'if' part of the assertion. If F is fully faithful, then it induces an equivalence of categories $\mathcal{D}' \xrightarrow{\cong} \mathrm{Im}(F) =: \mathcal{Y}$. The fact that F has both a left and a right adjoint implies that also the inclusion functor $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{D}$ has both a left and a right adjoint. By proposition 2.11 and its dual, we get that $(\mathcal{Y}, \mathcal{Y}^{\perp})$ and $({}^{\perp}\mathcal{Y}, \mathcal{Y})$ are semi-orthogonal decompositions of $\mathcal{D}$. Therefore $({}^{\perp}\mathcal{Y}, \mathcal{Y}, \mathcal{Y}^{\perp})$ is a TTF triple in $\mathcal{D}$ and, by proposition 2.17, we have a recollement $\mathcal{Y} \equiv \mathcal{D} \equiv {}^{\perp}\mathcal{Y}$, with $i_* = i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{D}$ the inclusion functor. Using now the equivalence $\mathcal{D}' \xrightarrow{\cong} \mathcal{Y}$ given by F, we immediately get a recollement $\mathcal{D}' \equiv \mathcal{D} \equiv {}^{\perp}\mathcal{Y}$, with $i_* = F$.

(2) Again, we just need to prove the 'if' part. If $G : \mathcal{D} \longrightarrow \mathcal{D}''$ is a triangulated functor as stated and we denote by L and R its left and right adjoint, respectively, then lemma 2.16 tells us that L and R are both fully faithful. Then, by proposition 2.12, we get that $(\mathrm{Im}(L), \mathrm{Ker}(G), \mathrm{Im}(R))$ is a TTF triple in $\mathcal{D}$.

Since L gives an equivalence of triangulated categories $\tilde{L} : \mathcal{D}'' \xrightarrow{\cong} \mathrm{Im}(L) = \mathcal{X}$, we easily get that the left truncation functor $\tau_{\mathcal{X}}$ with respect to the semi-orthogonal decomposition $(\mathcal{X}, \mathcal{X}^{\perp}) = (\mathrm{Im}(L), \mathrm{Ker}(G))$ is naturally isomorphic to $\tilde{L} \circ G$. Using proposition 2.17, we then get a recollement $\mathrm{Ker}(G) \equiv \mathcal{D} \equiv D''$, where $j^* = G$.

□

2.6. **The derived category of an abelian category.** In this subsection $\mathcal{A}$ will be an abelian category. A morphism $f : X^{\bullet} \longrightarrow Y^{\bullet}$ in $\mathcal{C}(\mathcal{A})$ is called a *quasi-isomorphism* when the morphism $H^k(f) : H^k(X^{\bullet}) \longrightarrow H^k(Y^{\bullet})$ is an isomorphism in $\mathcal{A}$, for each $k \in \mathbb{Z}$. Our main object of interest is the following category.

Definition 10. *The* derived category of $\mathcal{A}$*, denoted $\mathcal{D}(\mathcal{A})$, is the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms.*

Note that, in general, $\mathcal{D}(\mathcal{A})$ is a big category. Moreover, defined as above, we have the problem of the intractability of its morphisms. But, fortunately, this latter obstacle is overcome:

Proposition 2.20 (Verdier). *Let $\mathcal{Q}$ be the class of quasi-isomoprhisms in $\mathcal{C}(\mathcal{A})$. The following assertions hold:*

1. *The canonical functor $q : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ factors through the projection functor $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$. More concretely, there is a unique functor, up to natural isomorphism, $\bar{q} : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ such that $\bar{q} \circ p = q$.*
2. *$\overline{\mathcal{Q}} := p(\mathcal{Q})$ is a multiplicative system in $\mathcal{H}(\mathcal{A})$ compatible with the triangulation and the functor $\bar{q}$ induces an equivalence of categories $\mathcal{H}(\mathcal{A})[\overline{Q}^{-1}] \xrightarrow{\cong} \mathcal{D}(\mathcal{A})$.*

Proof. See Définition 1.2.2, Proposition 1.3.5 and Remarque 1.3.7 in [38, Chapitre III]. □

Corollary 2.21. *$\mathcal{D}(\mathcal{A})$ admits a unique structure of triangulated category such that the functor $\bar{q} : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ is triangulated. Moreover, for each $k \in \mathbf{Z}$, the cohomological functor $H^k : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{A}$ factors thorough $\bar{q}$ in a unique way.*

It remains to settle the set-theoretical problem that $\mathcal{D}(\mathcal{A})$ is a big category. Led by remark 2.7(2), we will characterize when the functor $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ has an adjoint. Recall that an object $X^\bullet \in \mathcal{C}(\mathcal{A})$ is called an *acyclic complex* when it has zero homology, i.e., when $H^k(X^\bullet) = 0$, for all $k \in \mathbb{Z}$. Note that, when $\mathcal{A}$ is AB4, the k-th homology functor $H^k : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ preserves coproducts. In this case, if $(X_i^\bullet)_{i\in I}$ is a family of acyclic complexes which has a coproduct in $\mathcal{C}(\mathcal{A})$ (equivalently, in $\mathcal{H}(\mathcal{A})$), then $\coprod_{i\in I} X_i^\bullet$ is also an acyclic complex. Viewed as a full subcategory of $\mathcal{H}(\mathcal{A})$, it follows that the class $\mathcal{Z}$ of acyclic complexes is a triangulated subcategory closed under taking coproducts. The dual fact applies to products when $\mathcal{A}$ is AB4*. The following result (resp. its dual), which is a direct consequence of proposition 2.13, gives a criterion for the canonical functor $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ to have a right (resp. left) adjoint.

Proposition 2.22. *Let $\mathcal{A}$ be an AB4 abelian category and denote by $\mathcal{Z}$ the full subcategory of $\mathcal{H}(\mathcal{A})$ whose objects are the (images by the quotient functor $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ of) acyclic complexes. The following assertions are equivalent:*

(1) *The pair $(\mathcal{Z}, \mathcal{Z}^\perp)$ is a semi-orthogonal decomposition in $\mathcal{H}(\mathcal{A})$.*
(2) *The canonical functor $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ has a right adjoint.*

In such case, there is an equivalence of categories $\mathcal{D}(\mathcal{A}) \cong \mathcal{Z}^\perp$, so that $\mathcal{D}(\mathcal{A})$ is a real category and not just a big one.

Definition 11. *A chain complex $Q^\bullet \in \mathcal{C}(\mathcal{A})$ is called* homotopically injective *(resp.* homotopically projective*) when $p(Q^\bullet) \in \mathcal{Z}^\perp$ (resp. $p(Q^\bullet) \in {}^\perp\mathcal{Z}$), where $\mathcal{Z}$ is as in the previous proposition and $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ is the projection functor. A chain complex $X^\bullet$ is said to have a* homotopically injective resolution *(resp.* homotopically projective resolution*) in $\mathcal{H}(\mathcal{A})$ when there is quasi-isomorphism $\iota : X^\bullet \longrightarrow I_X^\bullet$ (resp. $\pi : P_X^\bullet \longrightarrow X^\bullet$), where $I_X^\bullet$ (resp. $P_X^\bullet$) is a homotopically injective (resp. homotopically projective) complex.*

Note that $X^\bullet$ has a homotopically injective resolution if, and only if, there is a triangle $Z^\bullet \longrightarrow X^\bullet \xrightarrow{\iota} I^\bullet \xrightarrow{+}$ in $\mathcal{H}(\mathcal{A})$ such that $Z^\bullet \in \mathcal{Z}$ and $I^\bullet \in \mathcal{Z}^\perp$ (i.e. $Z^\bullet$ is acyclic and $I^\bullet$ is homotopically injective). A dual fact is true about the existence of a homotopically projective resolution.

As a direct consequence of the definition of semi-orthogonal decomposition and of proposition 2.22, we get the following result. The statement of the dual result is left to the reader.

Corollary 2.23. *Let $\mathcal{A}$ be an AB4 abelian category. The canonical functor $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ has a right adjoint if, and only if, each chain complex $X^\bullet$ has a homotopically injective resolution $\iota_X : X^\bullet \longrightarrow I_X^\bullet$. In such case $I_X^\bullet$ is uniquely determined, up to isomorphism in $\mathcal{H}(\mathcal{A})$, and the assignment $X \rightsquigarrow I_X^\bullet$ is the definition on objects of a triangulated functor $\mathbf{i}_\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ which is right adjoint to q.*

Definition 12. *In the situation of the previous corollary,* $\mathbf{i}_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ *is called the* homotopically injective resolution functor*. When the dual result holds, one defines the* homotopically projective resolution functor $\mathbf{p}_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$*, which is left adjoint to* $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(A)$*.*

Remark 2.24. *When the homotopically projective (resp. homotopically injective) resolution functor is defined, the counit* $q \circ \mathbf{i}_{\mathcal{A}} \longrightarrow 1_{\mathcal{D}(A)}$ *(resp. the unit* $1_{\mathcal{D}(\mathcal{A})} \longrightarrow q \circ \mathbf{p}_{\mathcal{A}}$*) of the adjunction pair* $(q, \mathbf{i}_{\mathcal{A}})$ *(resp.* $(\mathbf{p}_{\mathcal{A}}, q)$*) is a natural isomorphism. We will denote by* $\iota : 1_{\mathcal{H}(\mathcal{A})} \longrightarrow \mathbf{i}_{\mathcal{A}} \circ q$ *(resp.* $\pi : \mathbf{p}_{\mathcal{A}} \circ q \longrightarrow 1_{\mathcal{H}(A)}$*) the unit (resp. counit) of the first (resp. second) adjoint pair. But since q 'is' the identity on objects, we will simply write* $\iota = \iota_{X^\bullet} : X^\bullet \longrightarrow \mathbf{i}_{\mathcal{A}} X^\bullet$ *and* $\pi = \pi_{X^\bullet} : \mathbf{p}_{\mathcal{A}} X^\bullet \longrightarrow X^\bullet$*. Both of them are quasi-isomorphisms.*

The canonical examples that we should keep in mind are the following:

Examples 2.25.
(1) *If* $\mathcal{A} = \mathcal{G}$ *is a Grothendieck category, then each chain complex of objects of* $\mathcal{G}$ *admits a homotopically injective resolution (*[2, Theorem 5.4]*, see also* [37, Theorem D]*).*
(2) *If* $\mathcal{A} = Mod - A$ *is the category of (right) modules over a k-algebra A, then each chain complex of A-modules admits both a homotopically projective and a homotopically injective resolution (see* [37, Theorem C]*). Note that if A and B are k-algebras, then a* $B - A-$*bimodule is the same as a right* $B^{op} \otimes A$*-module, by the rule* $m(b^o \otimes a) = bma$*, for all* $a \in A$*,* $b \in B$ *and* $m \in M$*. Therefore the existence of homotopically injective and homotopically projective resolutions also applies when taking* $\mathcal{A} = Mod - (B^{op} \otimes A)$ *to be the category of* $B - A-$*bimodules.*
(3) *Since we have canonical equivalences* $\mathcal{C}(\mathcal{A}^{op}) \cong \mathcal{C}(\mathcal{A})^{op}$ *and* $\mathcal{H}(\mathcal{A}^{op}) \cong \mathcal{H}(\mathcal{A})^{op}$*, the opposite category of a (bi)module category is another example of abelian category* $\mathcal{A}$ *over which every complex admits both a homotopically projective resolution and a homotopically injective one.*

A consequence of Brown representability theorem is now the following result.

Proposition 2.26. *Let A be an algebra and* $\mathcal{S}$ *be any* <u>set</u> *of objects of* $\mathcal{D}(A)$*. The following assertions hold:*
(1) $\mathcal{U} := Susp_{\mathcal{D}(A)}(\mathcal{S})$ *is the aisle of a t-structure in* $\mathcal{D}(A)$*. The co-aisle* $\mathcal{U}^\perp$ *consists of the complexes* $Y^\bullet$ *such that* $\mathcal{D}(A)(S^\bullet[k], Y^\bullet) = 0$*, for all* $S^\bullet \in \mathcal{S}$ *and and all integers* $k \geq 0$*;*
(2) $\mathcal{T} := Tria_{\mathcal{D}(A)}(\mathcal{S})$ *is the aisle of a semi-orthogonal decomposition in* $\mathcal{D}(A)$*. In this case* $\mathcal{T}^\perp$ *consists of the complexes* $Y^\bullet$ *such that* $\mathcal{D}(A)(S^\bullet[k], Y^\bullet) = 0$*, for all* $S^\bullet \in \mathcal{S}$ *and* $k \in \mathbb{Z}$*.*

Proof. Assertion 1 is proved in [3, Proposition 3.2] (see also [36] and [22, Theorem 12.1] for more general versions). Assertion 2 follows from 1 since $\mathrm{Tria}_{\mathcal{D}(A)}(\mathcal{S}) = \mathrm{Susp}_{\mathcal{D}(A)}(\bigcup_{k \in \mathbb{Z}} \mathcal{S}[k])$. □

2.7. **Derived functors and adjunctions.**

Lemma 2.27. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be Frobenius exact categories and let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *and* $G : \mathcal{D} \longrightarrow \mathcal{C}$ *be functors which take conflations to conflations. Then the following statements hold true:*

(1) *If F takes projective objects to projective objects, then there is a triangulated functor $\underline{F} : \underline{C} \longrightarrow \underline{D}$, unique up to natural isomorphism, such that $p_{\mathcal{D}} \circ F = \underline{F} \circ p_{\mathcal{C}}$.*

(2) *If (F, G) is an adjoint pair, then the following assertions hold:*

   (a) *Both F and G preserve projective objects.*

   (b) *The induced triangulated functors $\underline{F} : \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{D}}$ and $\underline{G} : \underline{\mathcal{D}} \longrightarrow \underline{\mathcal{C}}$ form an adjoint pair $(\underline{F}, \underline{G})$.*

Proof. (1) The existence of the functor is immediate and the fact that it is triangulated is due to the fact that all triangles in $\underline{C}$ and $\underline{D}$ are 'image' of conflations in $\mathcal{C}$ and $\mathcal{D}$ by the respective projection functors.

(2) (a) The proof is identical to the one which proves, for arbitrary categories, that each left (resp. right) adjoint of a functor which preserves epimorphisms (resp. monomorphisms) preserves projective (resp. injective) objects. Then use the fact that the injective and projective objects coincide.

(b) Let us fix an isomorphism $\eta : \mathcal{D}(F(?), ?) \xrightarrow{\cong} \mathcal{C}(?, G(?))$ natural on both variables and let $C \in \mathcal{C}$ and $D \in \mathcal{D}$ be any objects. If $\alpha \in \mathcal{D}(F(C), D)$ is a morphism which factors through a projective object Q of $\mathcal{D}$, then we have a decomposition $F(C) \xrightarrow{\beta} Q \xrightarrow{\gamma} D$. Due to the naturality of η, we then have a commutative diagram:

$$\begin{tikzcd}
\mathcal{D}(F(C), Q) \arrow[r, "\eta_{C,Q}"] \arrow[d, "\gamma_*"] & \mathcal{C}(C, G(Q)) \arrow[d, "G(\gamma)_*"] \\
\mathcal{D}(F(C), D) \arrow[r, "\eta_{C,D}"] & \mathcal{C}(C, G(D))
\end{tikzcd}$$

with the obvious meaning of the vertical arrows. It follows that

$$\eta_{C,D}(\alpha) = \eta_{C,D}(\gamma \circ \beta) = (\eta_{C,D} \circ \gamma_*)(\beta) = G(\gamma)_* \circ \eta_{C,Q}(\beta) = G(\gamma) \circ \eta_{C,Q}(\beta),$$

which proves that $\eta_{C,D}(\alpha)$ factors through the projective object $G(Q)$ of $\mathcal{C}$. We then get and induced map $\bar{\eta}_{C,D} : \underline{D}(\underline{F}(C), D) \longrightarrow \underline{C}(C, \underline{G}(D))$, which is natural on both variables. Applying the symmetric argument to η^{-1}, we obtain a map $\overline{\eta^{-1}}_{C,D} : \underline{C}(C, \underline{G}(D)) \longrightarrow \underline{D}(\underline{F}(C), D)$, natural on both variables, which is clearly inverse to $\bar{\eta}$. Then $(\underline{F}, \underline{G})$ is an adjoint pair, as desired. $\square$

Bearing in mind that $\underline{\mathcal{C}(\mathcal{A})} = \mathcal{H}(\mathcal{A})$, for any abelian category category $\mathcal{A}$, the following definition makes sense.

Definition 13. *Let $\mathcal{A}$ be an AB4 abelian category such that every chain complex of objects of $\mathcal{A}$ admits a homotopically injective resolution, and let $\mathbf{i}_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ be the homotopically injective resolution functor. If $\mathcal{B}$ is another abelian category and $F : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{B})$ is a k-linear functor which takes conflations to conflations and contractible complexes to contractible complexes, then the composition of triangulated funtors*

$$\mathcal{D}(\mathcal{A}) \xrightarrow{\mathbf{i}_{\mathcal{A}}} \mathcal{H}(\mathcal{A}) \xrightarrow{\underline{F}} \mathcal{H}(\mathcal{B}) \xrightarrow{q_{\mathcal{B}}} \mathcal{D}(\mathcal{B})$$

is called the right derived functor of F, *usually denoted* $\mathbf{R}F$.

Dually, one defines the the left derived functor of F, denoted $\mathbf{L}F$, whenever $\mathcal{A}$ is an AB4 abelian category on which every chain complex admits a homotopically projective resolution.*

We now get:

Proposition 2.28. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Suppose that $\mathcal{A}$ is AB4* on which each chain complex has a homotopically projective resolution and $\mathcal{B}$ is AB4 on which each chain complex has a homotopically injective resolution. If $F : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{B})$ and $G : \mathcal{C}(\mathcal{B}) \longrightarrow \mathcal{C}(\mathcal{A})$ are k-linear functors which take conflations to conflations and form an adjoint pair (F, G), then the derived functors $\mathbf{L}F : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$ and $\mathbf{R}F : \mathcal{D}(\mathcal{B}) \longrightarrow \mathcal{D}(\mathcal{A})$ form an adjoint pair $(\mathbf{L}F, \mathbf{R}G)$.*

Proof. By definition, we have

$$\mathbf{L}F : \mathcal{D}(\mathcal{A}) \xrightarrow{\mathbf{p}_{\mathcal{A}}} \mathcal{H}(\mathcal{A}) \xrightarrow{\underline{F}} \mathcal{H}(\mathcal{B}) \xrightarrow{q_{\mathcal{B}}} \mathcal{D}(\mathcal{B})$$
and
$$\mathbf{R}G : \mathcal{D}(\mathcal{B}) \xrightarrow{\mathbf{i}_{\mathcal{B}}} \mathcal{H}(\mathcal{A}) \xrightarrow{\underline{G}} \mathcal{H}(\mathcal{A}) \xrightarrow{q_{\mathcal{A}}} \mathcal{D}(\mathcal{A}).$$

The result is an immediate consequence of the fact that $(\mathbf{p}_{\mathcal{A}}, q_{\mathcal{A}})$, $(\underline{F}, \underline{G})$ and $(q_{\mathcal{B}}, \mathbf{i}_{\mathcal{B}})$ are adjoint pairs. □

3. Classical derived functors defined by complexes of bimodules

3.1. **Definition and main adjunction properties.** If A is a k-algebra, we will write $\mathcal{C}(A)$, $\mathcal{H}(A)$ and $\mathcal{D}(A)$ instead of $\mathcal{C}(\text{Mod}A)$, $\mathcal{H}(\text{Mod}A)$ and $\mathcal{D}(\text{Mod}A)$, respectively. This rule applies also to the algebra $B^{op} \otimes A$, for any algebras A and B.

Let A, B and C be k-algebras. Given complexes $T^\bullet$ and $M^\bullet$, of $B - A-$bimodules and $A - C-$bimodules respectively, we shall associate to them several functors between categories of complexes of bimodules. The material can be found in [40], but the reader is warned on the difference of indization of complexes with respect to that book. The *total tensor product of $T^\bullet$ and $M^\bullet$*, denoted $T^\bullet \otimes_A^\bullet M^\bullet$, is the complex of $B - C-$bimodules given as follows (see [40, Section 2.7]):

1. $(T^\bullet \otimes_A^\bullet M^\bullet)^n = \oplus_{i+j=n} T^i \otimes_A M^j$, for each $n \in \mathbb{Z}$.
2. The differential $d^n : (T^\bullet \otimes_A^\bullet M^\bullet)^n \longrightarrow (T^\bullet \otimes_A^\bullet M^\bullet)^{n+1}$ takes $t \otimes m \rightsquigarrow d_T(t) \otimes m + (-1)^i t \otimes d_M(m)$, whenever $t \in T^i$ and $m \in M^j$

When $T^\bullet$ is fixed, the assignment $M^\bullet \rightsquigarrow T^\bullet \otimes_A^\bullet M^\bullet$ is the definition on objects of a k-linear functor $T^\bullet \otimes_A^\bullet ? : \mathcal{C}(A^{op} \otimes C) \longrightarrow \mathcal{C}(B^{op} \otimes C)$. It acts as $f \rightsquigarrow 1_T \otimes f$ on morphisms, where $(1_T \otimes f)(t \otimes m) = t \otimes f(m)$, whenever $t \in T^\bullet$ and $m \in M^\bullet$ are homogeneous elements. Note that, with a symmetric argument, when $M^\bullet \in \mathcal{C}(A^{op} \otimes C)$ is fixed, we also get another k-linear functor $? \otimes_A^\bullet M^\bullet : \mathcal{C}(B^{op} \otimes A) \longrightarrow \mathcal{C}(B^{op} \otimes C)$ which takes $T^\bullet \rightsquigarrow T^\bullet \otimes_A^\bullet M^\bullet$ and $g \rightsquigarrow g \otimes 1_M$. In this way, we get a bifunctor $? \otimes_A^\bullet ? : \mathcal{C}(B^{op} \otimes A) \times \mathcal{C}(A^{op} \otimes C) \longrightarrow \mathcal{C}(B^{op} \otimes C)$.

Suppose now that $N^\bullet$ is a complex of $C - A-$bimodules. The *total Hom complex $Hom_A^\bullet(T^\bullet, N^\bullet)$* is the complex of $C - B-$bimodules given as follows (see [40, Section 2.7] or, in a more general context, [19, Section 1.2]):

1. $\text{Hom}_A^\bullet(T^\bullet, N^\bullet)^n = \prod_{j-i=n} \text{Hom}_A(T^i, N^j)$, for each $n \in \mathbb{Z}$.
2. The differential $\text{Hom}_A^\bullet(T^\bullet, N^\bullet)^n \longrightarrow \text{Hom}_A^\bullet(T^\bullet, N^\bullet)^{n+1}$ takes $f \rightsquigarrow d_N \circ f - (-1)^n f \circ d_M$

The assigment $N^\bullet \rightsquigarrow \text{Hom}_A^\bullet(T^\bullet, N^\bullet)$ is the definition on objects of a k-linear functor $\text{Hom}_A^\bullet(T^\bullet, ?) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$. The functor acts on morphism as $\alpha \rightsquigarrow \alpha_*$, where $\alpha_*(f) = \alpha \circ f$, for each homogeneous element $f \in \mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet)$. Symmetrically, if $N^\bullet \in \mathcal{C}(C^{op} \otimes A)$ is fixed, then the assignment $T^\bullet \rightsquigarrow \mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet)$ is the definition on objects of a functor $\mathrm{Hom}_A^\bullet(?, N^\bullet) : \mathcal{C}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{C}(C^{op} \otimes B)$. It acts on morphisms as $\beta \rightsquigarrow \mathrm{Hom}_A^\bullet(\beta, N^\bullet) =: \beta^*$, where $\beta^*(f) = f \circ \beta$, for each homogeneous element $f \in \mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet)$. In this way, we get a bifunctor $\mathrm{Hom}_A^\bullet(?, ?) : \mathcal{C}(B^{op} \otimes A) \times \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$.

Obviously, we also get bifunctors:

$$?\otimes_B^\bullet ? : \mathcal{C}(C^{op} \otimes B) \times \mathcal{C}(B^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes A), \qquad (X^\bullet, T^\bullet) \rightsquigarrow X^\bullet \otimes_B^\bullet T^\bullet;$$
$$\mathrm{Hom}_{B^{op}}^\bullet(?, ?) : \mathcal{C}(B^{op} \otimes C) \times \mathcal{C}(B^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes A),$$
$$(Y^\bullet, T^\bullet) \rightsquigarrow \mathrm{Hom}_{B^{op}}^\bullet(Y^\bullet, T^\bullet).$$

Proposition 3.1. *Let A, B and C be k-algebras. All the bifunctors $?\otimes_A^\bullet ?$, $?\otimes_B^\bullet ?$, $Hom_A^\bullet(?, ?)$ and $Hom_{B^{op}}^\bullet(?, ?)$ defined above preserve conflations (i.e. semi-split exact sequences) and contractible complexes on each variable. Moreover, if $T^\bullet$ is a complex of $B-A-$bimodules, then the following assertions hold:*

1. *The pairs $(? \otimes_B^\bullet T^\bullet, Hom_A^\bullet(T^\bullet, ?))$ and $(T^\bullet \otimes_A^\bullet ?, Hom_{B^{op}}^\bullet(T^\bullet, ?))$ are adjoint pairs.*
2. *The pair $(Hom_{B^{op}}^\bullet(?, T^\bullet) : \mathcal{C}(B^{op} \otimes C) \longrightarrow \mathcal{C}(C^{op} \otimes A)^{op}, Hom_A^\bullet(?, T^\bullet) : \mathcal{C}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{C}(B^{op} \otimes C))$ is an adjoint pair.*

Proof. It is clear that the k-linear functors $T^\bullet \otimes_A^\bullet ? : \mathcal{C}(A^{op} \otimes C) \longrightarrow \mathcal{C}(B^{op} \otimes C)$, $\mathrm{Hom}_A^\bullet(T^\bullet, ?) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$ and $\mathrm{Hom}_A^\bullet(?, T^\bullet) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(B^{op} \otimes C)^{op}$, after forgetting about the differentials, induce corresponding functors between the associated graded categories. For instance, we have an induced k-linear functor $? \otimes_B^\bullet T^\bullet : (\mathrm{Mod}(C^{op} \otimes B))^{\mathbb{Z}} \longrightarrow (\mathrm{Mod}(C^{op} \otimes A))^{\mathbb{Z}}$. But then this latter functor preserves split exact sequences. This is exactly saying that $? \otimes_B^\bullet T^\bullet : \mathcal{C}(C^{op} \otimes B) \longrightarrow \mathcal{C}(C^{op} \otimes A)$ takes conflations to conflations. The corresponding argument works for all the other functors.

So all the bifunctors in the list preserve conflations on each component. That they also preserve contractible complexes on each variable, will follow from lemma 2.27 once we prove the adjunctions of assertions (1) and (2).

(1) This goes as in the usual adjunction between the tensor product and the Hom functor in module categories. Concretely, we define

$$\eta = \eta_{M,Y} : \mathcal{C}(C^{op} \otimes A)(M^\bullet \otimes_A^\bullet T^\bullet, Y^\bullet) \longrightarrow \mathcal{C}(C^{op} \otimes B)(M^\bullet, \mathrm{Hom}_A^\bullet(T^\bullet, Y^\bullet))$$

by the rule: $[\eta(f)(m)](t) = f(m \otimes t)$, whenever $m \in M^i$ and $t \in T^j$, for some $i, j \in \mathbb{Z}$. We leave to the reader the routine task of checking that $\eta(f)$ is a chain map $M^\bullet \longrightarrow \mathrm{Hom}_A^\bullet(T^\bullet, Y^\bullet)$. The naturallty of η is then proved as in modules.

(2) We need to define an isomorphism

$$\xi = \xi_{M,Y} : \mathcal{C}(B^{op} \otimes C)(Y^\bullet, \mathrm{Hom}_A^\bullet(M^\bullet, T^\bullet)) \longrightarrow \mathcal{C}(C^{op} \otimes A)^{op}(\mathrm{Hom}_{B^{op}}^\bullet(Y^\bullet, T^\bullet), M^\bullet) =$$
$$\mathcal{C}(C^{op} \otimes A)(M^\bullet, \mathrm{Hom}_{B^{op}}^\bullet(Y^\bullet, T^\bullet)),$$

natural on $M^\bullet \in \mathcal{C}(C^{op} \otimes A)$ and $Y^\bullet \in \mathcal{C}(B^{op} \otimes C)$. Our choice of ξ is identified by the equality $[\xi(f)(m)](y) = (-1)^{|m||y|} f(y)(m)$, for all homogeneous elements $f \in \mathcal{C}(B^{op} \otimes C)(Y^\bullet, \mathrm{Hom}_A^\bullet(M^\bullet, T^\bullet))$, $m \in M^\bullet$ and $y \in Y^\bullet$. □

With the notation of last proposition, we adopt the following notation:

a) $? \otimes^{\mathbf{L}}_B T : \mathcal{D}(C^{op} \otimes B) \longrightarrow \mathcal{D}(C^{op} \otimes A)$ will denote the left derived functor of $? \otimes^{\bullet}_B T^{\bullet} : \mathcal{C}(C^{op} \otimes B) \longrightarrow \mathcal{C}(C^{op} \otimes A)$

b) $\mathbf{R}\mathrm{Hom}_A(T^{\bullet}, ?) : \mathcal{D}(C^{op} \otimes A) \longrightarrow \mathcal{D}(C^{op} \otimes B)$ will denote the right derived functor of $\mathrm{Hom}^{\bullet}_A(T^{\bullet}, ?) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$

c) $\mathbf{R}\mathrm{Hom}_A(?, T^{\bullet}) : \mathcal{D}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes C)$ will denote the right derived functor of $\mathrm{Hom}^{\bullet}_A(?, T^{\bullet}) : \mathcal{C}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes C)$ or, equivalently, the left derived functor of $\mathrm{Hom}^{\bullet}_A(?, T^{\bullet}) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{D}(B^{op} \otimes C)^{op}$.

d) $\mathbf{T}_A(?, ?) : \mathcal{D}(B^{op} \otimes A) \times \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(B^{op} \otimes C)$ will denote the composition

$$\mathcal{D}(B^{op} \otimes A) \times \mathcal{D}(A^{op} \otimes C) \stackrel{\mathbf{p}_{B^{op}\otimes A} \times \mathbf{p}_{B^{op}\otimes A}}{\longrightarrow} \mathcal{H}(B^{op} \otimes A) \times \mathcal{H}(A^{op} \otimes C) \stackrel{? \otimes^{\bullet}_A ?}{\longrightarrow}$$
$$\mathcal{H}(B^{op} \otimes C) \stackrel{q}{\longrightarrow} \mathcal{D}(B^{op} \otimes C),$$

where the central arrow is induced by the bifunctor $\mathcal{C}(B^{op} \otimes A) \times \mathcal{C}(A^{op} \otimes C) \stackrel{? \otimes^{\bullet}_A ?}{\longrightarrow} \mathcal{C}(B^{op} \otimes C)$ and is well-defined due to proposition 3.1. The bifunctor $\mathbf{T}_A(?, ?)$ is triangulated on each variable.

e) $\mathbf{H}_A(?, ?) : \mathcal{D}(B^{op} \otimes A)^{op} \times \mathcal{D}(C^{op} \otimes A) \longrightarrow \mathcal{D}(C^{op} \otimes B)$ is the composition

$$\mathcal{D}(B^{op} \otimes A)^{op} \times \mathcal{D}(C^{op} \otimes A) \stackrel{\mathbf{p}_{B^{op}\otimes A} \times \mathbf{i}_{C^{op}\otimes A}}{\longrightarrow} \mathcal{H}(B^{op} \otimes A)^{op} \times \mathcal{H}(C^{op} \otimes A) \stackrel{\mathrm{Hom}^{\bullet}_A(?,?)}{\longrightarrow}$$
$$\mathcal{H}(C^{op} \otimes B) \stackrel{q}{\longrightarrow} \mathcal{D}(C^{op} \otimes B),$$

which is a bifunctor which is triangulated on each variable.

e') $\mathbf{H}_{B^{op}}(?, ?) : \mathcal{D}(B^{op} \otimes C)^{op} \times \mathcal{D}(B^{op} \otimes A) \longrightarrow \mathcal{D}(C^{op} \otimes A)$ is a bifunctor, triangulated on both variables, which is defined in a similar way as that in e).

Of course, there are left-right symmetric versions $T^{\bullet} \otimes^{\mathbf{L}}_A ? : \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(B^{op} \otimes C)$, $\mathbf{R}\mathrm{Hom}_{B^{op}}(T^{\bullet}, ?) : \mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(A^{op} \otimes C)$ and $\mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^{\bullet}) : \mathcal{D}(B^{op} \otimes C)^{op} \longrightarrow \mathcal{D}(C^{op} \otimes A)$ of the functors in a), b) and c). Their precise definition is left to the reader.

As a direct consequence of propositions 2.28 and 3.1, we get:

Corollary 3.2. *Let A, B and C be k-algebras and let $T^{\bullet}$ be a complex of $B-A-$bimodules. With the notation above, the following pairs of triangulated functors are adjoint pairs:*

(1) $(? \otimes^{\mathbf{L}}_B T^{\bullet}, \mathbf{R}Hom_A(T^{\bullet}, ?))$;
(2) $(T^{\bullet} \otimes^{\mathbf{L}}_A ?, \mathbf{R}Hom_{B^{op}}(T^{\bullet}, ?))$;
(3) $(\mathbf{R}Hom_{B^{op}}(?, T^{\bullet}) : \mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(C^{op} \otimes A)^{op}, \mathbf{R}Hom_A(?, T^{\bullet}) : \mathcal{D}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes C))$.

Definition 14. *We will adopt the following terminology, referred to the adjunctions of last corollary:*

(1) $\lambda : 1_{\mathcal{D}(C^{op}\otimes B)} \longrightarrow RHom_A(T^{\bullet}, ?) \circ (? \otimes^{\mathbf{L}}_B T)$ *and* $\delta : (? \otimes^{\mathbf{L}}_B T) \circ RHom_A(T^{\bullet}, ?) \longrightarrow 1_{\mathcal{D}(C^{op}\otimes A)}$ *will be the unit and the counit of the first adjunction.*
(2) $\rho : 1_{\mathcal{D}(A^{op}\otimes C)} \longrightarrow RHom_{B^{op}}(T^{\bullet}, ?) \circ (T^{\bullet} \otimes^{\mathbf{L}}_A ?)$ *and* $\phi : (T^{\bullet} \otimes^{\mathbf{L}}_A ?) \circ RHom_{B^{op}}(T^{\bullet}, ?) \longrightarrow 1_{\mathcal{D}(B^{op}\otimes C)}$ *are the unit and counit of the second adjunction.*
(3) $\sigma : 1_{\mathcal{D}(B^{op}\otimes C)} \longrightarrow \mathbf{R}Hom_A(?, T^{\bullet}) \circ \mathbf{R}Hom_{B^{op}}(?, T^{\bullet})$ *and* $\tau : 1_{\mathcal{D}(C^{op}\otimes A)} \longrightarrow \mathbf{R}Hom_{B^{op}}(?, T^{\bullet}) \circ \mathbf{R}Hom_A(?, T^{\bullet})$ *are the unit and the counit of the third adjunction (note that the last one is an arrow in the opposite direction when the functors are considered as endofunctors of $\mathcal{D}(C^{op} \otimes A)^{op}$).*

The following lemma is very useful:

Lemma 3.3. *In the situation of last definition, let us take $C = k$ and consider the following assertions:*

(1) $\lambda_B : B \longrightarrow \mathbf{R}Hom_A(T^\bullet, ? \otimes_B^{\mathbf{L}} T^\bullet)(B) \cong \mathbf{R}Hom_A(T^\bullet, ?)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(B)$;

(2) $\delta_{T^\bullet} : [\mathbf{R}Hom_A(T^\bullet, ?) \otimes_B^{\mathbf{L}} T^\bullet](T^\bullet) \longrightarrow T^\bullet$ *is an isomorphism in* $\mathcal{D}(A)$;

(3) $\sigma_B : B \longrightarrow [\mathbf{R}Hom_A(\mathbf{R}Hom_{B^{op}}(?, T^\bullet), T^\bullet)](B) \cong \mathbf{R}Hom_A(?, T^\bullet)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(B^{op})$;

(4) $\tau_{T^\bullet} : T^\bullet \longrightarrow [\mathbf{R}Hom_{B^{op}}(\mathbf{R}Hom_A(?, T^\bullet), T^\bullet)](T^\bullet)$ *is an isomorphism in* $\mathcal{D}(A)$.

Then the implications $(2) \Longleftarrow (1) \Longleftrightarrow (3) \Longrightarrow (4)$ *hold true.*

Proof. $(1) \Longrightarrow (2)$ Putting $F = ? \otimes_B^{\mathbf{L}} T^\bullet$ and $G = \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$, the truth of the implication is a consequence of the adjunction equation $1_{F(B)} = \delta_{F(B)} \circ F(\lambda_B)$ and the fact that $F(B) \cong T^\bullet$.

$(3) \Longrightarrow (4)$ also follows from the equations of the adjunction $(\mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^\bullet), \mathbf{R}\mathrm{Hom}_A(?, T^\bullet))$ (see corollary 3.2).

$(1) \Longleftrightarrow (3)$ Let $\mathbf{p}_A, \mathbf{i}_A : \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ be the homotopically projective resolution functor and the homotopically injective resolution functor, respectively. By definition of the derived functors, we have $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)(T^\bullet) = \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_A T^\bullet)$ and $\mathbf{R}\mathrm{Hom}_A(?, T^\bullet)(T^\bullet) = \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, T^\bullet)$. Let then $\pi : \mathbf{p}_A T^\bullet \longrightarrow T^\bullet$ and $\iota : T^\bullet \longrightarrow \mathbf{i}_A T^\bullet$ be the canonical quasi-isomorphisms. We then have quasi-isomorphisms in $\mathcal{C}(k)$:

$$\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)(T^\bullet) = \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_A T^\bullet) \xrightarrow{\pi^*} \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, \mathbf{i}_A T^\bullet) \xleftarrow{\iota_*} \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, T^\bullet) \cong \mathbf{R}\mathrm{Hom}_A(?, T^\bullet)(T^\bullet)$$

Note that λ_B and σ_B are the compositions:

$$\lambda_B : B \longrightarrow \mathrm{Hom}_A^\bullet(T^\bullet, T^\bullet) \xrightarrow{\tilde{\iota}} \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_A T^\bullet) = \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)(T^\bullet),$$

and

$$\sigma_B : B \longrightarrow \mathrm{Hom}_A^\bullet(T^\bullet, T^\bullet) \xrightarrow{\tilde{\pi}} \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, T^\bullet) = \mathbf{R}\mathrm{Hom}_A(?, T^\bullet)(T^\bullet),$$

where the first arrow, in both cases, takes $b \rightsquigarrow \lambda_b : t \rightsquigarrow bt$, and the other arrows are $\tilde{\iota} = \mathrm{Hom}_A^\bullet(T^\bullet, \iota)$ and $\tilde{\pi} = \mathrm{Hom}_A^\bullet(\pi, T^\bullet)$.

A direct easy calculation shows that the equality $\pi^* \circ \lambda_B = \iota_* \circ \sigma_B$ holds in $\mathcal{C}(k)$. As a consequence, λ_B is a quasi-isomorphism if, and only if, so is σ_B. □

We explicitly state the left-right symmetric of the previous lemma since it will be important for us:

Lemma 3.4. *In the situation of last definition, let us take $C = k$ and consider the following assertions:*

(1) $\rho_A : A \longrightarrow \mathbf{R}Hom_{B^{op}}(T^\bullet, T^\bullet \otimes_A^{\mathbf{L}} ?)(A) \cong \mathbf{R}Hom_{B^{op}}(T^\bullet, ?)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(A^{op})$;

(2) $\phi_T : [T^\bullet \otimes_A^{\mathbf{L}} \otimes \mathbf{R}Hom_A(T^\bullet, ?)](T^\bullet) \longrightarrow T^\bullet$ *is an isomorphism in* $\mathcal{D}(B^{op})$;

(3) $\tau_A : A \longrightarrow [\mathbf{R}Hom_{B^{op}}(\mathbf{R}Hom_A(?, T^\bullet), T^\bullet)](A) \cong \mathbf{R}Hom_{B^{op}}(?, T^\bullet)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(A)$;

(4) $\sigma_T : T^\bullet \longrightarrow [\mathbf{R}Hom_A(\mathbf{R}Hom_{B^{op}}(?, T^\bullet), T^\bullet)](T^\bullet)$ *is an isomorphism in* $\mathcal{D}(B^{op})$.

Then the implications $(2) \Longleftarrow (1) \Longleftrightarrow (3) \Longrightarrow (4)$ *hold true.*

3.2. **Homotopically flat complexes. Restrictions.** In order to deal with the derived tensor product, it is convenient to introduce a class of chain complexes which is wider than that of the homotopically projective ones.

Definition 15. *Let B be any algebra. A complex $F^\bullet \in \mathcal{C}(B)$ is called* homotopically flat *when the functor $F^\bullet \otimes_B^\bullet ? : \mathcal{C}(B) \longrightarrow \mathcal{C}(k)$ preserves acyclic complexes.*

The key points are assertion (3) and (4) of the following result.

Lemma 3.5. *Let A and B be k-algebras. The following assertions hold:*

1. *If $Z^\bullet \in \mathcal{C}(B)$ is acyclic and homotopically flat, then the essential image of $Z^\bullet \otimes_B^\bullet ? : \mathcal{H}(B^{op}) \longrightarrow \mathcal{H}(k)$ consists of acyclic complexes;*
2. *Each homotopically projective object of $\mathcal{H}(B)$ is homotopically flat;*
3. *If $F^\bullet \xrightarrow{s} M^\bullet$ is a quasi-isomorphism in $\mathcal{C}(B)$, where $F^\bullet$ is homotopically flat, then, for each $N^\bullet \in \mathcal{C}(B^{op} \otimes A)$, we have an isomorphicm $(? \otimes_B^{\mathbf{L}} N^\bullet)(M^\bullet) \cong F^\bullet \otimes_B^\bullet N^\bullet$ in $\mathcal{D}(A)$;*
4. *If $F^\bullet$ is homotopically flat in $\mathcal{H}(B)$ and $N^\bullet$ is a complex of $B-A$-bimodules, then $(F^\bullet \otimes_B^{\mathbf{L}} ?)(N^\bullet) = F^\bullet \otimes_B^\bullet N^\bullet$ in $\mathcal{D}(A)$.*

Proof. All throughout the proof we will use the fact that $\mathcal{D}(B)$ (resp. $\mathcal{D}(B^{op})$) is compactly generated by $\{B\}$ (see proposition 3.13 below).

(1) The given functor $T := Z^\bullet \otimes_B^\bullet ?$ is triangulated and take acyclic complexes to acyclic complexes. Then it preserves quasi-isomorphisms. By construction of the derived category, we then get a unique triangulated functor $\bar{T} : \mathcal{D}(B^{op}) \longrightarrow \mathcal{D}(k)$ such that $\bar{T} \circ q_B = q_k \circ T$, where $q_B : \mathcal{H}(B^{op}) \longrightarrow \mathcal{D}(B^{op})$ and $q_k : \mathcal{H}(k) \longrightarrow \mathcal{D}(k)$ are the canonical functor.

We will prove that $\bar{T}$ is the zero functor and this will prove the assertion. We consider the full subcategory $\mathcal{X}$ of $\mathcal{D}(B^{op})$ consisting of the complexes $M^\bullet$ such that $\bar{T}(M^\bullet) = 0$. It is a triangulated subcategory, closed under taking arbitrary coproducts, which contains B. We then have $\mathcal{X} = \mathcal{D}(B)$.

(2) By [19, Theorem P], each homotopically projective complex is isomorphic in $\mathcal{H}(A)$ to a complex $P^\bullet$ which admits a countable filtration

$$P_0^\bullet \subset P_1^\bullet \subset ... \subset P_n^\bullet \subset ...,$$

in $\mathcal{C}(B)$, where the inclusions are inflations and where $P_0^\bullet$ and all the factors $P_n^\bullet / P_{n-1}^\bullet$ are direct summands of direct sums of stalk complexes of the form $B[k]$, with $k \in \mathbb{Z}$. We just need to check that this $P^\bullet$ is homotopically flat. Due to the fact that the bifunctor $? \otimes_B^\bullet ? : \mathcal{C}(B) \times \mathcal{C}(B^{op}) \longrightarrow \mathcal{C}(k)$ preserves direct limits on both variables, with an evident induction argument, the proof is easily reduced to the case when $P^\bullet = B[r]$, for some $r \in \mathbb{Z}$, in which case it is trivial.

(3) Let $P^\bullet := \mathbf{p}M^\bullet \xrightarrow{\pi} M^\bullet$ be the homotopically projective resolution. Then $s^{-1} \circ \pi \in \mathcal{D}(B)(P^\bullet, F^\bullet) \cong \mathcal{H}(B)(P^\bullet, F^\bullet)$. Then we have a chain map $f : P^\bullet \longrightarrow F^\bullet$ such that $s \circ f = \pi$ in $\mathcal{H}(B)$. In particular, f is a quasi-isomoprhism between homotopically flat objects. Its cone is then an acyclic and homotopically flat complex. By assertion (1), we conclude that $f \otimes_B^\bullet 1_{N^\bullet}$ has an acyclic cone and, hence, it is a quasi-isomorphism, for each $N^\bullet \in \mathcal{C}(B^{op} \otimes A)$. We then have an isomorphism $(? \otimes_B^{\mathbf{L}} N^\bullet)(M^\bullet) = P^\bullet \otimes_B^\bullet N^\bullet \xrightarrow{f \otimes^\bullet 1_{N^\bullet}} F^\bullet \otimes_B^\bullet N^\bullet$ in $\mathcal{D}(A)$.

(4) Let us consider the triangle in $\mathcal{H}(B^{op} \otimes A)$

$$\mathbf{p}_{B^{op}\otimes A}N^\bullet \xrightarrow{\pi} N^\bullet \longrightarrow Z^\bullet \xrightarrow{+}$$

afforded by the homotopically projective resolution of $N^\bullet$. Then $Z^\bullet$ is acyclic and the homotopically flat condition of $F^\bullet$ tells us that $(F^\bullet \otimes^{\mathbf{L}}_B ?)(N^\bullet) = F^\bullet \otimes^\bullet_B \mathbf{p}_{B^{op}\otimes A}N^\bullet \xrightarrow{1\otimes\pi} F^\bullet \otimes^\bullet_B N^\bullet$ is a quasi-isomorphism and, therefore, an isomorphism in $\mathcal{D}(A)$. □

In order to understand triangulated functors between derived categories of bimodules as 'one-sided' triangulated functors, it is important to know how homotopically projective or injective resolutions behave with respect to the restriction functors. The following result and its left-right symmetric are important in this sense.

Lemma 3.6. *Let A and B be algebras. The following assertions hold:*

1. *If A is k-flat, then the forgetful functor $\mathcal{H}(A^{op} \otimes B) \longrightarrow \mathcal{H}(B)$ preserves homotopically injective complexes and takes homotopically projective complexes to homotopically flat ones.*
2. *If A is k-projective, then the forgetful functor $\mathcal{H}(A^{op} \otimes B) \longrightarrow \mathcal{H}(B)$ preserves homotopically projective complexes.*

Proof. Assertion (2) and the part of assertion (1) concerning the preservation of homotopically injective complexes are proved in [32] in a much more general contex. Suppose now that A is k-flat and let $P^\bullet \in \mathcal{H}(A^{op} \otimes B)$ be homotopically projective. Using [19, Theorem P], we can assume that $P^\bullet$ admits a countable filtration

$$P^\bullet_0 \subset P^\bullet_1 \subset ... \subset P^\bullet_n \subset ...,$$

where the inclusions are inflations and where $P^\bullet_0$ and all quotients $P^\bullet_n/P^\bullet_{n-1}$ are direct summands of coproducts of stalk complexes of the form $A \otimes B[r]$. That A is k-flat implies that it is the direct limit in $\text{Mod}-k$ of a direct system of finitely generated free k-modules (see [26, Théoréme 1.2]). It follows that, in $\mathcal{C}(B)$, $A \otimes B[r]$ is a direct limit stalk complexes of the form $B^{(k)}[r]$, all of which are homotopically flat in $\mathcal{C}(B)$. Bearing in mind that the bifunctor $?\otimes^\bullet_B ?$ preserves direct limits on both variables, one gets that each stalk complex $A \otimes B[r]$ is homotopically flat in $\mathcal{C}(B)$, and then one easily proves by induction that each $P^\bullet_n$ in the filtration is homotopically flat in $\mathcal{C}(B)$. But, by definition, the direct limit in $\mathcal{C}(B)$ of homotopically flat complexes is homotopically flat. □

3.3. **Classical derived functors as components of a bifunctor.** It would be a natural temptation to believe that if $T^\bullet \in \mathcal{C}(B^{op} \otimes A)$ and $M^\bullet \in \mathcal{C}(A^{op} \otimes C)$, then we have isomorphisms $\mathbf{T}_A(T^\bullet, M^\bullet) \cong (T^\bullet \otimes^{\mathbf{L}}_A ?)(M^\bullet)$ and $\mathbf{T}_A(T^\bullet, M^\bullet) \cong (? \otimes^{\mathbf{L}}_A M^\bullet)(T^\bullet)$ in $\mathcal{D}(B^{op} \otimes C)$. Similarly, one could be tempted to believe that if $T^\bullet$ is as above and $N^\bullet \in \mathcal{C}(C^{op} \otimes A)$, then one has isomorphisms $\mathbf{H}_A(T^\bullet, N^\bullet) \cong \mathbf{R}\text{Hom}_A(T^\bullet, ?)(N^\bullet)$ and $\mathbf{H}_A(T^\bullet, N^\bullet) \cong \mathbf{R}\text{Hom}_A(?, N^\bullet)(T^\bullet)$ in $\mathcal{D}(C^{op} \otimes B)$. However, we need extra hypotheses to guarantee that.

Proposition 3.7. *Let A, B and C be k-algebras and let ${}_BT^\bullet_A$, ${}_AM^\bullet_C$ and ${}_CN^\bullet_A$ complexes of bimodules over the indicated algebras. There exist canonical natural transformations of triangulated functors:*

1. $\mathbf{T}_A(T^\bullet, ?) \longrightarrow T^\bullet \otimes^{\mathbf{L}}_A ? : \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(B^{op} \otimes C)$;

(2) $\mathbf{T}_A(?,M) \longrightarrow ?\otimes_A^{\mathbf{L}} M^\bullet : \mathcal{D}(B^{op}\otimes A) \longrightarrow \mathcal{D}(B^{op}\otimes C)$;
(3) $\mathbf{R}Hom_A(T^\bullet,?) \longrightarrow \mathbf{H}_A(T^\bullet,?) : \mathcal{D}(C^{op}\otimes A) \longrightarrow \mathcal{D}(C^{op}\otimes B)$;
(4) $\mathbf{R}Hom_A(?,N^\bullet) \longrightarrow \mathbf{H}_A(?,N^\bullet) : \mathcal{D}(B^{op}\otimes A)^{op} \longrightarrow \mathcal{D}(C^{op}\otimes B)$

Moreover, the following assertions hold

a) *If C is k-flat, then the natural transformations 1 and 3 are isomorphism;*
b) *if B is k-flat, then the natural transformation 2 is an isomorphism;*
c) *if B is k-projective, then the natural transformation 4 is an isomorphism.*

Proof. (1) By definition, we have $\mathbf{T}_A(T^\bullet,M^\bullet) = (\mathbf{p}_{B^{op}\otimes A}T^\bullet)\otimes_A^\bullet(\mathbf{p}_{A^{op}\otimes C}M^\bullet)$ and $(T^\bullet\otimes_A^{\mathbf{L}}?)(M^\bullet) = T^\bullet\otimes_A^\bullet(\mathbf{p}_{A^{op}\otimes C}M^\bullet)$. If $\pi_T : (\mathbf{p}_{B^{op}\otimes A}T^\bullet) \longrightarrow T^\bullet$ is the homotopically projective resolution, we clearly have a chain map

$$\mathbf{T}_A(T^\bullet,M^\bullet) = (\mathbf{p}_{B^{op}\otimes A}T^\bullet)\otimes_A^\bullet(\mathbf{p}_{A^{op}\otimes C}M^\bullet) \stackrel{\pi_T\otimes^\bullet 1}{\longrightarrow} T^\bullet\otimes_A^\bullet(\mathbf{p}_{A^{op}\otimes C}M^\bullet) = (T^\bullet\otimes_A^{\mathbf{L}}?)(M^\bullet)$$

That this map defines a natural transformation $\mathbf{T}_A(T^\bullet,?) \longrightarrow T^\bullet\otimes_A^{\mathbf{L}}?$ is routine.

(2) follows as (1), by applying a left-right symmetric argument.

(3), (4) By definition again, we have $\mathbf{H}_A(T^\bullet,N^\bullet) = \mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet, \mathbf{i}_{C^{op}\otimes A}N^\bullet)$, $\mathbf{R}\mathrm{Hom}_A(T^\bullet,?)(N^\bullet) = \mathrm{Hom}_A^\bullet(T^\bullet,\mathbf{i}_{C^{op}\otimes A}N^\bullet)$ and $\mathbf{R}\mathrm{Hom}_A(?,N^\bullet)(T^\bullet) = \mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet,N^\bullet)$. If π_T is as above and $\iota_N : N^\bullet \longrightarrow \mathbf{i}_{C^{op}\otimes A}N^\bullet$ is the homotopically injective resolution, we then have obvious chain maps

$$\mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet,N^\bullet) \stackrel{(\iota_N)_*}{\longrightarrow} \mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet,\mathbf{i}_{C^{op}\otimes A}N^\bullet) \stackrel{\pi_T^*}{\longleftarrow} \mathrm{Hom}_A^\bullet(T^\bullet,\mathbf{i}_{C^{op}\otimes A}N^\bullet).$$

It is again routine to see that they induce natural transformations of triangulated functors

$$\mathbf{R}\mathrm{Hom}_A(?,N^\bullet) \longrightarrow \mathbf{H}_A(?,N^\bullet) \text{ and } \mathbf{H}_A(T^\bullet,?) \longleftarrow \mathbf{R}\mathrm{Hom}_A(T^\bullet,?).$$

On the other hand, when C is k-flat, by lemma 3.6, we know that the forgetful functor $\mathcal{H}(A^{op}\otimes C) \longrightarrow \mathcal{H}(A^{op})$ (resp. $\mathcal{H}(C^{op}\otimes A) \longrightarrow \mathcal{H}(A)$) takes homotopically projective objects to homotopically flat objects (resp. preserves homotopically injective objects). In particular, the morphisms $\pi_T\otimes^\bullet 1$ and π_T^* considered above, are both quasi-isomorphisms, which gives assertion a). Assertion b) follows from the part of a) concerning the derived tensor product by a symmetric argument.

Finally, if B is k-projective, then, by lemma 3.6, the forgetful functor $\mathcal{H}(B^{op}\otimes A) \longrightarrow \mathcal{H}(A)$ preserves homotopically projective objects. Then the map $(\iota_N)_*$ above is a quasi-isomorphism.

□

Lemma 3.8. *Let A, B and C be k-algebras. The following assertions hold:*

(1) *The assignment $(T^\bullet,X^\bullet) \rightsquigarrow Hom_{B^{op}}^\bullet(T^\bullet,B)\otimes_B^\bullet X^\bullet$ is the definition on objects of a bifunctor*
$$Hom_{B^{op}}^\bullet(?,B)\otimes_B^\bullet ? : \mathcal{C}(B^{op}\otimes A)^{op}\times\mathcal{C}(B^{op}\otimes C) \longrightarrow \mathcal{C}(A^{op}\otimes C)$$
which preserves conflations and contractible complexes on each variable.

(2) *There is a natural transformation of bifunctors*
$$\psi : Hom_{B^{op}}^\bullet(?,B)\otimes_B^\bullet ? \longrightarrow Hom_{B^{op}}^\bullet(?,?).$$

Proof. Assertion (1) is routine and left to the reader. As for assertion (2), let us fix $T^\bullet\in\mathcal{C}(B^{op}\otimes A)$ and $X^\bullet\in\mathcal{C}(B^{op}\otimes C)$. We need to define a map

$$\psi = \psi_{T,X} : \mathrm{Hom}^\bullet_{B^{op}}(T^\bullet, B) \otimes^\bullet_B X^\bullet \longrightarrow \mathrm{Hom}^\bullet_{B^{op}}(T^\bullet, X^\bullet).$$

Let $f \in \mathrm{Hom}^\bullet_{B^{op}}(T^\bullet, B)$ and $x \in X^\bullet$ be homogeneous elements, whose degrees are denoted by $|f|$ and $|x|$. We define $\psi(f \otimes x)(t) = (-1)^{|f|\cdot|x|} f(t)x$, for all homogeneous elements $f \in \mathrm{Hom}^\bullet_{B^{op}}(T^\bullet, B)$, $x \in X^\bullet$ and $t \in T^\bullet$, and leave to the reader the routine task of checking that ψ is a chain map of complexes of $A - C-$bimodules which is natural on both variables. □

Assertion 1 in the previous lemma allows us to define the following bifunctor, which is triangulated on both variables:

$$\mathbf{TH}(?,?) : \mathcal{D}(B^{op} \otimes A)^{op} \times \mathcal{D}(B^{op} \otimes C) \overset{\mathbf{p}_{B^{op}\otimes A} \times \mathbf{p}_{B^{op}\otimes C}}{\longrightarrow}$$
$$\mathcal{H}(B^{op} \otimes A)^{op} \times \mathcal{H}(B^{op} \otimes C) \overset{\mathrm{Hom}^\bullet_{B^{op}}(?,B)\otimes^\bullet_B ?}{\longrightarrow} \mathcal{H}(A^{op} \otimes C) \overset{q}{\longrightarrow} \mathcal{D}(A^{op} \otimes C).$$

We have the following property of this bifunctor.

Proposition 3.9. *Let A, B and C be k-algebras and let $T^\bullet \in \mathcal{C}(B^{op} \otimes A)$ and $Y^\bullet \in \mathcal{C}(B^{op} \otimes C)$ be complexes of bimodules. The natural transformation of lemma 3.8 induces:*

1. *A natural transformation of bifunctors $\theta : \mathbf{TH}(?,?) \longrightarrow \mathbf{H}_{B^{op}}(?,?)$ and natural transformation $\theta_{Y^\bullet} : (? \otimes^{\mathbf{L}}_B Y^\bullet) \circ \mathbf{R}Hom_{B^{op}}(?, B) \longrightarrow \mathbf{R}Hom_{B^{op}}(?, Y^\bullet)$ of triangulated functors $\mathcal{D}(B^{op} \otimes A) \longrightarrow \mathcal{D}(A^{op} \otimes C)$. When A is k-projective, there is a natural transformation $\upsilon_{Y^\bullet} : (? \otimes^{\mathbf{L}}_B Y^\bullet) \circ \mathbf{R}Hom_{B^{op}}(?, B) \longrightarrow \mathbf{TH}(?, Y^\bullet)$ such that $\theta_{Y^\bullet} \cong \theta(?, Y^\bullet) \circ \upsilon_{Y^\bullet}$. If, in addition, C is k-flat, then $\upsilon_{Y^\bullet}$ is a natural isomorphism.*
2. *When C is k-flat, θ induces a natural transformation $\theta^{T^\bullet} : \mathbf{R}Hom_{B^{op}}(T^\bullet, B) \otimes^{\mathbf{L}}_B ? \cong \mathbf{TH}(T^\bullet, ?) \longrightarrow \mathbf{H}_{B^{op}}(T^\bullet, ?) \cong \mathbf{R}Hom_{B^{op}}(T^\bullet, ?)$ of triangulated functors $\mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(A^{op} \otimes C)$.*

Proof. All throughout the proof, for simplicity, whenever $M^\bullet$ is a complex of bimodules, we will denote by $\pi = \pi_M : \mathbf{p}M^\bullet \longrightarrow M^\bullet$ and $\iota = \iota_M : M^\bullet \longrightarrow \mathbf{i}M^\bullet$, respectively, the homotopically projective resolution and the homotopically injective resolution, without explicitly mentioning over which algebras it is a complex of bimodules. In each case, it should be clear from the context which algebras they are.

(1) We define $\theta_{(T,Y)}$ as the composition

$$\mathbf{TH}(T^\bullet, Y^\bullet) = \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes^\bullet_B \mathbf{p}Y^\bullet \overset{\psi}{\longrightarrow} \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, \mathbf{p}Y^\bullet) \overset{(\pi_Y)_*}{\longrightarrow}$$
$$\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, Y^\bullet) \overset{(\iota_Y)_*}{\longrightarrow} \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, \mathbf{i}Y^\bullet) = \mathbf{H}_{B^{op}}(T^\bullet, Y^\bullet).$$

Checking its naturality on both variables is left to the reader.

As definition of $\theta_{Y^\bullet}$, we take the composition:

$$[(? \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)](T^\bullet) = \mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes^\bullet_B Y^\bullet \overset{\pi\otimes^\bullet 1}{\longrightarrow}$$
$$\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes^\bullet_B Y^\bullet \overset{\psi}{\longrightarrow} \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, Y^\bullet) = \mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet)(T^\bullet).$$

When A is k-projective, we have a natural isomorphism $\mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet) \cong \mathbf{H}_{B^{op}}(?, Y^\bullet)$ (see proposition 3.7 c)). On the other hand, we have morphisms in $\mathcal{H}(A^{op} \otimes B)$

$$\mathbf{TH}(T^\bullet, Y^\bullet) = \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes^\bullet_B \mathbf{p}Y^\bullet \overset{\pi_H\otimes^\bullet 1}{\longleftarrow} \mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes^\bullet_B \mathbf{p}Y^\bullet \overset{1\otimes^\bullet \pi_Y}{\longrightarrow}$$
$$\mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes^\bullet_B Y^\bullet = [(? \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)](T^\bullet),$$

where π_H is taken for $H^\bullet := \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B)$. The right morphism is a quasi-isomorphism since the k-projectivity of A guarantees that $\mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B)$ is homotopically projective in $\mathcal{H}(B)$. The evaluation of the natural transformation υ_Y at $T^\bullet$ is $\upsilon_{Y,T} = (\pi_H \otimes^\bullet 1) \circ (1 \otimes^\bullet \pi_Y)^{-1} : [(? \otimes \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)](T^\bullet) \longrightarrow \mathbf{TH}(T^\bullet, Y^\bullet)$. This is an isomorphism if, and only if, $\pi_H \otimes^\bullet 1$ is an isomorphism in $\mathcal{D}(A^{op} \otimes C)$. For this to happen it is enough that C be k-flat, for then $\mathbf{p}Y^\bullet$ is homotopically flat in $\mathcal{H}(B^{op})$.

The fact that, when A is k-projective, we have an equality $\theta_Y = \theta(?, Y) \circ \upsilon_Y$ follows from the commutativity in $\mathcal{H}(A^{op} \otimes B)$ of the following diagram:

$$
\begin{tikzcd}
 & \left[(? \otimes^{\mathbf{L}}_B Y^\bullet) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)\right](T^\bullet) \arrow[d, equal] \\
\mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes_B \mathbf{p}Y^\bullet \arrow[r, "1\otimes\pi_Y"] \arrow[d, "\pi_H\otimes 1"] & \mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B \otimes_B Y^\bullet \arrow[d, "\pi_H\otimes 1"] \\
\mathbf{TH}(T^\bullet, Y^\bullet) = \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes \mathbf{p}Y^\bullet \arrow[r, "1\otimes\pi_Y"] \arrow[d, "\psi"] & \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes Y^\bullet \arrow[d, "\psi"] \\
\mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, \mathbf{p}Y^\bullet) \arrow[r, "(\pi_Y)_*"] & \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, Y^\bullet) = \mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet)(T^\bullet) \arrow[d, "(\iota_Y)_*"] \\
 & \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, \iota Y^\bullet) = \mathbf{H}_{B^{op}}(T^\bullet, Y^\bullet)
\end{tikzcd}
$$

(2) Note that, by definition of the involved functors, we have an equality $\mathbf{R}\mathrm{Hom}_{B^{op}}(T, B) \otimes^{\mathbf{L}}_B ? = \mathbf{TH}(T^\bullet, ?)$. Then assertion (2) is a consequence of assertion (1) and proposition 3.7 a). $\square$

Definition 16. *Let A and B be k-algebras. We shall denote by $(?)^*$ both functors $\mathbf{R}Hom_B(?, B) : \mathcal{D}(A^{op} \otimes B)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes A)$ and $\mathbf{R}Hom_{B^{op}}(?, B) : \mathcal{D}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(A^{op} \otimes B)$.*

Note that, with the appropriate interpretation, the functors $(?)^*$ are adjoint to each other due to corollary 3.2. In particular, we have the corresponding unit $\sigma : 1 \longrightarrow (?)^{**}$ for the two possible compositions. The following result is now crucial for us:

Proposition 3.10. *Let A, B and C be k-algebras, let $T^\bullet$ be a complex of $B-A-$bimodules such that ${}_BT^\bullet$ is compact in $\mathcal{D}(B^{op})$ and suppose that A is k-projective. The following assertions hold:*

1. *For each complex $Y^\bullet$ of $B - C-$bimodules, the map $\theta_Y(T^\bullet) : (? \otimes^{\mathbf{L}}_B Y)(T^{\bullet *}) = [(? \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}Hom_{B^{op}}(?, B)](T^\bullet) \longrightarrow \mathbf{R}Hom_{B^{op}}(?, Y^\bullet)(T^\bullet)$ is an isomorphism in $\mathcal{D}(A^{op} \otimes C)$;*
2. *$T^{\bullet *}_B$ is compact in $\mathcal{D}(B)$ and he map $\sigma_T : T^\bullet \longrightarrow T^{\bullet **}$ is an isomophism in $\mathcal{D}(B^{op} \otimes A)$;*
3. *When in addition C is k-flat, the following assertions hold:*
   (a) *There is a natural isomorphism of triangulated functors $T^{\bullet *} \otimes^{\mathbf{L}}_B ? \cong \mathbf{R}Hom_{B^{op}}(T^\bullet, ?) : \mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(A^{op} \otimes C)$;*

(b) *There is a natural isomorphism of triangulated functors* $?\otimes_B^{\mathbf{L}} T^\bullet \cong \mathbf{R}Hom_B(T^{\bullet *}, ?) : \mathcal{D}(C^{op} \otimes B) \longrightarrow \mathcal{D}(A^{op} \otimes C)$.

Proof. All throughout the proof we use the fact that $\mathrm{per}(B) = \mathrm{thick}_{\mathcal{D}(B)}(B)$ and similarly for B^{op} (see proposition 3.14 below).

(1) For any k-projective algebra A, let us put $F_A := (? \otimes_B^{\mathbf{L}} Y^\bullet) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)$ and $G_A := \mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet)$, which are triangulated functors $\mathcal{D}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(A^{op} \otimes C)$. Since the forgetful functors $\mathcal{H}(B^{op} \otimes A) \longrightarrow \mathcal{H}(B^{op}) = \mathcal{H}(B^{op} \otimes k)$ and $\mathcal{H}(A^{op} \otimes C) \longrightarrow \mathcal{H}(C) = \mathcal{H}(C \otimes k)$ preserve homotopically projective objects, we have the following commutative diagrams (one for F and another one for G), where the vertical arrows are the forgetful (or restriction of scalars) functors:

$$\begin{tikzcd}
\mathcal{D}(B^{op} \otimes A)^{op} \arrow[r, shift left, "F_A"] \arrow[r, shift right, "G_A"'] \arrow[d, "U_{B^{op}}"] & \mathcal{D}(A^{op} \otimes C) \arrow[d, "U_C"] \\
\mathcal{D}(B^{op})^{op} \arrow[r, shift left, "F_k"] \arrow[r, shift right, "G_k"'] & \mathcal{D}(C)
\end{tikzcd}$$

We shall denote $\theta_{Y^\bullet}(T^\bullet)$ by $\theta^A_{T^\bullet, Y^\bullet} : F_A(T^\bullet) \longrightarrow G_A(T^\bullet)$ to emphasize that there is one for each choice of a k-projective algebra A. Note that, by proposition 3.9, if C is not k-flat we cannot guarantee that $\theta^A_{?,?}$ is a natural transformation of bifunctors. Recall that $\theta^A_{T^\bullet, Y^\bullet}$ is the composition

$$F_A(T^\bullet) = [(? \otimes_B^{\mathbf{L}} Y) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)](T^\bullet) = \mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes_B^\bullet Y^\bullet \overset{\pi \otimes^\bullet 1}{\longrightarrow}$$
$$\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes_B^\bullet Y^\bullet \overset{\psi}{\longrightarrow} \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, Y^\bullet) = \mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet)(T^\bullet) = G_A(T^\bullet).$$

When applying the forgetful functor $U_C : \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(C)$, we obtain $U_C(\theta^A_{T,Y}) := (U_C \circ F_A)(T^\bullet) \longrightarrow (U_C \circ G_A)(T^\bullet)$. Due to the commutativity of the above diagram, this last morphism can be identified with the morphism $\theta^k_{T,Y}$, i.e., the version of θ_Y obtained when taking $A = k$. But observe that $\theta^k_{B,Y}$ is an isomorphism. This implies that $\theta^k_{X,Y}$ is an isomorphism, for each $X^\bullet \in \mathrm{thick}_{\mathcal{D}(B^{op})}(B) = \mathrm{per}(B^{op})$. It follows that $\theta^k_{T,Y} = U_C(\theta^A_{T,Y})$ is an isomorphism since ${}_BT^\bullet \in \mathrm{per}(B^{op})$. But then $\theta^A_{T,Y}$ is an isomorphism because U_C reflects isomorphisms.

(2) Due to the k-projectivity of A, we have the following commutative diagram, where the lower horizontal arrow is the version of $(?)^*$ when $A = k$.

$$\begin{tikzcd}
\mathcal{D}(B^{op} \otimes A)^{op} \arrow[r, "(?)^*"] \arrow[d, "U_{B^{op}}"] & \mathcal{D}(A^{op} \otimes B) \arrow[d, "U_B"] \\
\mathcal{D}(B^{op})^{op} \arrow[r, "(?)^*"] & \mathcal{D}(B)
\end{tikzcd}$$

The task is hence reduced to check that the lower horizontal arrow preserves compact objects. That is a direct consequence of the fact that $({}_BB)^* = \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)(B) \cong B_B$ and that the full subcategory of $\mathcal{D}(B^{op})$ consisting of the $X^\bullet$ such that $X^{\bullet *} \in \mathrm{per}(B)$ is a thick subcategory of $\mathcal{D}(B^{op})$.

In order to prove that $\sigma = \sigma_{T^\bullet}$ is an isomorphism there is no loss of generality in assuming that $A = k$. Note that the full subcategory $\mathcal{X}$ of $\mathcal{D}(B^{op})$ consisting of the $X^\bullet$ for which $\sigma_{X^\bullet} : X^\bullet \longrightarrow X^{\bullet**}$ is an isomorphism is a thick subcategory. We just need to prove that ${}_BB \in \mathcal{X}$ since ${}_BT^\bullet \in \text{per}(B^{op})$. We do it by applying lemma 3.3 with ${}_BT^\bullet_A = {}_BB_B$. Indeed, $\lambda_B : B \longrightarrow \mathbf{R}\text{Hom}_B(B, ? \otimes^{\mathbf{L}}_B B)(B)$ is an isomorphism in $\mathcal{D}(B)$.

(3) Assume now that C is k-flat. Due to assertion (2), assertions (3)(a) and (3)(b) are left-right symmetric. We then prove (3)(a). Proposition 3.9 allows us to identify $\theta_{Y^\bullet} = \theta_{?,Y^\bullet}$ and $\theta^{T^\cdot} = \theta_{T^\bullet,?}$, where $\theta : \mathbf{TH}(?,?) \longrightarrow \mathbf{H}_{B^{op}}(?,?)$ is the natural transformation of bifunctors given in that proposition. We then have the following chain of double implications:

$$\theta^{T^\bullet} \text{ natural isomorphism} \Longleftrightarrow \theta_{T^\bullet,Y^\bullet} \text{ isomorphism, for each } Y^\bullet \in \mathcal{D}(B^{op} \otimes C), \Longleftrightarrow \theta^{Y^\bullet}(T^\bullet) \text{ isomorphism, for each } Y^\bullet \in \mathcal{D}(B^{op} \otimes C).$$

Now use assertion (1). □

As a direct consequence of the previous proposition, we have:

Corollary 3.11. *Let A and B be k-algebras, where A is k-projective, and denote by $\mathcal{L}_{B,A}$ (resp $\mathcal{R}_{A,B}$) the full subcategory of $\mathcal{D}(B^{op} \otimes A)$ (resp. $\mathcal{D}(A^{op} \otimes B)$) consisting of the objects $X^\cdot$ such that ${}_BX^\bullet$ (resp. $X^\bullet_B$) is compact in $\mathcal{D}(B^{op})$ (resp. $\mathcal{D}(B)$). Then $\mathcal{L}_{B,A}$ and $\mathcal{R}_{A,B}$ are thick subcategories of $\mathcal{D}(B^{op} \otimes A)$ and $\mathcal{D}(A^{op} \otimes B)$, respectively. Moreover, the assignments $X^\bullet \rightsquigarrow X^{\bullet *}$ define quasi-inverse triangulated dualities (=contravariant equivalences) $\mathcal{L}_{B,A} \overset{\cong}{\longleftrightarrow} \mathcal{R}_{A,B}$.*

3.4. **A few remarks on dg algebras and their derived categories.** In a few places in this paper, we will need some background on dg algebras and their derived categories. We just give an outline of the basic facts that we need. Interpreting dg algebras as dg categories with just one object, the material is a particular case of the development in [19] or [20].

A *dg algebra* is a ($\mathbb{Z}$-)graded algebra $A = \oplus_{p\in\mathbb{Z}} A^p$ together with a *differential*. This is a graded k-linear map $d : A \longrightarrow A$ of degree 1 such that $d(ab) = d(a)b + (-1)^{|a|}ad(b)$, for all homogeneous elements $a, b \in A$, where $|a|$ denotes the degree of a, and such that $d \circ d = 0$. In such case a (right) *dg A-module* is a graded A-module $M^\bullet = \oplus_{p\in\mathbb{Z}} M^p$ together with a k-linear map $d_M : M^\bullet \longrightarrow M^\bullet$ of degree $+1$, called the differential of $M^\bullet$, such that $d_M(xa) = d_M(x)a + (-1)^{|x|}xd(a)$, for all homogeneous elements $x \in M^\bullet$ and $a \in A$, and such that $d_M \circ d_M = 0$. It is useful to look at each dg A-module as a complex $...M^p \xrightarrow{d^p_M} M^{p+1} \longrightarrow ..$ of k-modules with some extra properties. Note that an ordinary algebra is just a dg algebra with grading concentrated in degree 0, i.e. $A^p = 0$ for $p \neq 0$. In that case a dg A-module is just a complex of A-modules.

Let A be a dg algebra in the rest of this paragraph. We denote by $\text{Gr} - A$ the category of graded A-modules (and morphisms of degree 0). Note that, when A is an ordinary algebra, we have $\text{Gr} - A = (\text{Mod} - A)^{\mathbb{Z}}$, with the notation of subsection 2.2. We next define a category $\mathcal{C}(A)$ whose objects are the dg A-modules as follows. A morphism $f : M^\bullet \longrightarrow N^\bullet$ in $\mathcal{C}(A)$ is a morphism in $\text{Gr} - A$ which is a chain map of complex of k-modules, i.e., such that $f \circ d_M = d_M \circ f$. This category is abelian and comes with a canonical shifting $?[1] : \mathcal{C}(A) \longrightarrow \mathcal{C}(A)$ which comes from the canonical shifting of

$\mathrm{Gr}-A$, by defining $d^n_{M[1]}=-d^{n+1}_M$, for each $n\in\mathbb{Z}$. Note that we have an obvious faithful forgetful functor $\mathcal{C}(A)\longrightarrow\mathrm{Gr}-A$, which is also dense since we can interpret each graded A-module as an object of $\mathcal{C}(A)$ with zero differential. Viewing the objects of $C(A)$ as complexes of k-modules, we clearly have, for each $p\in\mathbb{Z}$, the *p-th homology functor* $H^p:\mathcal{C}(A)\longrightarrow\mathrm{Mod}-k$. A morphism $f:M^\bullet\longrightarrow N^\bullet$ in $\mathcal{C}(A)$ is called a *quasi-isomorphism* when $H^p(f)$ is an isomorphism, for all $p\in\mathbb{Z}$. A dg A-module $M^\bullet$ is called *acyclic* when $H^p(M)=0$, for all $p\in\mathbb{Z}$.

Given any dg A-module $X^\bullet$, we denote by $P^\bullet_X$ the dg A-module which, as a graded A-module, is equal to $X^\bullet\oplus X^\bullet[1]$, and where the differential, viewed as a morphism $X^\bullet\oplus X^\bullet[1]=P^\bullet_X\longrightarrow P^\bullet_X[1]=X^\bullet[1]\oplus X^\bullet[2]$ in $\mathrm{Gr}-k$, is the 'matrix' $\begin{pmatrix} d_X & 1_{X[1]}\\ 0 & d_{X[1]}\end{pmatrix}$. Note that we have a canonical exact sequence $0\to X^\bullet\longrightarrow P^\bullet_X\longrightarrow X^\bullet[1]\to 0$ in $\mathcal{C}(A)$, which splits in $\mathrm{Gr}-A$ but not in $\mathcal{C}(A)$. A morphism $f:M^\bullet\longrightarrow N^\bullet$ in $\mathcal{C}(A)$ is called *null-homotopic* when there exists a morphism $\sigma:M^\bullet\longrightarrow N^\bullet[-1]$ in $\mathrm{Gr}-A$ such that $\sigma\circ d_M+d_N\circ\sigma=f$.

The following is the fundamental fact (see [19] and [20]):

Proposition 3.12. *Let A be a dg algebra. The following assertions hold:*

1. *$\mathcal{C}(A)$ has a structure of exact category where the conflations are those exact sequences which become split when applying the forgetful functor $\mathcal{C}(A)\longrightarrow Gr-A$;*
2. *The projective objects with respect to this exact structure coincide with the injective ones, and they are the direct sums of dg A-modules of the form $P^\bullet_X$. In particular $\mathcal{C}(A)$ is a Frobenius exact category;*
3. *A morphism $f:M^\bullet\longrightarrow N^\bullet$ in $\mathcal{C}(A)$ factors through a projective object if, an only it, it is* null-homotopic. *The stable category $\underline{\mathcal{C}(A)}$ with respect to the given exact structure is denoted by $\mathcal{H}(A)$ and called the* homotopy category of *A. It is a triangulated category, where $?[1]$ is the suspension functor and where the triangles are the images of conflations by the projection functor $p:\mathcal{C}(A)\longrightarrow\mathcal{H}(A)$;*
4. *If Q denotes the class of quasi-isomorphisms in $\mathcal{C}(A)$ and $\Sigma:=p(Q)$, then Σ is a multiplicative system in $\mathcal{H}(A)$ compatible with the triangulation and $\mathcal{C}(A)[Q^{-1}]\cong\mathcal{H}(A)[\Sigma^{-1}]$. In particular, this latter category is triangulated. It is denoted by $\mathcal{D}(A)$ and called the* derived category of *A;*
5. *If $\mathcal{Z}$ denotes the full subcategory of $\mathcal{H}(A)$ consisting of the acyclic dg A-modules, then $\mathcal{Z}$ is a triangulated subcategory closed under taking coproducts and products in $\mathcal{H}(A)$. The category $\mathcal{D}(A)$ is equivalent, as a triangulated category, to the quotient category $\mathcal{H}(A)/\mathcal{Z}$.*

As in the case of an ordinary algebra, we do not have set-theoretical problems with the just defined derived category. The reason is the following:

Proposition 3.13. *Let A be a dg algebra and let $\mathcal{Z}$ be the full subcategory of $\mathcal{H}(A)$ consisting of acyclic dg A-modules. Then the pairs $(\mathcal{Z},\mathcal{Z}^\perp)$ and $({}^\perp\mathcal{Z},\mathcal{Z})$ are semi-orthogonal decompositions of $\mathcal{H}(A)$. In particular, the canonical functor $q:\mathcal{H}(A)\longrightarrow\mathcal{D}(A)$ has both a left adjoint $p_A:\mathcal{D}(A)\longrightarrow\mathcal{H}(A)$ and a right adjoint $i_A:\mathcal{D}(A)\longrightarrow\mathcal{H}(A)$.*

Moreover, the category $\mathcal{D}(A)$ is compactly generated by $\{A\}$.

As in the case of an ordinary algebra, we call $\mathbf{p}_A$ and $\mathbf{i}_A$ the *homotopically projective resolution functor* and the *homotopically injective resolution functor*, respectively. Also, the objects of ${}^{\perp}\mathcal{Z}$ are called *homotopically projective* and those of $\mathcal{Z}^{\perp}$ are called *homotopically injective.*

When A is a dg algebra, its *opposite dg algebra* A^{op} is equal to A as a graded k-module, but the multiplication is given by $b^o \cdot a^o = (-1)^{|a||b|}(ab)^o$. Here we denote by x^o any homogeneous element $x \in A$ when viewed as an element of A^{op}. Then, given dg algebras A and B, a dg $B-A-$module T is just a dg right $B^{op} \otimes A-$module, and these dg $B-A-$modules form the category $\mathcal{C}(B^{op} \otimes A)$.

With some care on the use of signs, the reader will have no difficulty in extending to dg algebras and their derived categories the results in the earlier paragraphs of this section, simply by defining correctly the total tensor and total Hom functors bifunctors. Indeed, if A, B and C are dg algebras and ${}_BT^{\bullet}_A$, ${}_AM^{\bullet}_C$ and ${}_CN^{\bullet}_A$ are dg bimodules, then $T^{\bullet} \otimes^{\bullet}_A M^{\bullet}$ is just the classical tensor product of a graded right A-module and a graded left A-module and the differential $d = d_{T\otimes^{\bullet}_A M} : (T \otimes^{\bullet}_A M) \longrightarrow (T \otimes^{\bullet}_A M)$ takes $t \otimes m \rightsquigarrow d_T(t) \otimes m + (-1)^{|t|} t \otimes d_M(m)$, for all homogeneous elements $t \in T^{\bullet}$ and $m \in M^{\bullet}$.

We define $\mathrm{Hom}^{\bullet}_A(T^{\bullet}, N^{\bullet}) = \oplus_{p\in\mathbb{Z}} \mathrm{Gr}(A)(M^{\bullet}, N^{\bullet}[p])$. This has an obvious structure of graded $C-B-$bimodule, where the homogeneous component of degree p is $\mathrm{Hom}^{\bullet}_A(T^{\bullet}, N^{\bullet})^p = \mathrm{Gr}(A)(M^{\bullet}, N^{\bullet}[p])$, for each $p \in \mathbb{Z}$. We then define the differential $d : \mathrm{Hom}^{\bullet}_A(T^{\bullet}, N^{\bullet}) \longrightarrow \mathrm{Hom}^{\bullet}_A(T^{\bullet}, N^{\bullet})$ by taking $f \rightsquigarrow d_N \circ f - (-1)^{|f|} f \circ d_M$, for each homogeneous element $f \in \mathrm{Hom}^{\bullet}_A(T^{\bullet}, N^{\bullet})$.

The following is a very well-known fact (see [19, Theorem 5.3] and [34]):

Proposition 3.14. *Let A be a dg algebra. The compact objects of $\mathcal{D}(A)$ are precisely the objects of $thick_{\mathcal{D}(A)}(A) =: per(A)$. When A is an ordinary algebra, those are precisely the complexes which are quasi-isomorphic to bounded complexes of finitely generated projective A-modules.*

3.5. **Some special objects and a generalization of Rickard theorem.**

Definition 17. *Let $\mathcal{D}$ be a triangulated category with coproducts. An object T of $\mathcal{D}$ is called:*

(1) exceptional *when $\mathcal{D}(T, T[p]) = 0$, for all $p \neq 0$;*
(2) classical tilting *when T is exceptional and $\{T\}$ is a set of compact generators of $\mathcal{D}$;*
(3) self-compact *when T is a compact object of $Tria_{\mathcal{D}}(T)$.*

The following result is a generalization of Rickard theorem ([34]), which, as can be seen in the proof, is an adaptation of an argument of Keller that the authors used in [31].

Theorem 3.15. *Let A be a k-flat dg algebra and $T^{\bullet}$ be a self-compact and exceptional object of $\mathcal{D}(A)$. If $B = End_{\mathcal{D}(A)}(T^{\bullet})$ is the endomorphism algebra then, up to replacement of $T^{\bullet}$ by an isomorphic object in $\mathcal{D}(A)$, we can view $T^{\bullet}$ as a dg B-A-bimodules such that the restriction of $\mathbf{R}Hom_A(T^{\bullet}, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ induces an equivalence of triangulated categories $Tria_{\mathcal{D}(A)}(T^{\bullet}) \xrightarrow{\cong} \mathcal{D}(B)$.*

Proof. We can assume that $T^{\bullet}$ is homotopically injective in $\mathcal{H}(A)$. Then $\hat{B} = \mathrm{End}^{\bullet}_A(T^{\bullet}) := \mathrm{Hom}^{\bullet}_A(T^{\bullet}, T^{\bullet})$ is a dg algebra and $T^{\bullet}$ becomes a dg $\hat{B} - A-$bimodule in a natural way.

Moreover, by [31, Corollary 2.5], we get that $? \otimes^{\mathbf{L}}_{\hat{B}} T^\bullet : \mathcal{D}(\hat{B}) \longrightarrow \mathcal{D}(A)$ induces an equivalence of triangulated categories $\mathcal{D}(\hat{B}) \xrightarrow{\cong} \mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\cdot)$.

On the other hand, at the level of homology, we have:

$$H^n(\hat{B}) = \mathcal{H}(\hat{B})(\hat{B}, \hat{B}[n]) \cong \mathcal{D}(\hat{B})(\hat{B}, \hat{B}[n]) \cong \mathcal{T}(T^\bullet, T^\bullet[n]) \cong \mathcal{D}(A)(T^\bullet, T^\bullet[n]),$$

for all $n \in \mathbb{Z}$. It follows that $H^n(\hat{B}) = 0$, for $n \neq 0$, while $H^0(\hat{B}) \cong B$. We take the canonical truncation of $\hat{B}$ at 0, i.e., the dg subalgebra $\tau^{\leq 0}\hat{B}$ of $\hat{B}$ given, as a complex of k-modules, by

$$...\hat{B}^{-n} \longrightarrow ... \longrightarrow \hat{B}^{-1} \longrightarrow \mathrm{Ker}(d^0) \longrightarrow 0...,$$

where $d^0 : \hat{B}^0 \longrightarrow \hat{B}^1$ is the 0-th differential of $\hat{B}$. Then $B = H^0(\tau^{\leq 0}\hat{B}) = H^0(\hat{B})$ and we have quasi-isomorphism of dg algebras $B \xleftarrow{p} \tau^{\leq 0}\hat{B} \overset{j}{\hookrightarrow} \hat{B}$, where p and j are the projection and inclusion, respectively. Replacing $\hat{B}$ by $\tau^{\leq 0}\hat{B}$ and the dg bimodule ${}_{\hat{B}}T^\bullet_A$ by ${}_{\tau^{\leq 0}\hat{B}}T^\bullet_A$, we can assume, without loss of generality, that $\hat{B}^p = 0$, for all $p > 0$. We assume this in the sequel.

It is convenient now to take the homotopically projective resolution $\pi : \mathbf{p}_{\hat{B}^{op} \otimes A}T^\bullet \longrightarrow T^\bullet$ in $\mathcal{H}(\hat{B}^{op} \otimes A)$. We then replace $T^\bullet$ by $\mathbf{p}T^\bullet := \mathbf{p}_{\hat{B}^{op}\otimes A}T^\bullet$. In this way we lose the homotopically injective condition of $T^\bullet_A$ in $\mathcal{H}(A)$, but we win that ${}_{\hat{B}}T^\bullet$ is homotopically flat in $\mathcal{H}(\hat{B}^{op})$ (this follows by the extension of lemma 3.6 to dg algebras). Note, however, that $\pi : \mathbf{p}T^\bullet \longrightarrow T^\bullet$ induces a natural isomorphism $? \otimes^{\mathbf{L}}_{\hat{B}} \mathbf{p}T^\bullet \cong ? \otimes^{\mathbf{L}}_{\hat{B}} T^\bullet$ of triangulated functors $\mathcal{D}(\hat{B}) \longrightarrow \mathcal{D}(A)$.

We can view $p : \hat{B} \longrightarrow B$ as a quasi-isomorphism in $\mathcal{H}(\hat{B})$ and then $T^\bullet \cong \hat{B} \otimes^\bullet_{\hat{B}} T^\bullet \xrightarrow{p \otimes 1_T} B \otimes^\bullet_{\hat{B}} T^\bullet$ is a quasi-isomorphism since ${}_{\hat{B}}T^\bullet$ is homotopically flat. By this same reason, we have that $B \otimes^\bullet_{\hat{B}} T^\bullet \cong B \otimes^{\mathbf{L}}_{\hat{B}} T^\bullet$ (see lemma 3.5). Then we get a composition of triangulated equivalences

$$\mathcal{D}(B) \xrightarrow{p_*} \mathcal{D}(\hat{B}) \xrightarrow{? \otimes^{\mathbf{L}}_{\hat{B}} T^\bullet} \mathcal{T},$$

where the first arrow is the restriction of scalars along the projection $p : \hat{B} \longrightarrow B$. This composition of equivalences is clearly identified with the functor $? \otimes^{\mathbf{L}}_B (B \otimes^{\mathbf{L}}_{\hat{B}} T^\bullet) = ? \otimes^{\mathbf{L}}_B (B \otimes^\bullet_{\hat{B}} T^\bullet)$.

Replacing $\hat{T}^\bullet$ by $B \otimes^\bullet_{\hat{B}} T^\bullet$, we can then assume that $T^\bullet$ is a dg B-A-bimodule such that $? \otimes^{\mathbf{L}}_B T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ induces an equivalence $\mathcal{D}(B) \xrightarrow{\cong} \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. A quasi-inverse of this equivalence is then the restriction of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ to $\mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. $\square$

4. Main results

All throughout this section A and B are arbitrary k-algebras and $T^\bullet$ is a complex of $B - A-$bimodules. We want to give necessary and sufficient conditions of the functors $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ or $? \otimes^{\mathbf{L}}_B T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ to be fully faithful. We start with two very helpful auxiliary results.

The following is the final result of [32]. Note that condition 2 of the proposition was given only for $\mathcal{D}(B^{op})$, but this condition holds if, and only if, it holds for $\mathcal{D}(k)$ since the forgetful functor $\mathcal{D}(B^{op}) \longrightarrow \mathcal{D}(k)$ reflects isomorphisms.

Proposition 4.1. *Suppose that $H^p(T^\bullet) = 0$, for $p >> 0$. The following assertions are equivalent:*

1. *$T^\bullet$ is isomorphic in $\mathcal{D}(B^{op})$ to an upper bounded complex of finitely generated projective left B-modules;*
2. *The canonical morphism $B^\alpha \otimes_B^{\mathbf{L}} T^\bullet \longrightarrow (B \otimes_B^{\mathbf{L}} T^\bullet)^\alpha = T^{\bullet\alpha}$ is an isomorphism (in $\mathcal{D}(A)$, $\mathcal{D}(B^{op}$ or $\mathcal{D}(k)$), for each cardinal α.*

We can go further in the direction of the previous proposition. Recall that if $X^\bullet = (X^\bullet, d) \in \mathcal{C}(\mathcal{A})$ is any complex, where $\mathcal{A}$ is an abelian category, then its *left (resp. right) stupid truncation at m* is the complex $\sigma^{\leq m}X^\bullet = \sigma^{<m+1}X^\bullet$: $...X^k \xrightarrow{d^k} ...X^{m-1} \xrightarrow{d^{m-1}} X^m \longrightarrow 0 \longrightarrow 0...$ (resp. $\sigma^{\geq m}X^\bullet = \sigma^{>m-1}X^\bullet$: $...0 \longrightarrow 0 \longrightarrow X^m \xrightarrow{d^m} X^{m+1}...X^n \xrightarrow{d^n} ...$). Note that we have a conflation $0 \to \sigma^{>m}X^\bullet \hookrightarrow X^\bullet \longrightarrow \sigma^{\leq m}X^\bullet \to 0$ in $\mathcal{C}(\mathcal{A})$, and, hence, an induced triangle

$$\sigma^{\geq m}X^\bullet \longrightarrow X^\bullet \longrightarrow \sigma^{<m}X^\bullet \xrightarrow{+}$$

in $\mathcal{H}(\mathcal{A})$ and in $\mathcal{D}(\mathcal{A})$.

Proposition 4.2. *The following assertions are equivalent:*

1. *${}_BT^\bullet$ is compact in $\mathcal{D}(B^{op})$;*
2. *The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ preserves products;*
3. *The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a left adjoint.*

Proof. (2) $\Longleftrightarrow$ (3) is a direct consequence of corollary 2.15.

(1) $\Longrightarrow$ (2) It is enough to prove that the composition $\mathcal{D}(B) \xrightarrow{?\otimes_B^{\mathbf{L}} T^\bullet} \mathcal{D}(A) \xrightarrow{\text{forgetful}} \mathcal{D}(k)$ preserves products since the forgetful functor preserves products and reflects isomorphisms.

Abusing of notation, we still denote by $? \otimes_B^{\mathbf{L}} T^\bullet$ the mentioned composition and note that, when doing so, we have isomorphisms $(?\otimes_B^{\mathbf{L}} N^\bullet)(X^\bullet) \cong \mathbf{T}_B(X^\bullet, N^\bullet) \cong (X^\bullet \otimes_B^{\mathbf{L}} ?)(N^\bullet)$ in $\mathcal{D}(k)$, for all $X^\bullet \in \mathcal{D}(B)$ and $N^\bullet \in \mathcal{D}(B^{op})$ (see proposition 3.7). If $(X_i^\bullet)_{i\in I}$ is any family of objects of $\mathcal{D}(B)$, we consider the full subcategory $\mathcal{T}$ of $\mathcal{D}(B^{op})$ consisting of the $N^\bullet$ such that the canonical morphism $(\prod_{i\in I} X_i^\bullet) \otimes_B^{\mathbf{L}} N^\bullet \longrightarrow \prod_{i\in I} X_i^\bullet \otimes_B^{\mathbf{L}} N^\bullet$ is an isomorphism. It is a thick subcategory of $\mathcal{D}(B^{op})$ which contains ${}_BB$. Then it contains $\text{per}(B^{op})$ and, in particular, it contains ${}_BT^\bullet$.

(2) $\Longrightarrow$ (1) Without loss of generality, we can assume that $A = k$. The canonical morphism $\coprod_{p\in\mathbb{Z}} B[p] \longrightarrow \prod_{p\in\mathbb{Z}} B[p]$ is an isomorphism in $\mathcal{D}(B^{op})$. By applying $? \otimes_B^{\mathbf{L}} T^\bullet$ and using the hypothesis, we then have an isomorphism in $\mathcal{D}(k)$

$$\coprod_{p\in\mathbb{Z}} T^\bullet[p] \cong (? \otimes_B^{\mathbf{L}} T^\bullet)(\coprod_{p\in\mathbb{Z}} B[p]) \xrightarrow{\cong} (? \otimes_B^{\mathbf{L}} T^\bullet)(\prod_{p\in\mathbb{Z}} B[p]) \cong \prod_{p\in\mathbb{Z}}(B[p] \otimes_B^{\mathbf{L}} T^\bullet) \cong \prod_{p\in\mathbb{Z}} T^\bullet[p],$$

which can be identified with the canonical morphism from the coproduct to the product. It follows that the canonical map

$$\coprod_{p\in\mathbb{Z}} H^p(T^\bullet) \cong H^0(\coprod_{p\in\mathbb{Z}} T^\bullet[p]) \longrightarrow H^0(\prod_{p\in\mathbb{Z}} T^\bullet[p]) \cong \prod_{p\in\mathbb{Z}} H^p(T^\bullet)$$

is an isomorphism. This implies that $H^p(T^\bullet) = 0$, for all but finitely many $p \in \mathbb{Z}$.

By proposition 4.1, replacing $T^\bullet$ by its homotopically projective resolution in $\mathcal{H}(B^{op})$, we can assume without loss of generality that $T^\bullet$ is an upper bounded complex of finitely generated projective left B-modules. Let us put $m := \min\{p \in \mathbb{Z} : H^p(T) \neq 0\}$. We then consider the triangle in $\mathcal{D}(B^{op})$ induced by the stupid truncation at m

$$\sigma^{\geq m} T^\bullet \longrightarrow T^\bullet \longrightarrow \sigma^{<m} T^\bullet \xrightarrow{+}.$$

Then $\sigma^{\geq m} T^\bullet$ is compact in $\mathcal{D}(B^{op})$ while $\sigma^{<m} T^\bullet$ has homology concentrated in degree $m-1$. Then we have an isomorphism $\sigma^{<m} T^\bullet \cong M[1-m]$ in $\mathcal{D}(B^{op})$, where $M = H^{m-1}(\sigma^{<m} T^\bullet)$. By the implication 1) $\Longrightarrow$ 2) we know that $? \otimes_B^{\mathbf{L}} \sigma^{\geq m} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ preserves products and, by hypothesis, also $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ does. It follows that $? \otimes_B^{\mathbf{L}} \sigma^{<m} T^\bullet \cong ? \otimes_B^{\mathbf{L}} M[1-m] : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ preserves products.

Note that M admits admits a projective resolution with finitely generated terms, namely, the canonical quasi-isomorphism $P^\bullet := \sigma^{<m} T^\bullet[m-1] \longrightarrow M = M[0]$. Therefore our task reduces to check that if M is a left B module which admits a projective resolution with finitely generated terms and such that $? \otimes_B^{\mathbf{L}} M : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ preserves products, then M has finite projective dimension. For that it is enough to prove that there is an integer $n \geq 0$ such that $\mathrm{Tor}_{n+1}^B(?, M) \equiv 0$. Indeed, if this is proved then $\Omega^n(M) := \mathrm{Im}(d^{-n} : P^{-n} \longrightarrow P^{-n+1})$ will be a flat module, and hence projective (see [26, Corollaire 1.3]), thus ending the proof.

Let us assume by way of contradiction that $\mathrm{Tor}_n^B(?, M) \not\equiv 0$, for all $n > 0$. For each such n, choose a right B-module X_n such that $\mathrm{Tor}_n^B(X_n, M) \neq 0$. Then the canonical morphism $\coprod_{n>0} X_n[-n] \longrightarrow \prod_{n>0} X_n[-n]$ is an isomorphism in $\mathcal{D}(B)$. Our hypothesis then guarantees that the canonical morphism

$$\coprod_{n>0}(X_n[-n] \otimes_B^{\mathbf{L}} M) \cong (\coprod_{n>0} X_n[-n]) \otimes_B^{\mathbf{L}} M \cong (\prod_{n>0} X_n[-n]) \otimes_B^{\mathbf{L}} M \longrightarrow \prod_{n>0}(X_n[-n] \otimes_B^{\mathbf{L}} M)$$

is an isomorphism. When applying the 0-homology functor H^0, we obtain an isomorphism

$$\coprod_{n>0} \mathrm{Tor}_n^B(X_n, M) \cong \coprod_{n>0} H^0(X_n[-n] \otimes_B^{\mathbf{L}} M) \longrightarrow \prod_{n>0} H^0(X_n[-n] \otimes_B^{\mathbf{L}} M) \cong \prod_{n>0} \mathrm{Tor}_n^B(X_n, M)$$

which is identified with the canonical morphism from the coproduct to the product in $\mathrm{Mod}-K$. It follows that $\mathrm{Tor}_n^B(X_n, M) = 0$, for almost all $n > 0$, which is a contradiction. □

Remark 4.3. *The argument in the last two paragraphs of the proof of proposition 4.2 was communicated to us by Rickard, to whom we deeply thank for it. When passing to the context of dg algebras or even dg categories, the implication* (1) $\Longrightarrow$ (2) *in that proposition still holds (see* [32]*), essentially with the same proof. However, we do not know if* (2) $\Longrightarrow$ (1) *holds for dg algebras A and B.*

4.1. **Statements and proofs.**

Proposition 4.4. *Let $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ be the counit of the adjoint pair $(? \otimes_B^{\mathbf{L}} T^\bullet, \mathbf{R}Hom_A(T^\bullet, ?))$. The following assertions are equivalent:*

1. $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ *is fully faithful;*
2. *The map $\delta_A : [(? \otimes_B^{\mathbf{L}} T) \circ \mathbf{R}Hom_A(T, ?)](A) \longrightarrow A$ is an isomorphism in $\mathcal{D}(A)$ and the functor $(? \otimes_B^{\mathbf{L}} T) \circ \mathbf{R}Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ preserves coproducts.*

In that case, the functor $? \otimes_B^{\mathbf{L}} T^\bullet$ induces a triangulated equivalence $\mathcal{D}(B)/Ker(? \otimes_B^{\mathbf{L}} T^\bullet) \xrightarrow{\cong} \mathcal{D}(A)$.

Proof. Assertion (1) is equivalent to saying that $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism (see the dual of [17, Proposition II.7.5]). As a consequence, the implication (1) $\Longrightarrow$ (2) is automatic. Conversely, if assertion (2) holds, then the full subcategory $\mathcal{T}$ of $\mathcal{D}(A)$ consisting of the $M^\bullet \in \mathcal{D}(A)$ such that $\delta_{M^\bullet}$ is an isomorphism is a triangulated subcategory closed under taking coproducts and containing A. It follows that $\mathcal{T} = \mathcal{D}(A)$, so that assertion (1) holds.

For the final statement, note that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ gives an equivalence of triangulated categories $\mathcal{D}(A) \xrightarrow{\cong} \mathrm{Im}(\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?))$. On the other hand, by proposition 2.12, we know that $(\mathrm{Ker}(? \otimes_B^{\mathbf{L}} T^\bullet), \mathrm{Im}(\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)))$ is a semi-orthogonal decomposition of $\mathcal{D}(B)$. Then, by proposition 2.13, we have a triangulated equivalence $\mathcal{D}(B)/\mathrm{Ker}(? \otimes_B^{\mathbf{L}} T^\bullet) \cong \mathrm{Im}(\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?))$. We then get a triangulated equivalence $\mathcal{D}(B)/\mathrm{Ker}(? \otimes_B^{\mathbf{L}} T^\bullet) \xrightarrow{\cong} \mathcal{D}(A)$, which is easily seen to be induced by $? \otimes_B^{\mathbf{L}} T^\bullet$. ☐

We now pass to study the recollement situations where one of the fully faithful functors is $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$.

Corollary 4.5. *Let $T^\bullet$ a complex of $B - A-$bimodules. The following assertions hold:*

1. *There is a triangulated category $\mathcal{D}'$ and a recollement $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, with $i_* = \mathbf{R}Hom_A(T^\bullet, ?)$;*
2. *There is a triangulated category $\mathcal{D}'$ and a recollement $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, with $i^* = ? \otimes_B^{\mathbf{L}} T^\bullet$;*
3. *$T_A^\bullet$ is compact in $\mathcal{D}(A)$ and $\delta_A : [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?)](A) \longrightarrow A$ is an isomorphism in $\mathcal{D}(A)$, where δ is the counit of the adjoint pair $(? \otimes_B^{\mathbf{L}} T^\bullet, \mathbf{R}Hom_A(T^\bullet, ?))$.*

Proof. (1) $\Longleftrightarrow$ (2) is clear.

(1) $\Longrightarrow$ (3) If the recollement exists, then $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, so that δ_A is an isomorphism. Moreover, $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ preserves coproducts since it it is a left adjoint functor. This preservation of coproducts is equivalent to having $T_A^\bullet \in \mathrm{per}(A)$.

(3) $\Longrightarrow$ (1) We clearly have that the functor $(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ preserves coproducts. Then assertion (2) of proposition 4.4 holds, so that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ is fully faithful. On the other hand, by proposition 2.14, we get that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ has a right adjoint, so that assertion (1) holds. ☐

Theorem 4.6. *Let ${}_BT_A^\bullet$ be a complex of $B - A-$bimodules. Consider the following assertions:*

(1) *There is a recollement* $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, *with* $j_* = \mathbf{R}Hom_A(T^\bullet, ?)$, *for some triangulated category (which is equivalent to* $\mathcal{D}(C)$, *where* C *is a dg algebra);*
(2) *There is a recollement* $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, *with* $j^* = j^! = ? \otimes_B^{\mathbf{L}} T^\bullet$, *for some triangulated category (which is equivalent to* $\mathcal{D}(C)$, *where* C *is a dg algebra);*
(3) *The following three conditions hold:*
  (a) *The counit map* $\delta_A : [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?)](A) \longrightarrow A$ *is an isomorphism;*
  (b) *The functor* $(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ *preserves coproducts;*
  (c) *The functor* $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ *preserves products.*
(4) ${}_BT^\bullet$ *is compact and exceptional in* $\mathcal{D}(B^{op})$ *and the canonical algebra morphism* $A \longrightarrow End_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ *is an isomorphism.*

The implications $(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftarrow (4)$ *hold true and, when* A *is* k*-projective, all assertions are equivalent. Moreover, if* B *is* k*-flat, then the dg algebra* C *can be chosen together with a homological epimorphism* $f : B \longrightarrow C$ *such that* i_* *is the restriction of scalars* $f_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$.

Proof. $(1) \Longleftrightarrow (2)$ is clear.

$(1) \Longleftrightarrow (3)$ By proposition 2.19, we know that the recollement in (3) exists if, and only if, $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a left adjoint and $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful. Apply now corollary 2.15 and proposition 4.4.

$(4) \Longrightarrow (3)$ Condition (3)(c) follows from proposition 4.2. By proposition 4.4, proving conditions (3)(a) and (3)(b) is equivalent to proving that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ is fully faithful. This is in turn equivalent to proving that the counit $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism.

In order to prove this, we apply proposition 3.10 to $T^\bullet$, when viewed as a complex of left B-modules (equivalently, of $B-k-$bimodules). Then $T^{\bullet *}$ is obtained from $T^\bullet$ by applying $\mathbf{R}\mathrm{Hom}_{B^{op}}(?, B) : \mathcal{D}(B^{op})^{op} \longrightarrow \mathcal{D}(B)$ and, similarly, we obtain $T^{\bullet **}$ from $T^{\bullet *}$. By the mentioned proposition, we know that $T^\bullet \cong T^{\bullet **}$ in $\mathcal{D}(B^{op})$. Moreover, applying assertion (1) of that proposition, with A and C replaced by k and A, respectively, and putting $Y^\bullet = T^\bullet \in \mathcal{C}(B^{op} \otimes A)$, we obtain isomorphisms in $\mathcal{D}(A)$:

$$T^{\bullet *} \otimes_B^{\mathbf{L}} T^\bullet := [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)](T^\bullet) \cong \mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^\bullet)(T^\bullet) \cong A_A,$$

where the last isomorphism follows from the exceptionality of ${}_BT^\bullet$ in $\mathcal{D}(B^{op})$ and the fact that the canonical algebra morphism $A \longrightarrow \mathrm{End}_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ is an isomorphism.

Using the previous paragraph, proposition 3.10 and adjunction, for each object $M^\bullet \in \mathcal{D}(A)$ we get a chain of isomorphisms in $\mathcal{D}(k)$:

$$[(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)](M^\bullet) = (? \otimes_B^{\mathbf{L}} T^\bullet)(\mathbf{R}\mathrm{Hom}_A(T^\bullet, M^\bullet)) \xrightarrow{\cong}$$
$$\mathbf{R}\mathrm{Hom}_B(T^{\bullet *}, \mathbf{R}\mathrm{Hom}_A(T^\bullet, M^\bullet)) \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_A(T^{\bullet *} \otimes_B^{\mathbf{L}} T^\bullet, M^\bullet) \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_A(A, M^\bullet) \xrightarrow{\cong} M^\bullet.$$

It is routine now to see that the composition of these isomorphisms is obtained from the counit map $\delta_{M^\bullet} : [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)](M^\bullet) \longrightarrow M^\bullet$ by applying the forgetful functor $\mathcal{D}(A) \longrightarrow \mathcal{D}(k)$. Since this last functor reflects isomorphisms we get that δ is a natural isomorphism, so that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is a fully faithful functor.

$(1) - (3) \Longrightarrow (4)$ (Assuming that A is k-projective). By proposition 4.2, we know that ${}_BT^\bullet \in \mathrm{per}(B^{op})$. Then, by proposition 3.10, we have a natural isomorphism $? \otimes_B^{\mathbf{L}} T^\bullet \cong$

$\mathbf{R}\mathrm{Hom}_B(T^{\bullet *}, ?)$ of triangulated functors $D(B) \longrightarrow \mathcal{D}(A)$, where now we are considering $T^{\bullet *}$ as obtained from $T^\bullet$ by applying $(?)^* := \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B) : \mathcal{D}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(A^{op} \otimes B)$. By lemma 2.16, the fully faithful condition of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ implies the same condition for $? \otimes_A^{\mathbf{L}} T^{\bullet *}$. Then corollary 4.14 below says that $T_B^{\bullet *}$ is compact and exceptional in $\mathcal{D}(B)$ and the canonical algebra morphism $A \longrightarrow \mathrm{End}_{\mathcal{D}(B)}(T^{\bullet *})$ is an isomorphism.

Note now that, due to the k-projectivity of A, when applying to ${}_BT^\bullet$ the functor $\mathbf{R}\mathrm{Hom}_{B^{op}}(?, B) : \mathcal{D}(B^{op})^{op} \longrightarrow \mathcal{D}(B)$, we obtain an object isomorphic to $T_B^{\bullet *}$ in $\mathcal{D}(B)$, and conversely. Applying now corollary 3.11 with $A = k$, we have that ${}_BT^\bullet \cong {}_BT^{\bullet **}$ is exceptional in $\mathcal{D}(B^{op})$ and that the algebra map $A \longrightarrow \mathrm{End}_{\mathcal{D}(B^{op})}(T^\bullet)^{op} \cong \mathrm{End}_{\mathcal{D}(B)}(T^{\bullet *})$ is an isomorphism.

For the final statement note that, when the recollement of assertions (1) or (2) exists, $\mathcal{D}'$ has a compact generator, namely $i^!(A)$. Then, by [19, Theorem 4.3], we know that $\mathcal{D}' \cong \mathcal{D}(C)$, for some dg algebra C. When B is k-flat, the fact that this dg algebra can be chosen together with a homological epimorphism $f : B \longrightarrow C$ satisfying the requirements is a direct consequence of [31, Theorem 4]. $\square$

Remark 4.7. *Due to the results in* [32], *except for the implication* $(1) - (3) \Longrightarrow (4)$, *theorem 4.6 is also true in the context of dg categories, with the proof adapted. In that case its implication* $(4) \Longrightarrow (3)$ *partially generalizes* [6, Theorem 4.3] *in the sense that we explicitly prove that the recollement exists with* $j^* = j^! =? \otimes_B^{\mathbf{L}} T^\bullet$. *Note, however, that if* $T^{\bullet *}$ *is the dg* $A - B-$*module obtained from* $T^\bullet$ *by application of the functor* $\mathbf{R}Hom_{B^{op}}(?, B) : \mathcal{D}(B^{op} \otimes A) \longrightarrow \mathcal{D}(A^{op} \otimes B)$, *we cannot guarantee that the left adjoint of* $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ *is (naturally isomorphic to)* $? \otimes_A^{\mathbf{L}} T^{\bullet *}$. *Due to the version of proposition 3.10 for dg algebras, we can guarantee that when* A *is assumed to be* k*-projective. Note that, for the entire theorem 4.6 to be true in the context of dg algebras(or even dg categories), one only needs to prove that the implication* $(2) \Longrightarrow (1)$ *of proposition 4.2 holds in this more general context. The rest of the proof of theorem 4.6 can be extended without problems.*

Example 4.8. *Let* A *be a hereditary Artin algebra and let* S *be a non-projective simple module. Then* $T = A \oplus S$ *is a right* A*-module, so that* T *becomes a* $B - A-$*bimodule, where* $B = End(T_A) \cong \begin{pmatrix} A & 0 \\ S & D \end{pmatrix}$, *where* $D = End(S_A)$. *There are a recollement* $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, *with* $i_* = \mathbf{R}Hom_A(T, ?)$, *and a recollement* $\mathcal{D}'' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, *with* $j_* = \mathbf{R}Hom_A(T, ?)$, *for some triangulated categories* $\mathcal{D}'$ *and* $\mathcal{D}''$. *However* T_A *is not exceptional in* $\mathcal{D}(A)$.

Proof. It is well-known that $\mathrm{Ext}_A^1(S, S) = 0$, which implies that $\mathrm{Ext}_A^1(T, T) \cong \mathrm{Ext}_A^1(S, A) \neq 0$ and, hence, that T_A is not exceptional in $\mathcal{D}(A)$.

We denote by e_i $(i = 1, 2)$ the canonical idempotents of B. We readily see that ${}_BT \cong Be_1$, that $\mathrm{Hom}_A(T, A) \cong e_1B$ and that $\mathrm{Ext}_A^1(T, A)$ is isomorphic to $\begin{pmatrix} 0 & \mathrm{Ext}_A^1(S, A) \end{pmatrix}$, when viewed as a right B-module in the usual way (see, e.g., [4, Proposition III.2.2]). In particular, we have $\mathrm{Ext}_A^1(T, A)e_1 = 0$. We then get a triangle in $\mathcal{D}(B)$:

$$e_1B[0] \longrightarrow \mathbf{R}\mathrm{Hom}_A(T, A) \longrightarrow \mathrm{Ext}_A^1(T, A)[-1] \xrightarrow{+}.$$

When applying $? \otimes_B^{\mathbf{L}} T =? \otimes_B^{\mathbf{L}} Be_1 =? \otimes_B Be_1$, we get a triangle in $\mathcal{D}(A)$:

$$A = e_1 B e_1[0] \longrightarrow \mathbf{R}\mathrm{Hom}_A(T, A) \otimes_B^{\mathbf{L}} T \longrightarrow \mathrm{Ext}_A^1(T, A)e_1[-1] = 0 \xrightarrow{+}.$$

We then get an isomorphism $A \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_A(T, A) \otimes_B^{\mathbf{L}} T$ in $\mathcal{D}(A)$, which is easily seen to be inverse to δ_A. Then assertion (3) of corollary 4.5 holds.

On the other hand, also condition (4) of theorem 4.6 holds. $\square$

Although the exceptionality property is not needed, it helps to extract more information about $T^\bullet$, when $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ is fully faithful. The following is an example:

Proposition 4.9. *Let $T^\bullet$ be a complex of $B-A-$ bimodules such that $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful. The following assertions hold:*

1. *If $\mathcal{D}(A)(T^\bullet, T^\bullet[n]) = 0$, for all but finitely many $n \in \mathbb{Z}$, then $H^p(T^\bullet) = 0$, for all but finitely many $p \in \mathbb{Z}$;*
2. *If $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$ and the algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism, then ${}_BT^\bullet$ is isomorphic in $\mathcal{D}(B^{op})$ to an upper bounded complex of finitely generated projective left B-modules (with bounded homology).*

Proof. (1) Let us put $X^\bullet = \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)$. The hypothesis says that $H^n(X^\bullet) = 0$, for all but finitely many $n \in \mathbb{Z}$. In particular the canonical morphism $f : \coprod_{n\in\mathbb{Z}} X^\bullet[n] \longrightarrow \prod_{n\in\mathbb{Z}} X^\bullet[n]$ is an isomorphism in $\mathcal{D}(B)$ since $H^p(f)$ is an isomorphism, for all $p \in \mathbb{Z}$. On the other hand, due to the adjunction equations and the fact the the counit $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism, the unit $\lambda : 1_{\mathcal{D}(B)} \longrightarrow \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)$ satisfies that $\lambda_{M^\bullet} \otimes_B^{\mathbf{L}} 1_{T^\bullet} := (? \otimes_B^{\mathbf{L}} T^\bullet)(\lambda_{M^\bullet})$ is an isomorphism, for each $M^\bullet \in \mathcal{D}(B)$. Applying this to the map $\lambda_B : B \longrightarrow [\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)](B) = X^\bullet$, we conclude that $\lambda_B \otimes_B^{\mathbf{L}} 1_{T^\bullet} : T^\bullet \cong B \otimes_B^{\mathbf{L}} T^\bullet \longrightarrow X^\bullet \otimes_B^{\mathbf{L}} T^\bullet$ is an isomorphism in $\mathcal{D}(A)$.

We now have the following chain of isomorphisms in $\mathcal{D}(A)$:

$$\coprod_{n\in\mathbb{Z}} T^\bullet[n] \xrightarrow{\cong} \coprod_{n\in\mathbb{Z}} (X^\bullet \otimes_B^{\mathbf{L}} T^\bullet[n]) \cong (\coprod_{n\in\mathbb{Z}} X^\bullet[n]) \otimes_B^{\mathbf{L}} T^\bullet \xrightarrow{f\otimes 1} (\prod_{n\in\mathbb{Z}} X^\bullet[n]) \otimes_B^{\mathbf{L}} T^\bullet \cong$$
$$(\prod_{n\in\mathbb{Z}} \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet[n])) \otimes_B^{\mathbf{L}} T^\bullet \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_A(T^\bullet, \prod_{n\in\mathbb{Z}} T^\bullet[n]) \otimes_B^{\mathbf{L}} T^\bullet \xrightarrow{\delta} \prod_{n\in\mathbb{Z}} T^\bullet[n].$$

It is routine to check that the composition of these isomormorphisms is the canonical morphism from the coproduct to the product. Arguing now as in the proof of proposition 4.2, we conclude that $H^p(T^\bullet) = 0$, for all but finitely many $p \in \mathbb{Z}$.

(2) The counit gives an isomorphism

$$B^\alpha \otimes_B^{\mathbf{L}} T^\bullet \cong \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)^\alpha \otimes_B^{\mathbf{L}} T^\bullet \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^{\bullet\alpha}) \otimes_B^{\mathbf{L}} T^\bullet \xrightarrow{\delta} T^{\bullet\alpha},$$

for each cardinal α. Now apply proposition 4.1 to end the proof. $\square$

Unlike the case of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$, one-sided exceptionality is a consequence of the fully faithfulness of $? \otimes_B^{\mathbf{L}} T^\bullet$.

Proposition 4.10. *Let $T^\bullet$ be a complex of $B-A-$bimodules. Consider the following assertions:*

1. *$? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful;*

(2) $T_A^\bullet$ *is exceptional in* $\mathcal{D}(A)$*, the canonical algebra morphism* $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ *is an isomorphism, and the functor* $\mathbf{R}Hom_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet) : \mathcal{D}(B) \longrightarrow \mathcal{D}(B)$ *preserves coproducts.*

(3) $T_A^\bullet$ *is exceptional and self-compact in* $\mathcal{D}(A)$*, and the canonical algebra morphism* $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ *is an isomorphism*

(4) $T_A^\bullet$ *satisfies the following conditions:*
   (a) *The canonical morphism of algebras* $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ *is an isomorphism;*
   (b) *For each cardinal* α*, the canonical morphism* $\mathcal{D}(A)(T^\bullet, T^\bullet)^{(\alpha)} \longrightarrow \mathcal{D}(A)(T^\bullet, T^{\bullet(\alpha)})$ *is an isomorphism and* $\mathcal{D}(A)(T^\bullet, T^{\bullet(\alpha)}[p]) = 0$*, for all* $p \in \mathbf{Z} \setminus \{0\}$*;*
   (c) $Susp_{\mathcal{D}(A)}(T^\bullet)^\perp \cap Tria_{\mathcal{D}(A)}(T^\bullet)$ *is closed under taking coproducts.*

The implications (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) *hold true. When* $T_A^\bullet$ *is quasi-isomorphic to a bounded complex of projective* A*-modules, all assertions are equivalent.*

Proof. Let $\lambda : 1_{\mathcal{D}(B)} \longrightarrow \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)$ be the unit of the adjoint pair $(? \otimes_B^{\mathbf{L}} T^\bullet, \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?))$. Assertion (1) is equivalent to saying that λ is a natural isomorphism.

(1) $\Longrightarrow$ (2) In particular, $\lambda_B : B \longrightarrow [\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)](B) =: \mathbf{R}\mathrm{Hom}_A(T^\bullet, B \otimes_B^{\mathbf{L}} T^\bullet) \cong \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)$ is an isomorphism. This implies that $T^\bullet$ is exceptional and the 0-homology map $B = H^0(B) \longrightarrow H^0(\mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)) = \mathcal{D}(A)(T^\bullet, T^\bullet)$ is an isomorphism. But this latter map is easily identified with the canonical algebra morphism $B \longrightarrow \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$. That $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet) : \mathcal{D}(B) \longrightarrow \mathcal{D}(B)$ preserves coproducts is automatic since this functor is naturally isomorphic to the identity.

(2) $\Longrightarrow$ (1) The full subcategory $\mathcal{D}$ of $\mathcal{D}(B)$ consisting of the objects $X^\bullet$ such that $\lambda_{X^\bullet}$ is an isomorphism is a triangulated subcategory, closed under coproducts, which contains B_B. Then we have $\mathcal{D} = \mathcal{D}(B)$.

(1), (2) $\Longrightarrow$ (3) The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ induces a triangulated equivalence $\mathcal{D}(B) \xrightarrow{\cong} \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. The exceptionality (in $\mathcal{D}(A)$ or in $\mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$) and the compactness of $T^\bullet$ in $\mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$ follow from the exceptionality and compactness of B in $\mathcal{D}(B)$. Moreover, we get an algebra isomorphism:

$$B \cong \mathrm{End}_{\mathcal{D}(B)}(B) \xrightarrow{\cong} \mathrm{End}_{\mathcal{D}(A)}(B \otimes_B^{\mathbf{L}} T^\bullet) \cong \mathrm{End}_{\mathcal{D}(A)}(T^\bullet).$$

(3) $\Longrightarrow$ (2) The functor $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet) : \mathcal{D}(B) \longrightarrow \mathcal{D}(B)$ coincides with the following composition:

$$\mathcal{D}(B) \stackrel{? \otimes_B^{\mathbf{L}} T^\bullet}{\longrightarrow} \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet) \stackrel{\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)}{\longrightarrow} \mathcal{D}(B).$$

The self-compactness of $T^\bullet$ implies that the second functor in this composition preserves coproducts. It then follows that the composition itself preserves coproducts since so does $? \otimes_B^{\mathbf{L}} T^\bullet$.

(3) $\Longrightarrow$ (4) Condition (4)(a) is in the hypthesis, and from the self-compactness and the exceptionality of $T^\bullet$ conditions (4)(b) and (4)(c) follow immediately.

(4) $\Longrightarrow$ (3) Without loss of generality, we assume that that $T^\bullet$ is a bounded complex of projective A-modules in $\mathcal{C}^{\leq 0}(A)$. We just need to prove the self-compactness of $T^\bullet$ in $\mathcal{D}(A)$. We put $\mathcal{U} := \mathrm{Susp}_{\mathcal{D}(A)}(T^\bullet)$ and $\mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. Note that $(\mathcal{U}, \mathcal{U}^\perp[1])$ is a t-structure in $\mathcal{D}(A)$ and that $\mathcal{U}^\perp$ consists of the $Y^\bullet \in \mathcal{D}(A)$ such that $\mathcal{D}(A)(T^\bullet[k], Y^\bullet) = 0$, for all $k \geq 0$ (see proposition 2.26). Let $(X_i^\bullet)_{i \in I}$ be any family of objects in $\mathcal{T}$. We want

to check that the canonical morphism $\coprod_{i\in I}\mathcal{D}(A)(T^\bullet, X_i^\bullet) \longrightarrow \mathcal{D}(A)(T^\bullet, \coprod_{i\in I} X_i^\bullet)$ is an isomorphism, which is equivalent to proving that it is an epimorphism. We consider the triangle associated to the t-structure $(\mathcal{U}, \mathcal{U}^\perp[1])$

$$\coprod_{i\in I}\tau_{\mathcal{U}}(X_i) \longrightarrow \coprod_{i\in I} X_i \longrightarrow \coprod_{i\in I}\tau^{\mathcal{U}^\perp}(X_i) \xrightarrow{+}.$$

This triangle is in $\mathcal{T}$ because its central and left terms are in $\mathcal{T}$. Note also that we have $\mathcal{U} \subseteq \mathcal{D}^{\leq 0}(A)$. In particular, $\tau_{\mathcal{U}}(X_i)$ is in $\mathcal{D}^{\leq 0}(A)$, for each $i \in I$.

From [30, Theorem 3 and its proof], we know that the inclusion $\mathcal{T} \cap \mathcal{D}^-(A) \hookrightarrow \mathcal{D}^-(A)$ has a right adjoint and that $T^\bullet$ is a compact object of $\mathcal{T} \cap \mathcal{D}^-(A)$. That is, the functor $\mathcal{D}(A)(T^\bullet, ?)$ preserves coproducts of objects in $\mathcal{T} \cap \mathcal{D}^-(A)$ whenever the coproduct is in $\mathcal{D}^-(A)$. In our case, this implies that the canonical morphism $\coprod_{i\in I}\mathcal{D}(A)(T^\bullet, \tau_{\mathcal{U}}(X_i^\bullet)) \longrightarrow \mathcal{D}(A)(T^\bullet, \coprod_{i\in I}\tau_{\mathcal{U}}(X_i))$ is an isomorphism. On the other hand, condition 4.c says that $\mathcal{D}(A)(T^\bullet, \coprod_{i\in I}\tau^{\mathcal{U}^\perp}(X_i^\bullet)) = 0$. It follows that the induced map $\mathcal{D}(A)(T, \coprod_{i\in I}\tau_{\mathcal{U}}(X_i)) \longrightarrow \mathcal{D}(A)(T, \coprod_{i\in I} X_i)$ is an epimorphism. We then get the following commutative square:

$$\begin{tikzcd}
\coprod_{i\in I}\mathcal{D}(A)(T^\bullet, \tau_{\mathcal{U}}X_i^\bullet) \arrow[r, "\sim"] \arrow[d, "\wr"] & \coprod_{i\in I}\mathcal{D}(A)(T^\bullet, X_i^\bullet) \arrow[d] \\
\mathcal{D}(A)(T^\bullet, \coprod_{i\in I}\tau_{\mathcal{U}}X_i^\bullet) \arrow[r, two heads] & \mathcal{D}(A)(T^\bullet, \coprod_{i\in I} X_i^\bullet)
\end{tikzcd}$$

Its upper horizontal and its left vertical arrows are isomorphism, while the lower horizontal one is an epimorphism. It follows that the right vertical arrow is an epimorphism. $\square$

Recall that if A and B are dg algebras and $f : B \longrightarrow A$ is morphism of dg algebras, then f is called a *homological epimorphism* when the morphism $(? \otimes_B^{\mathbf{L}} A)(A) \longrightarrow A$ in $\mathcal{D}(A)$, defined by the multiplication map $A \otimes_B^\bullet A \longrightarrow A$, is an isomorphism. This is also equivalent to saying that the left-right symmetric version $(A \otimes_B^{\mathbf{L}} ?)(A) \longrightarrow A$ is an isomorphism $\mathcal{D}(A^{op})$ (see [33]). When A and B are ordinary algebras, f is a homological epimorphism if, and only if, $\mathrm{Tor}_i^B(A, A) = 0$, for $i \neq 0$, and the multiplication map $A \otimes_B A \longrightarrow A$ is an isomorphism.

Remark 4.11. *Under the hypothesis that $T_A^\bullet$ is quasi-isomorphic to a bounded complex of projective modules, we do not know if condition (4)(c) is needed for the implication* (4) $\Longrightarrow$ (3) *to hold (see the questions in the next subsection). On the other hand, if in that same assertion one replaces $Susp_{\mathcal{D}(A)}(T^\bullet)^\perp \cap Tria_{\mathcal{D}(A)}(T^\bullet)$ by just $Susp_{\mathcal{D}(A)}(T^\bullet)^\perp$, then the implication* (3) $\Longrightarrow$ (4) *need not be true, as the following example shows.*

Example 4.12. *The functor $? \otimes_{\mathbb{Q}}^{\mathbf{L}} \mathbb{Q} : \mathcal{D}(\mathbb{Q}) \longrightarrow \mathcal{D}(\mathbb{Z})$ is fully faithful, but $Susp_{\mathcal{D}(\mathbb{Z})}(Q)^\perp$ is not closed under taking coproducts in $\mathcal{D}(\mathbb{Z})$.*

Proof. The fully faithful condition of $? \otimes_{\mathbb{Q}}^{\mathbf{L}} \mathbb{Q}$ follows from theorem 4.13 below since the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a homological epimorphism. On the other hand, if I is the set of prime natural numbers and we consider the family of stalk complexes $(\mathbb{Z}_p[1])_{p\in I}$, we clearly have that $\mathcal{D}(\mathbb{Z})(Q[k], \mathbb{Z}_p[1]) = 0$, for all $p \in I$ and integers $k \geq 0$, so that $\mathbb{Z}_p[1] \in \mathrm{Susp}_{\mathcal{D}(\mathbb{Z})}(\mathbb{Q})^\perp$.

We claim that $\mathcal{D}(\mathbb{Z})(\mathbb{Q}, \coprod_{p\in I}\mathbb{Z}_p[1]) = \mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \coprod_{p\in I}\mathbb{Z}_p) \neq 0$. Indeed, consider the map $\epsilon : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ given by multiplication by the 'infinite product' of all prime natural numbers. Concretely, if $\frac{a}{p_1^{m_1}\cdot\ldots\cdot p_t^{m_t}}$ is a fraction of integers, where $m_j > 0$ and $p_j \in I$, for all $j = 1, ..., t$, then $\epsilon(\frac{a}{b} + \mathbb{Z}) = \frac{a}{p_1^{m_1-1}\cdot\ldots\cdot p_t^{m_t-1}} + \mathbb{Z}$. It is clear that $\mathrm{Ker}(\epsilon)$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the elements $\frac{1}{p} + \mathbb{Z}$, with $p \in I$, which is clearly isomorphic to $\coprod_{p\in I}\mathbb{Z}_p$. In particular $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \coprod_{p\in I} Z_p)$ is isomorphic to the cokernel of the induced map $\epsilon_* : \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Q}/\mathbb{Z})$, which takes $\alpha \rightsquigarrow \epsilon \circ \alpha$. The reader is invited to check that the canonical projection $\pi : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is not in the image of ϵ_*, thus proving our claim. $\square$

We know pass to study the recollement situations in which one of the fully faithful functors is $? \otimes^{\mathbf{L}}_B T^\bullet$.

Theorem 4.13. *Let $T^\bullet$ be a complex of $B-A-$bimodules. Consider the following assertions:*

(1) *There is a recollement $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$, for some triangulated category $\mathcal{D}'$, where $i_* =? \otimes^{\mathbf{L}}_B T^\bullet$;*
(2) *There is a recollement $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$, for some triangulated category $\mathcal{D}'$, where $i^! = \mathbf{R}Hom_A(T^\bullet, ?)$;*
(3) *$T^\bullet_A$ is exceptional, self-compact, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and the functor $? \otimes^{\mathbf{L}}_B T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ preserves products;*
(4) *$T^\bullet_A$ is exceptional, self-compact, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and $Tria_{\mathcal{D}(A)}(T^\bullet)$ is closed under taking products in $\mathcal{D}(A)$.*
(5) *$T^\bullet_A$ is exceptional, self-compact, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and ${}_BT^\bullet \in per(B^{op})$.*
(6) *There is a dg algebra $\hat{A}$, a homological epimorphism of dg algebras $f : A \longrightarrow \hat{A}$ and a classical tilting object $\hat{T}^\bullet \in \mathcal{D}(\hat{A})$ such that the following conditions hold:*
    (a) *$T^\bullet_A \cong f_*(\hat{T}^\bullet)$, where $f_* : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(A)$ is the restriction of scalars functor;*
    (b) *the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T) \cong End_{\mathcal{D}(\hat{A})}(\hat{T})$ is an isomorphism.*

Then the implications (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) $\Longleftarrow$ (6) *hold true and, when A is k-flat, all assertions are equivalent.*

Proof. (1) $\Longleftrightarrow$ (2) is clear.

Note that the recollement of assertions (1) or (2) exists if, and only if, the functor $? \otimes^{\mathbf{L}}_B T^\bullet$ is fully faithful and has a left adjoint.

(1) $\Longleftrightarrow$ (3) is then a direct consequence of theorem 4.10 and corollary 2.15.

(1), (3) $\Longrightarrow$ (4) The functor $? \otimes^{\mathbf{L}}_B T^\bullet$ induces an equivalence of triangulated categories $\mathcal{D}(B) \xrightarrow{\cong} \mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. The fact that $? \otimes^{\mathbf{L}}_B T^\bullet$ has a left adjoint functor implies that also the inclusion functor $j : \mathcal{T} \hookrightarrow \mathcal{D}(A)$ has a left adjoint. But then j preserves products, so that $\mathcal{T}$ is closed under taking products in $\mathcal{D}(A)$.

(4) $\Longrightarrow$ (3) By proposition 4.10 and its proof, we know that $? \otimes^{\mathbf{L}}_B T^\bullet$ is a fully faithful functor which establishes an equivalence of triangulated categories $\mathcal{D}(B) \xrightarrow{\cong} \mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. In particular, this equivalence of categories preserves products. This, together with the fact that $\mathcal{T}$ is closed under taking products in $\mathcal{D}(A)$, implies that $? \otimes^{\mathbf{L}}_B T^\bullet$ preserves products.

(3) $\Longleftrightarrow$ (5) is a direct consequence of proposition 4.2.

(6) $\Longrightarrow$ (4) By the properties of homological epimorphisms (see [31, Section 4]), the functor $f_* : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(A)$ is fully faithful (and triangulated). It then follows that

$$\mathrm{Hom}_{\mathcal{D}(A)}(T^\bullet, T^\bullet[p]) = \mathrm{Hom}_{\mathcal{D}(A)}(f_*(\hat{T}^\bullet), f_*(\hat{T}^\bullet)[p]) \cong \mathrm{Hom}_{\mathcal{D}(\hat{A})}(\hat{T}^\bullet, \hat{T}^\bullet[p]) = 0,$$

for all integers $p \neq 0$. By condition (6)(b), it also follows that the canonical algebra morphism $B \longrightarrow \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism.

On the other hand, f_* defines an equivalence $\mathcal{D}(\hat{A}) \xrightarrow{\cong} \mathrm{Im}(f_*)$. Moreover $\hat{T}^\bullet$ is a compact generator of $\mathcal{D}(\hat{A})$, which implies that $T^\bullet_A$ is a compact generator of $\mathrm{Im}(f_*)$. Since the last one is a full triangulated subcategory of $\mathcal{D}(A)$ closed under taking coproducts we get that $\mathrm{Im}(f_*) = \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. But $\mathrm{Im}(f_*)$ is also closed under taking products in $\mathcal{D}(A)$ (see [31, Section 4]). Then $T^\bullet_A$ is self-compact and $\mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$ is closed under taking products in $\mathcal{D}(A)$.

(1), (4) $\Longrightarrow$ (6) (assuming that A is k-flat) We have seen above that the inclusion functor $\mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet) \hookrightarrow \mathcal{D}(A)$ has a left adjoint. By proposition 2.11, its dual and by proposition 2.26, we get that $({}^\perp\mathcal{T}, \mathcal{T}, \mathcal{T}^\perp)$ is a TTF triple in $\mathcal{D}(A)$. By [31, Theorem 4], there exists a dg algebra $\hat{A}$ and a homological epimorphism of dg algebras $f : A \longrightarrow \hat{A}$ such that $\mathcal{T} = \mathrm{Im}(f_*)$. Bearing in mind that f_* is fully faithful, it induces an equivalence of categories $f_* : \mathcal{D}(\hat{A}) \xrightarrow{\cong} \mathcal{T}$. If now $\hat{T}^\bullet \in \mathcal{D}(\hat{A})$ is an object such that $f_*(\hat{T}^\bullet) \cong T^\bullet_A$, then $\hat{T}^\bullet$ is a compact generator of $\mathcal{D}(\hat{A})$ such that $\mathcal{D}(\hat{A})(\hat{T}^\bullet, \hat{T}^\bullet[p]) \cong \mathcal{D}(A)(T^\bullet, T^\bullet[p])$, for each $p \in \mathbb{Z}$. Now all requirements for $\hat{T}^\bullet$ are easily verified.

□

The following result deeply extends [11, Lemma 4.2].

Corollary 4.14. *The following assertions are equivalent for a complex $T^\bullet$ of $B-A-$bimodules:*

1. *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$ such that $j_! = ? \otimes^{\mathbf{L}}_B T^\bullet$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, where C is a dg algebra);*
2. *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$ such that $j^! = j^* = \mathbf{R}Hom_A(T^\bullet, ?)$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, where C is a dg algebra)*
3. *$T^\bullet_A$ is compact and exceptional in $\mathcal{D}(A)$ and the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism.*

In such case, if A is k-flat, then C can be chosen together with a homological epimorphism $f : A \longrightarrow C$ such that i_* is the restriction of scalars $f_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(A)$.

Proof. (1) $\Longleftrightarrow$ (2) is clear.

(1) $\Longleftrightarrow$ (3) The mentioned recollement exists if, and only if, $? \otimes^{\mathbf{L}}_B T$ is fully faithful and $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ has a right adjoint. Using proposition 4.10 and corollary 2.15, the existence of such recollement is equivalent to assertion (3). This is because that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ preserves coproducts if, and only if, $T^\bullet_A$ is compact in $\mathcal{D}(A)$.

The statements about the dg algebra C follow as at the end of the proof of theorem 4.6.

□

We then get the following consequence (compare with [14, Theorem 2]).

Corollary 4.15. *Let $T^\bullet$ be a complex of $B-A-$bimodules, where A and B are ordinary algebras. If there is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$, with $j_! = ? \otimes_B^{\mathbf{L}} T^\bullet$, for some triangulated category $\mathcal{D}'$, then there is a recollement $\mathcal{D}'' \equiv \mathcal{D}(A^{op}) \equiv \mathcal{D}(B^{op})$, with $j^! = j^* = T^\bullet \otimes_A^{\mathbf{L}} ?$, for a triangulated category $\mathcal{D}''$. When A is k-projective, the converse is also true.*

Proof. It is a direct consequence of corollary 4.14 and the left-right symmetric version of theorem 4.6. □

The following is a rather general result on the existence of recollements.

Proposition 4.16. *Let Λ be any dg algebra and $U^\bullet$, $V^\bullet$ be exceptional objects of $\mathcal{D}(\Lambda)$. Put $A = End_{\mathcal{D}(\Lambda)}(U^\bullet)$ and $B = End_{\mathcal{D}(\Lambda)}(V^\bullet)$. If $V^\bullet \in thick_{\mathcal{D}(\Lambda)}(U^\bullet)$, then there is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, for some dg algebra C).*

Proof. We choose $U^\bullet$ and $V^\bullet$ to be homotopically projective. We then put $\hat{A} = \text{End}_\Lambda^\bullet(U^\bullet)$ and $\hat{B} = \text{End}_\Lambda^\bullet(V^\bullet)$, which are dg algebras whose associated homology algebras are concentrated in degree 0 and satisfy that $H^0(\hat{A}) \cong A$ and $H^0(\hat{B}) \cong B$. Then we know that $\mathcal{D}(\hat{A}) \cong \mathcal{D}(A)$ and $\mathcal{D}(\hat{B}) \cong \mathcal{D}(B)$.

We next put $T^\bullet = \text{Hom}_\Lambda^\bullet(U^\bullet, V^\bullet)$ which is a dg $\hat{B} - \hat{A}-$bimodule. It is easy to see that $\text{Hom}_\Lambda^\bullet(U^\bullet, ?) = \mathbf{R}\text{Hom}_\Lambda(U^\bullet, ?) : \mathcal{D}(\Lambda) \longrightarrow \mathcal{D}(\hat{A})$ induces an equivalence of of triangulated categories $\text{thick}_{\mathcal{D}(\Lambda)}(U^\bullet) \xrightarrow{\cong} \text{thick}_{\mathcal{D}(\hat{A})}(\hat{A}) = \text{per}(\hat{A})$. In particular, we get that $T^\bullet_{\hat{A}}$ is compact in $\mathcal{D}(\hat{A})$. We claim now that there is a recollement $\mathcal{D}' \equiv \mathcal{D}(\hat{A}) \equiv \mathcal{D}(\hat{B})$, where $j_! = ? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet$, an this will end the proof.

The desired recollement exists if, and only if, $? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet$ is fully faithful and $\mathbf{R}\text{Hom}_{\hat{A}}(T^\bullet, ?) : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(\hat{B})$ has a right adjoint. This second condition holds due to corollary 2.15. As for the first condition, we just need to check that the unit $\lambda : 1_{\mathcal{D}(\hat{B})} \longrightarrow \mathbf{R}\text{Hom}_{\hat{A}}(T^\bullet, ?) \circ (? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet)$ is a natural isomorphism. For that, we take the full subcategory $\mathcal{X}$ of $\mathcal{D}(\hat{B})$ consisting of the objects $X^\bullet$ such that $\lambda_{X^\bullet}$ is an isomorphism. It is clearly a triangulated subcategory closed under taking coproducts. But if we apply to $\lambda_{\hat{B}} : \hat{B} \longrightarrow [\mathbf{R}\text{Hom}_{\hat{A}}(T^\bullet, ?) \circ (? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet)](\hat{B}) \cong \mathbf{R}\text{Hom}_{\hat{A}}(T^\bullet, T^\bullet)$ the homology functor, we obtain a map

$$H^p(\lambda_{\hat{B}}) : 0 = H^p(\hat{B}) \longrightarrow H^p(\mathbf{R}\text{Hom}_{\hat{A}}(T^\bullet, T^\bullet)) \cong \mathcal{D}(\hat{A})(T^\bullet, T^\bullet[p]) \cong \mathcal{D}(\Lambda)(V^\bullet, V^\bullet[p]) = 0,$$

so that $H^p(\lambda_B)$ is an isomorphism, for each $p \in \mathbb{Z} \setminus \{0\}$. As for $p = 0$, we have

$$H^0(\lambda_{\hat{B}}) : B \cong H^0(\hat{B}) \longrightarrow H^0(\mathbf{R}\text{Hom}_{\hat{A}}(T^\bullet, T^\bullet)) \cong \mathcal{D}(\hat{A})(T^\bullet, T^\bullet) \cong \mathcal{D}(\Lambda)(V^\bullet, V^\bullet) \cong B,$$

which also an isomorphism. This proves $\lambda_{\hat{B}}$ is an isomorphism in $\mathcal{D}(\hat{B})$, so that $\hat{B} \in \mathcal{X}$. It follows that $\mathcal{X} = \mathcal{D}(\hat{B})$. □

The following is a particular case of last proposition. Compare with [11, Theorem 1.3]

Example 4.17. *Let Λ be an algebra, let V be an injective cogenerator of $Mod-\Lambda$ and let $B = End_\Lambda(V)$ be its endomorphism algebra. Suppose that U is a Λ-module satisfying the following two conditions:*

(1) *$Ext^p_\Lambda(U,U) = 0$, for all $p > 0$;*
(2) *There is an exact sequence $0 \to U^{-n} \longrightarrow ... \longrightarrow U^{-1} \longrightarrow U^0 \longrightarrow V \to 0$, where $U^k \in add_{Mod-\Lambda}(U)$, for each $k = -n, ..., -1, 0$.*

Then there exists a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$, for some triangulated category $\mathcal{D}'$, where $A = End_\Lambda(U)$. Moreover, a slight modification of the proof of last proposition shows that the recollement can be chosen in such a way that $j_! = ? \otimes^{\mathbf{L}}_B T$, where $T = Hom_\Lambda(U,V)$, which is a $B-A-$bimodule.

Note that when Λ is an algebra with Morita duality (e.g. an Artin algebra), the algebras Λ and A are Morita equivalent, and so $\mathcal{D}(A) \cong \mathcal{D}(\Lambda)$.

We will end the section by studying an interesting case of fully faithfulness of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$, where two recollement situations come at once.

Theorem 4.18. *Let $T^\bullet$ be a complex of $B-A-$bimodules such that $T^\bullet_A$ is exceptional in $\mathcal{D}(A)$ and the algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism. The following assertion are equivalent:*

(1) *${}_BT^\bullet$ is compact and exceptional in $\mathcal{D}(B^{op})$ and the canonical algebra morphism $A \longrightarrow End_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ is an isomorphism;*
(1′) *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(B^{op}) \equiv \mathcal{D}(A^{op})$, with $j_! = T^\bullet \otimes^{\mathbf{L}}_A ?$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, for some dg algebra C);*
(2) *$\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful and preserves compact objects;*
(3) *A_A is in the thick subcategory of $\mathcal{D}(A)$ generated by $T^\bullet_A$;*
(4) *$? \otimes^{\mathbf{L}}_B T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a fully faithful left adjoint;*
(5) *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, with $j_* = \mathbf{R}Hom_A(T^\bullet, ?)$, for triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, for some dg algebra C).*

When B is k-flat, the dg algebra in conditions (1') and (5) can be chosen together with a homological epimorphism of dg algebras $f : B \longrightarrow C$ such that i_* is the restriction of scalars $f_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$.

Proof. Note that the exceptionality of $T^\bullet_A$ plus the fact that the algebra morphism $B \longrightarrow \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism is equivalent to saying that the unit map $\lambda_B : B \longrightarrow [\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes^{\mathbf{L}}_B T^\bullet)](B) = \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)$ is an isomorphism in $\mathcal{D}(B)$.

(1) $\Longleftrightarrow$ (1′) follows from the left-right symmetric version of corollary 4.14.

(1) $\Longrightarrow$ (3) The hypothesis implies that the unit map $\rho_A : A \longrightarrow [\mathbf{R}\mathrm{Hom}_{B^{op}}(T^\bullet, ?) \circ (T^\bullet \otimes^{\mathbf{L}}_A ?)](A)$ is an isomorphism in $\mathcal{D}(A^{op})$. Then, using lemma 3.4 and its terminology, we know that τ_A is an isomorphism in $\mathcal{D}(A)$, which implies that $A_A \cong \mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^\bullet)(T^\bullet)$. The fact that ${}_BT^\bullet \in \mathrm{per}(B^{op}) = \mathrm{thick}_{\mathcal{D}(B^{op})}(B)$ implies then that $A_A \in \mathrm{thick}_{\mathcal{D}(A)}([\mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^\bullet)](B))$ $\mathrm{thick}_{\mathcal{D}(A)}(T^\bullet)$.

(3) $\Longrightarrow$ (1) Since λ_B is an isomorphism, using lemma 3.3 and its terminology, we get that $\tau_{T^\bullet} : T^\bullet \longrightarrow [\mathbf{R}\mathrm{Hom}_{B^{op}}(\mathbf{R}\mathrm{Hom}_A(?, T^\bullet), T^\bullet)](T^\bullet)$ is an isomorphism in $\mathcal{D}(A)$. Since $A_A \in \mathrm{thick}_{\mathcal{D}(A)}(T^\bullet)$ we get that τ_A is an isomorphism in $\mathcal{D}(A)$ which, by lemma 3.4, implies that ρ_A is an isomorphism. That is, ${}_BT^\bullet$ is exceptional in $\mathcal{D}(B^{op})$ and the algebra morphism $A \longrightarrow \mathrm{End}_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ is an isomorphism.

On the other hand, the fact that $A_A \in \text{thick}_{\mathcal{D}(A)}(T^\bullet)$ implies that ${}_BT^\bullet \cong \mathbf{R}\text{Hom}_A(?, T^\bullet)(A)$ is in $\text{thick}_{\mathcal{D}(B^{op})}(\mathbf{R}\text{Hom}_A(?, T^\bullet)(T^\bullet)) = \text{thick}_{\mathcal{D}(B^{op})}(B) = \text{per}(B^{op})$.

$(1') \Longrightarrow (5)$ is the left-right symmetric version of corollary 4.15.

$(4) \Longleftrightarrow (5)$ Apply proposition 2.19 to the functor $G =: ? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$.

$(4) \Longrightarrow (3)$ Let $L : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ be a fully faithful left adjoint of $? \otimes_B^{\mathbf{L}} T^\bullet$. One easily sees that L preserves compact objects. Moreover, the unit $1_{\mathcal{D}(A)} \longrightarrow (? \otimes_B^{\mathbf{L}} T^\bullet) \circ L$ of the adjunction $(L, ? \otimes_B^{\mathbf{L}} T^\bullet)$ is a natural isomorphism. It follows that $A \cong [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ L](A) = (? \otimes_B^{\mathbf{L}} T^\bullet)(L(A)) \in (? \otimes_B^{\mathbf{L}} T^\bullet)(\text{per}(B)) = (? \otimes_B^{\mathbf{L}} T^\bullet)(\text{thick}_{\mathcal{D}(B)}(B)) \subseteq \text{thick}_{\mathcal{D}(A)}((? \otimes_B^{\mathbf{L}} T^\bullet)(B)) = \text{thick}_{\mathcal{D}(A)}(T^\bullet)$.

$(5), (3) \Longrightarrow (2)$ From assertion (5) we get that $\mathbf{R}\text{Hom}_A(T^\bullet, ?)$ is fully faithful. On the other hand, the fact that $A_A \in \text{thick}_{\mathcal{D}(A)}(T^\bullet)$ and that λ_B is an isomorphism imply that $\mathbf{R}\text{Hom}_A(T^\bullet ?)(A) \in \text{thick}_{\mathcal{D}(B)}(\mathbf{R}\text{Hom}_A(T^\bullet, ?)(T^\bullet)) = \text{thick}_{\mathcal{D}(B)}(B) = \text{per}(B)$. It follows from this that $\mathbf{R}\text{Hom}_A(T^\bullet, ?)$ takes objects of $\text{per}(A) = \text{thick}_{\mathcal{D}(A)}(A)$ to objects of $\text{per}(B)$, thus proving assertion (2).

$(2) \Longrightarrow (3)$ Bearing in mind that the counit $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\text{Hom}_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism and that $\mathbf{R}\text{Hom}_A(T^\bullet, ?)(A)$ is a compact object of $\mathcal{D}(B)$, we get:

$$A \cong [? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\text{Hom}_A(T^\bullet, ?)](A) = (? \otimes_B^{\mathbf{L}} T^\bullet))(\mathbf{R}\text{Hom}_A(T^\bullet, A)) \in$$
$$(? \otimes_B^{\mathbf{L}} T^\bullet)(\text{per}(B)) = (? \otimes_B^{\mathbf{L}} T^\bullet)(\text{thick}_{\mathcal{D}(B)}(B)) \subseteq \text{thick}_{\mathcal{D}(A)}(B \otimes_B^{\mathbf{L}} T^\bullet) = \text{thick}_{\mathcal{D}(A)}(T^\bullet).$$

□

Remark 4.19. *The precursor of theorem 4.18 is* [5, Theorem 2.2]*, where the authors prove that if T_A is a good tilting module (see definition 18) and $B = End(T_A)$, then condition (2) in our theorem holds. It is a consequence of theorem 4.18 (see corollary 5.5 below) that the converse is also true when one assume that T_A has finite projective dimension and $Ext_A^p(T, T^{(\alpha)}) = 0$, for all integers $p > 0$ and all cardinals α. Another consequence (see corollary 5.5 and example 5.6(1)) is that there are right A-modules, other than the good tilting ones, for which the equivalent conditions of the theorem holds. In the case of good* 1*-tilting modules, it was proved in* [10, Theorem 1.1] *that the dg algebra C can be chosen to be an ordinary algebra.*

The corresponding of the implication (1) $\Longrightarrow$ (5) *in our theorem was proved in* [41, Theorem 1] *for dg algebras over field. This result and its converse is then covered by the extension of theorem 4.18 to the context of dg categories, which is proved in* [32].

4.2. **Some natural questions.** As usual, $T^\bullet$ is a complex of $B-A-$bimodules. After the previous subsection, some natural questions arise, starting with the questions 1.2 of the introduction. Our next list of examples gives negative answers to all questions 1.2.

For question 1.2(1)(a), the following is a counterexample:

Example 4.20. *Let T_A be a good tilting module (see definition 18) which is not finitely generated (e.g. $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ as $\mathbb{Z}$-module) and let $B = End(T_A)$ be its endomorphism algebra. The functor $\mathbf{R}Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, but there is no recollement $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, with $i_* = \mathbf{R}Hom_A(T, ?)$, for any triangulated category $\mathcal{D}'$.*

Proof. That $\mathbf{R}\text{Hom}_A(T, ?)$ is fully faithful follows from corollary 5.5 below. On the other hand, if the desired recollement $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$ existed, then, by corollary 4.5, we would have that T_A is compact in $\mathcal{D}(A)$, and this is not the case. □

For question 1.2(1.b), the following is a counterexample.

Example 4.21. *If $f : B \longrightarrow A$ is a homological epimorphism of algebras, and we take $T = {}_BA_A$, then $\mathbf{R}Hom_A(A, ?) = f_* : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, but there need not exist a recollement $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, with $j_* = \mathbf{R}Hom_A(A, ?)$, for any triangulated category $\mathcal{D}'$.*

Proof. That $\mathbf{R}\mathrm{Hom}_A(A, ?) = f_*$ is fully faithful follows from the properties of homological epimorphisms. If the mentioned recollement exists, then the functor $? \otimes^{\mathbf{L}}_B A : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ preserves products and, by proposition 4.2, ${}_BA$ is compact in $\mathcal{D}(B^{op})$. There are obvious homological epimorphisms which do not satisfy this last property. □

For question 1.2(2)(a), the following is a counterexample, inspired by theorem 4.13:

Example 4.22. *Let A be an algebra and let P be a finitely generated projective right A-module such that P is not finitely generated as a left module over $B := End_A(P)$. Then $? \otimes^{\mathbf{L}}_B P : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful, but there is no recollement $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$, with $i_* =? \otimes^{\mathbf{L}}_B P =? \otimes_B P$, for any triangulated category $\mathcal{D}'$.*

Concretely, if k is a field, V is an infinite dimensional k-vector space and $A = End_k(V)^{op}$, then $? \otimes^{\mathbf{L}}_k V =? \otimes_k V : \mathcal{D}(k) \longrightarrow \mathcal{D}(A)$ is fully faithful, but does not define the mentioned recollement.

Proof. The final statement follows directly from the first part since V is a simple projective right A-module such that $\mathrm{End}_A(V) \cong k$. On the other hand, we get from proposition 4.10 that $? \otimes^{\mathbf{L}}_B P : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful and, since ${}_BP$ is not compact in $\mathcal{D}(B^{op})$, theorem 4.13 implies that the recollement does not exist.

□

As a counter example to question 1.2(2)(b), we have:

Example 4.23. *The functor $? \otimes^{\mathbf{L}}_{\mathbb{Q}} \mathbb{Q} =? \otimes_{\mathbb{Q}} Q : \mathcal{D}(\mathbb{Q}) \longrightarrow \mathcal{D}(\mathbb{Z})$ is fully faithful (see example 4.12), but there is no recollement $\mathcal{D}' \equiv \mathcal{D}(\mathbb{Z}) \equiv \mathcal{D}(\mathbb{Q})$, with $j_! =? \otimes_{\mathbb{Q}} Q$, for any triangulated category $\mathcal{D}'$.*

Proof. If the recollement existed, then, by corollary 4.14, $\mathbb{Q}$ would be compact in $\mathcal{D}(\mathbb{Z})$, which is absurd. □

But, apart from questions 1.2, there are some other natural questions whose answer we do not know even in the case of a bimodule.

Questions 4.24. (1) *(Motivated by proposition 4.10) Suppose that $T^\bullet_A$ is isomorphic in $\mathcal{D}(A)$ to a bounded complex of projective right A-modules, that the canonical morphism $Hom_{\mathcal{D}(A)}(T^\bullet, T^\bullet)^{(\alpha)} \longrightarrow Hom_{\mathcal{D}(A)}(T^\bullet, T^{\bullet(\alpha)})$ is an isomoprhism and that $Hom_{\mathcal{D}(A)}(T^\bullet, T^{\bullet(\alpha)}[p]) = 0$, for all cardinals α and all integers $p \neq 0$. Is $T^\bullet_A$ self-compact in $\mathcal{D}(A)$?.*

(2) *(Motivated by proposition 4.10) Suppose that $? \otimes_B T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful. Is $H^p(T^\bullet) = 0$ for $p >> 0$?. Is $T^\bullet_A$ quasi-isomorphic to a bounded complex of projective right A-modules?*

Remark 4.25. *The converse of question 4.24(2) has a negative answer, even for a bounded complex of projective A-modules. For instance, if P is a projective generator of $Mod-A$ which is not finitely generated, then $Tria_{\mathcal{D}(A)}(P) = \mathcal{D}(A)$ and P is not compact in this category. It follows from proposition 4.10 that $? \otimes_B^{\mathbf{L}} P : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is not fully faithful, where $B = End(P_A)$.*

Our next question concerns the relationship between proposition 4.4 and theorem 4.18. That $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ be fully faithful does not imply that it preserves compact objects (see example 4.21). The correct question to answer, for which we do not have an aswer, is the following:

Question 4.26. *Suppose that $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and that $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful. Due to theorems 4.6 and 4.18, each of the following questions has an affirmative answer if, and only if, so do the other ones, but we do not know the answer:*

(1) *Does $\mathbf{R}Hom_A(T^\bullet, ?)$ preserve compact objects?;*
(2) *Is A_A in $thick_{\mathcal{D}(A)}(T^\bullet)$?;*
(3) *Is ${}_BT^\bullet$ compact in $\mathcal{D}(B^{op})$?*

Note that, by proposition 4.9, ${}_BT^\bullet$ is isomorphic in $\mathcal{D}(B^{op})$ to an upper bounded complex of finitely generated projective left B-modules with bounded homology.

In next section, we will show that, in case $T^\bullet$ is a $B-A-$bimodule, the question has connections with Wakamatsu tilting problem.

5. The case of a bimodule

5.1. **Re-statement of the main results.** For the convenience of the reader, we make explicit what some results of the previous section say in the particular case when $T^\bullet = T$ is just a $B-A-$bimodule. The statements show a close connection with the theory of (not necessarily finitely generated) tilting modules.

Recall:

Definition 18. *Consider the following conditions for an A-module T:*

a) *T has finite projective dimension;*
a') *T admits a finite projective resolution with finitely generated terms;*
b) *There is an exact sequence $0 \to A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow .... \longrightarrow T^m \to 0$ in $Mod-A$, with $T^i \in Add(T)$ for $i = 0, 1, ..., m$;*
b') *There is an exact sequence $0 \to A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow .... \longrightarrow T^m \to 0$ in $Mod-A$, with $T^i \in add(T)$ for $i = 0, 1, ..., m$;*
c) *$Ext_A^p(T, T^{(\alpha)}) = 0$, for all integers $p > 0$ and all cardinals α.*

T is called a n-tilting module *when conditions a), b) and c) hold and $pd_A(T) = n$. Such a tilting module is* classical n-tilting *when it satisfies a'), b') and c) and it is called a* good n-tilting module *when it satisfies conditions a), b') and c). We will simply say that T is tilting (resp. classical tilting, resp. good tilting) when it is n-tilting (resp. classical n-tilting, resp. good n-tilting), for some $n \in \mathbf{N}$.*

Remark 5.1. *Note that T is a classical n-tilting module if, and only if, it satisfies conditions a'), b') and $Ext_A^p(T, T) = 0$, for all integers $p > 0$.*

When $\mathrm{Ext}^p_A(T,T) = 0$, for all $p > 0$, it is proved in [32] that the condition that $A_A \in \mathrm{thick}_{\mathcal{D}(A)}(T)$ is equivalent condition b') in Definition 18.

In the rest of the subsection, unless otherwise stated, T will be a $B-A-$bimodule and all statements are given for it. The following result is then a direct consequence of proposition 4.10:

Corollary 5.2. *Consider the following assertions:*

(1) $\otimes^{\mathbf{L}}_B T : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ *is fully faithful;*

(2) $Ext^p_A(T,T) = 0$*, for all* $p > 0$*, the canonical algebra morphism* $B \longrightarrow End_A(T)$ *is an isomorphism and* T_A *is a compact object of* $Tria_{\mathcal{D}(A)}(T)$.

(3) *The following conditions hold:*
   (a) *the canonical map* $B^{(\alpha)} \longrightarrow Hom_A(T,T)^{(\alpha)} \longrightarrow Hom_A(T,T^{(\alpha)})$ *is an isomorphism, for all cardinals* α
   (b) $Ext^p_A(T,T^{(\alpha)}) = 0$*, for all cardinals* α *and integers* $p > 0$*;*
   (c) *for each family* $(X^\bullet_i)_{i\in I}$ *in* $Tria_{\mathcal{D}(A)}(T)$ *such that* $\mathcal{D}(A)(T[k],X^\bullet_i) = 0$*, for all* $k \geq 0$ *and all* $i \in I$*, one has that* $\mathcal{D}(A)(T,\coprod_{i\in I} X^\bullet_i) = 0$.

The implications $(1) \Longleftrightarrow (2) \Longrightarrow (3)$ *hold true. When* T_A *has finite projective dimension, all assertions are equivalent.*

The next result is then a consequence of theorem 4.13:

Corollary 5.3. *The following assertions are equivalent:*

(1) *There is a recollement* $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$ *with* $i_* =? \otimes^{\mathbf{L}}_B T$*, for some triangulated category* $\mathcal{D}'$*;*

(2) $Ext^p_A(T,T) = 0$*, for all* $p > 0$*, the canonical algebra morphism* $B \longrightarrow End_A(T)$ *is an isomorphism,* T *is a compact object of* $Tria_{\mathcal{D}(A)}(T)$ *and this subcategory is closed under taking products in* $\mathcal{D}(A)$.

(3) $Ext^p_A(T,T) = 0$*, for all* $p > 0$*, the canonical algebra morphism* $B \longrightarrow End_A(T)$ *is an isomorphism,* T *is a compact object of* $Tria_{\mathcal{D}(A)}(T)$ *and* T *admits a finite projective resolution with finitely generated terms as a left* B*-module.*

   When A *is* k*-flat, these conditions are also equivalent to:*

(4) *There is a dg algebra* $\hat{A}$*, a homological epimorphism of dg algebras* $f : A \longrightarrow \hat{A}$ *and a classical tilting object* $\hat{T}^\bullet \in \mathcal{D}(\hat{A})$ *such that*
   (a) $f_*(\hat{T}^\bullet) \cong T_A$*, where* $f_* : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(A)$ *is the restriction of scalars functor;*
   (b) *The canonical algebra morphism* $B \longrightarrow End_A(T) \cong End_{\mathcal{D}(\hat{A})}(\hat{T}^\bullet)$ *is an isomorphism.*

The next result is a direct consequence of corollaries 4.14 and 4.15:

Corollary 5.4. *Consider the following assertions for the* $B-A-$*bimodule* T*:*

(1) T_A *admits a finite projective resolution with finitely generated terms,* $Ext^p_A(T,T) = 0$*, for all* $p > 0$*, and the algebra morphism* $B \longrightarrow End(T_A)$ *is an isomorphism;*

(2) *There is recollement* $\mathcal{D}' \equiv D(A) \equiv \mathcal{D}(B)$*, with* $j_! =? \otimes^{\mathbf{L}}_B T$*, for some triangulated category* $\mathcal{D}'$ *(which is equivalent to* $\mathcal{D}(C)$*, for some dg algebra* C*);*

(3) *There is a recollement* $\mathcal{D}' \equiv D(A^{op}) \equiv \mathcal{D}(B^{op})$*, with* $j^! = j^* = T\otimes^{\mathbf{L}}_A ?$*, for some triangulated category* $\mathcal{D}'$ *(which is equivalent to* $\mathcal{D}(C^{op})$*, for some dg algebra* C*).*

Then the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) *hold true. When A is k-projective, all assertions are equivalent.*

The next result is a direct consequence of theorem 4.18 and the definition of good tilting module.

Corollary 5.5. *Let T be a right A-module such that $Ext_A^p(T,T) = 0$, for all $p > 0$, and let $B = End_A(T)$. The following assertions are equivalent:*

(1) *$Ext_{B^{op}}^p(T,T) = 0$, for all $p > 0$, the canonical algebra morphism $A \longrightarrow End_{B^{op}}(T)^{op}$ is an isomorphism and T admits a finite projective resolution with finitely generated terms as a left B-module;*
(2) *$\mathbf{R}Hom_A(T,?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful and preserves compact objects;*
(3) *There exists an exact sequence $0 \to A \longrightarrow T^0 \longrightarrow T^1 ... \longrightarrow T^n \to 0$ in $Mod - A$, with $T^k \in add(T)$ for each $k = 0, 1, ..., n$;*
(4) *$?\otimes_B^{\mathbf{L}} : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a fully faithful left adjoint.*

When in addition T_A has finite projective dimension and $Ext_A^p(T, T^{(\alpha)}) = 0$, for all cardinals α and all integers $p > 0$, the above conditions are also equivalent to:

(5) *T is a good tilting right A-module.*

The last results show that the fully faithful condition of the classical derived functors associated to an exceptional module is closely related to tilting theory. However, this relationship tends to be tricky, as the following examples show. They are explained in detail in the final part of [32].

Examples 5.6. (1) *If A is a non-Noetherian hereditary algebra and I is an injective cogenerator of $Mod - A$ containing an isomorphic copy of each cyclic module, then $T = E(A) \oplus \frac{E(A)}{A} \oplus I$ satisfies the conditions (1)-(5) of corollary 5.5, but $Ext_A^1(T, T^{(\aleph_0)}) \neq 0$. Hence T is not a tilting A-module.*

(2) *Let A be a right Noetherian right hereditary algebra such that $Hom_A(E(A/A, E(A))) = 0$ and $E(A)/A$ contains an indecomposable summand with infinite multiplicity. If I is the direct sum of one isomorphic copy of each indecomposable summand of $E(A)/A$, then $T = E(A) \oplus I$ is a 1-tilting module such that $\mathbf{R}Hom_A(T,?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is not fully faithful. The Weyl algebra $A_1(k) = k < x, y > /(xy - yx - 1)$ over the field k is an example where the situation occurs.*

(3) *If A is a hereditary Artin algebra, T is a finitely generated projective right A-module which is not a generator and $B = End_A(T)$, then T admits a finite projective resolution with finitely generated terms as a left B-module, but $\mathbf{R}Hom_A(T,?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is not fully faithful. Indeed $\mathbf{R}Hom_A(T,?)$ preserves compact objects, but condition (3) of last corollary does not hold.*

5.2. **Connection with Wakamatsu tilting problem.** In this subsection we show a connection of question 4.26 with a classical problem in Representation Theory.

Definition 19. *Let T_A be a module and $B = End(T_A)$. Consider the following conditions:*

(1) *T_A admits a projective resolution with finitely generated terms;*
(2) *$Ext_A^p(T,T) = 0$, for all $p > 0$;*
(3) *There exists an exact sequence $0 \to A \to T^0 \to T^1 \to ... \to T^n \to ...$ such that*

(a) $T^i \in add(T_A)$, *for all* $i \geq 0$;
(b) *The functor* $Hom_A(?, T)$ *leaves the sequence exact.*

(4) *There exists an exact sequence* $0 \to A \to T^0 \to \dots \to T^n \to 0$, *with* $T^i \in add(T_A)$, *for all* $i \geq 0$.

We shall say that T_A *is*

a) Wakamatsu tilting *when (1), (2) and (3) hold;*
b) semi-tilting *when (1), (2) and (4) hold;*
c) generalized Wakamatsu tilting *when (2) and (3) hold;*
d) generalized semi-tilting *when (2) and (4) hold.*

Remark 5.7. *Each classical tilting module is (generalized) semi-tilting and each (generalized) semi-tilting module is (generalized) Wakamatsu tilting.*

Proposition 5.8. *Let* T *be a Wakamatsu tilting right* A*-module. The following assertions are equivalent:*

(1) T *is classical tilting;*
(2) T *is semi-tilting of finite projective dimension;*
(3) T *has finite projective dimension, both as a right* A*-module and as left module over* $B = End(T_A)$.

Proof. (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (3) By hypothesis, we have that $\text{pd}(T_A) < \propto$. On the other hand, a finite projective resolution for ${}_BT$ is obtained by applying the functor $\text{Hom}_A(?, T)$ to the exact sequence $0 \to A \to T^0 \to \dots \to T^n \to 0$, with $T^i \in add(T_A)$ given in the definition of semi-tilting module.

(3) $\Longrightarrow$ (1) This is known (see [27, Section 4]). □

Question 5.9. 1. *Is statement (1) of last proposition true, for all Wakamatsu tilting modules?.*
2. *We can ask an intermediate question, namely: is each Wakamatsu tilting module a semi-tilting one?.*

Remark 5.10. *The answer to question 1 is negative in general (see* [39, Example 3.1]*). However it is still an open question, known as* Wakamatsu tilting problem, *whether each Wakamatsu tilting module of finite projective dimension is classical tilting. Note that, by proposition 5.8, an affirmative answer to question 2 above implies an affirmative answer to Wakamatsu problem and, conversely.*

It turns out that question 2 is related to question 4.26, as the following result shows:

Proposition 5.11. *Let us assume that* $Ext_A^p(T, T) = 0$, *for all* $p > 0$, *and that* $RHom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ *is fully faithful, where* $B = End(T_A)$. *Consider the following assertions:*

(1) $RHom_A(T, ?)$ *preserves compact objects;*
(2) T_A *is a generalized semi-tilting module;*
(3) T_A *is a generalized Wakamatsu tilting module;*
(4) *The structural algebra homomorphism* $A \longrightarrow End_{B^{op}}(T)^{op}$ *is an isomorphism and* $Ext^p_{B^{op}}(T, T) = 0$, *for all* $p > 0$.

Then the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) *hold true.*

Proof. (1) $\Longleftrightarrow$ (2) is a direct consequence of corollary 5.5 and the definition of generalized semi-tllting module.

(2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (4) Let us fix an exact sequence $0 \to A \longrightarrow T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} T^n \xrightarrow{d^n} ...$ (*), with $T^i \in \text{add}(T)$, for all $i \geq 0$. As shown in the proof of proposition 5.8, when we apply to it the functor $\text{Hom}_A(?, T) : \text{Mod} - A \longrightarrow B - \text{Mod}$, we obtain a projective resolution of ${}_BT \cong \text{Hom}_A(A, T)$. Bearing in mind that the canonical natural transformation $\sigma : 1_{\text{Mod}-A} \longrightarrow \text{Hom}_{B^{op}}(\text{Hom}_A(?, T), T)$ is an isomorphism, when evaluated at a module $T' \in \text{add}(T)$, when we apply $\text{Hom}_{B^{op}}(?, T)$ to that projective resolution of ${}_BT$, we obtain a sequence

$$0 \to \text{Hom}_{B^{op}}(T, T) \longrightarrow T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} T^n \xrightarrow{d^n} ....$$

This sequence is exact due to the left exactness of $\text{Hom}_{B^{op}}(?, T)$ and to the exactness of the sequence (*) and, hence, both sequences are isomorphic. Then assertion (4) holds.

(4) $\Longrightarrow$ (3) By proposition 4.9, we know that ${}_BT$ admits a projective resolution with finitely generated terms, say

$$...P^{-n} \longrightarrow ... \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow {}_BT \to 0. \qquad (**)$$

The hypotheses imply that, when we apply to it the functor $\text{Hom}_{B^{op}}(?, T)$, we obtain an exact sequence in $\text{Mod} - A$

$$0 \to A \longrightarrow \text{Hom}_{B^{op}}(P^0, T) \longrightarrow \text{Hom}_{B^{op}}(P^1, T) \longrightarrow ... \longrightarrow \text{Hom}_{B^{op}}(P^{-n}, T) \longrightarrow ....$$

Note that $\text{Hom}_{B^{op}}(P^{-i}, T)$ is a direct summand of $\text{Hom}_{B^{op}}(B^{(r)}, T) \cong T_A^{(r)}$, for some $r \in \mathbb{N}$, so that $\text{Hom}_{B^{op}}(P^{-i}, T) =: T^i$ is in $\text{add}(T_A)$, for each $i \geq 0$. Note also that the canonical natural transformation $\sigma : 1_{B-\text{Mod}} \longrightarrow \text{Hom}_A(\text{Hom}_{B^{op}}(?, T), T)$ is an isomorphism when evaluated at any finitely generated projective left B-module, because $\text{Hom}_A(T, T) \cong {}_BB$. It follows from this and the fact that $\text{Ext}_A^p(T, T) = 0$, for all $p > 0$, that when we apply $\text{Hom}_A(?, T)$ to the last exact sequence we obtain, up to isomorphism, the initial projective resolution (**). Then the exact sequence

$$0 \to A_A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow ... \longrightarrow T^n \longrightarrow ...$$

is kept exact when applying $\text{Hom}_A(?, T)$. Therefore T_A is a generalized Wakamatsu tilting module. ☐

As an immediate consequence, we get:

Corollary 5.12. *Each of the following statements is true if, and only if, so is the other:*

1. *If T_A is a generalized Wakamatsu tilting module such that $\mathbf{R}Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, where $B = End_A(T)$, then T_A is generalized semi-tilting.*
2. *Let ${}_BT_A$ be a bimodule such that $Ext_A^p(T, T) = 0 = Ext_{B^{op}}^p(T, T)$, for all $p > 0$ and the algebra morphisms $B \longrightarrow End_A(T)$ and $A \longrightarrow End_{B^{op}}(T)^{op}$ are isomorphisms. If the functor $\mathbf{R}Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, then it preserves compact objects.*

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ONE POINT EXTENSION OF A QUIVER ALGEBRA DEFINED BY TWO CYCLES AND A QUANTUM-LIKE RELATION

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Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Tokyo University of Science, Japan, 12–14 October 2013.

In this paper, we consider a one point extension algebra B of a quiver algebra A_q over a field k defined by two cycles and a quantum-like relation depending on a nonzero element q in k. We determine the Hochschild cohomology ring of B modulo nilpotence and show that if q is a root of unity then B negates Snashall-Solberg's conjecture.

Introduction

Let A be an indecomposable finite dimensional algebra over a field k. We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

Let q be a non-zero element in k and s, t integers with $s, t \geq 1$. Let Γ be the quiver with $s + t$ vertices and $s + t + 1$ arrows as follows:

$$
\begin{array}{ccccccccc}
 & a_3 & \xleftarrow{\alpha_2} & a_2 & & b_2 & \cdots\cdots & & \\
{}^{\alpha_3}\swarrow & & & & \nwarrow^{\alpha_1} \quad {}^{\beta_1}\nearrow & & & \ddots & \\
a_4 & & & & 1 & & & & b_{t-2} \\
\ddots & & & & {}^{\alpha_s}\nearrow \downarrow^{\gamma} \nwarrow^{\beta_t} & & & \swarrow_{\beta_{t-2}} & \\
 & \cdots\cdots & & a_s & 2 & b_t & \xleftarrow{\beta_{t-1}} & b_{t-1} &
\end{array}
$$

and $I_{q,v,u}$ the ideal of $k\Gamma$ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}, \gamma X^{sv+u}$$

for $a, b \geq 2$, $0 \leq v \leq a-1$, $0 \leq u \leq s-1$ and $(v, u) \neq (0, 0)$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$. Paths in Γ are written from right to left. In this paper, we consider the quiver algebra $B = k\Gamma/I_{q,v,u}$. We denote the trivial path at the vertex $a(i)$ and at the vertex $b(j)$ by $e_{a(i)}$ and by $e_{b(j)}$ respectively. We regard the numbers i in the subscripts of $e_{a(i)}$ modulo s and j in the subscripts of $e_{b(j)}$ modulo t.

In the case $s = t = 1$ and $a = b = 2$, B is a Koszul algebra. In this case, the Hochschild cohomology ring of B modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}$ is not finitely generated as a k-algebra (see [4]).

This algebra B is a one point extension of a quiver algebra $A_q = kQ/I_q$ where Q the following quiver:

$$\begin{tikzcd}
 & a_3 \arrow[dl, "\alpha_3"'] & a_2 \arrow[l, "\alpha_2"'] & & b_2 & \cdots & \ddots \\
a_4 & & & 1 \arrow[ul, "\alpha_1"] \arrow[ur, "\beta_1"'] & & & b_{t-2} \arrow[dl, "\beta_{t-2}"] \\
\ddots & \cdots & a_s \arrow[ur, "\alpha_s"] & & b_t \arrow[ul, "\beta_t"'] & b_{t-1} \arrow[l, "\beta_{t-1}"'] &
\end{tikzcd}$$

and I_q is the ideal of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}.$$

This algebra A_q is the quiver algebra defined by two cycles and a quantum-like relation. In [2] and [3], we described the minimal projective bimodule resolution of A_q and showed that if q is a root of unity then $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables and that if q is not a root of unity then $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the field k.

The Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ was used in [5] to define a support variety for any finitely generated module over a finite dimensional algebra A. In [5], Snashall and Solberg conjectured that if A is an artin k-algebra then $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra.

In this paper, we determine the Hochschild cohomology ring of B modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}$ and show that if q is a root of unity then $\mathrm{HH}^*(B)/\mathcal{N}$ is not finitely generated as an algebra. So B negates Snashall-Solberg's conjecture.

The content of the paper is organized as follows. In Section 1, we determine the minimal projective bimodule resolution of B. In Section 2 we determine the ring structure of $\mathrm{HH}^*(B)/\mathcal{N}$.

1. A projective bimodule resolution of B

In this section, we determine a minimal projective bimodule resolution of B.

Let k be a field, $q \in k$ a nonzero element and s, t integers with $s, t \geq 1$. Let $B = k\Gamma/I_{q,v,u}$ where Γ is the following quiver:

$$\begin{tikzcd}
 & a_3 \arrow[dl, "\alpha_3"'] & a_2 \arrow[l, "\alpha_2"'] & & b_2 & \cdots & \ddots \\
a_4 & & & 1 \arrow[ul, "\alpha_1"] \arrow[ur, "\beta_1"'] \arrow[d, "\gamma"] & & & b_{t-2} \arrow[dl, "\beta_{t-2}"] \\
\ddots & \cdots & a_s \arrow[ur, "\alpha_s"] & 2 & b_t \arrow[ul, "\beta_t"'] & b_{t-1} \arrow[l, "\beta_{t-1}"'] &
\end{tikzcd}$$

and $I_{q,v,u}$ is the ideal of $k\Gamma$ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}, \gamma X^{sv+u}$$

for $a, b \geq 2, 0 \leq v \leq a-1$, $0 \leq u \leq s-1$ and $(v, u) \neq (0, 0)$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$.

Then we note that the following elements in B form a k-basis of B.

$$
\begin{array}{ll}
X^{sl+l'} e_{a(i)} & \text{for } 2 \leq i \leq s, 0 \leq l \leq a-1, 0 \leq l' \leq s-1, \\
Y^{tl+l'} e_{b(j)} & \text{for } 1 \leq j \leq t, 0 \leq l \leq b-1, 0 \leq l' \leq t-1, \\
X^{si+l} Y^{tj+l'} & \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 0 \leq l' \leq t-1, \\
X^{si} Y^{tj+l'} & \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 0 \leq l' \leq t-1, \\
X^{si} Y^{tj} X^{l} & \text{for } 0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l \leq s-1, \\
Y^{l'} X^{si} Y^{tj} & \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l' \leq t-1, \\
Y^{l'} X^{si} Y^{tj} X^{l} & \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 1 \leq l' \leq t-1, \\
X^{si+l} Y^{tj} X^{l'} & \text{for } 0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l, l' \leq s-1, \\
Y^{l} X^{si} Y^{tj+l'} & \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l, l' \leq t-1. \\
e_2, & \\
\gamma X^{sl+l'} & \text{for } 0 \leq l \leq v-1, 0 \leq l' \leq s-1, \\
\gamma X^{sv+l'} & \text{for } 0 \leq l' \leq u-1 \text{ if } u \neq 0, \\
\gamma X^{sl} Y^{tl'+l''} & \text{for } \begin{cases} 0 \leq l \leq v, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u \neq 0, \\ 0 \leq l \leq v-1, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u = 0, \end{cases} \\
\gamma X^{sl} Y^{tl'} X^{l''} & \text{for } \begin{cases} 0 \leq l \leq v, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u \neq 0, \\ 0 \leq l \leq v-1, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u = 0. \end{cases}
\end{array}
$$

So we have the dimension of the algebra B as follows:

$$
\dim_k B = \begin{cases} ab(s+t-1)^2 + (v+1)b(s+t-1) - s + u + 1 & \text{if } u \neq 0, \\ ab(s+t-1)^2 + vb(s+t-1) + 1 & \text{if } u = 0. \end{cases}
$$

Let M be the right A_q-module with the following basis elements:

$$
\begin{array}{ll}
X^{sl+l'} & \text{for } 0 \leq l \leq v-1, 0 \leq l' \leq s-1, \\
X^{sv+l'} & \text{for } 0 \leq l' \leq u-1 \text{ if } u \neq 0, \\
X^{sl} Y^{tl'+l''} & \text{for } \begin{cases} 0 \leq l \leq v, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u \neq 0, \\ 0 \leq l \leq v-1, 0 \leq l' \leq b-1, 1 \leq l'' \leq t-1 \text{ if } u = 0, \end{cases} \\
X^{sl} Y^{tl'} X^{l''} & \text{for } \begin{cases} 0 \leq l \leq v, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u \neq 0, \\ 0 \leq l \leq v-1, 1 \leq l' \leq b-1, 0 \leq l'' \leq s-1 \text{ if } u = 0. \end{cases}
\end{array}
$$

Then we regard the algebra B as the one point extension $\begin{pmatrix} k & M \\ 0 & A_q \end{pmatrix}$ of A_q by the A_q-module M. Let $\mathcal{F}\colon \operatorname{Mod} A_q^e \to \operatorname{Mod} B^e$ and $\mathcal{G}\colon \operatorname{Mod} A_q \to \operatorname{Mod} B^e$ be the natural functors given by $\mathcal{F}(Q) = \begin{pmatrix} 0 & M \\ 0 & A_q \end{pmatrix} \otimes_{A_q} Q$ and $\mathcal{G}(L) = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$. We give an explicit projective bimodule resolution of a one point extension algebra by using the following Theorem in [1].

Theorem 1. *Let $\cdots \to Q_n \xrightarrow{\delta_n} \cdots \to Q_1 \xrightarrow{\delta_1} Q_0 \xrightarrow{\delta_0} A \to 0$ be an A^e-projective resolution of A and $\cdots \to L_n \xrightarrow{r_n} \cdots \to L_2 \xrightarrow{r_2} L_1 \xrightarrow{r_1} L_0 \xrightarrow{r_0} M \to 0$ a right A-projective resolution of M. Then we have a B^e-projective resolution of $B = \begin{pmatrix} k & M \\ 0 & A \end{pmatrix}$:*

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0,$$

where $P_0 = \mathcal{F}(Q_0) \oplus (Be' \otimes e'B)$, $d_0 = (\mathcal{F}(\delta_0), id_{Be' \otimes e'B})$, $P_n = \mathcal{F}(Q_n) \oplus \mathcal{G}(L_{n-1})$ and $d_n = \begin{pmatrix} \mathcal{F}(\delta_n) & \sigma_n \\ 0 & -\mathcal{G}(r_{n-1}) \end{pmatrix}$ for $n \geq 1$, where $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in B$ is new vertex and σ_n: $\mathcal{G}(L_{n-1}) \to \mathcal{F}(Q_{n-1})$ is a B^e-homomorphism such that $\mathcal{F}(\delta_n) \circ \sigma_{n+1} = \sigma_n \circ \mathcal{G}(\eta_n)$, where σ_0 is the natural monomorphism.

Remark 2. The following sequence is a minimal projective resolution of M.

$$\cdots \to L_{2n} \xrightarrow{r_{2n}} L_{2n-1} \xrightarrow{r_{2n-1}} \cdots \to L_1 \xrightarrow{r_1} L_0 \xrightarrow{r_0} M \to 0$$

where $L_{2n} = e_1 A_q$, $L_{2n+1} = e_{a(s+1-u)} A_q$ for $n \geq 0$, r_0 is a natural epimorphism and for $n \geq 1$,

$$r_{2n-1}(e_{a(s+1-u)}) = X^{sv+u} e_{a(s+1-u)},$$
$$r_{2n}(e_1) = X^{s(a-v-1)+s-u} e_1.$$

In [2], we gave the minimal projective bimodule resolution of A_q. Then, by Theorem 1 we have the minimal projective bimodule resolution of B.

For $n \geq 0$, we define left B^e-modules, equivalently B-bimodules

$$P_0 = Be_1 \otimes e_1 B \oplus \coprod_{i=2}^{s} Be_{a(i)} \otimes e_{a(i)} B \oplus \coprod_{j=2}^{t} Be_{b(j)} \otimes e_{b(j)} B \oplus Be_2 \otimes e_2 B,$$

$$P_{2n} = \coprod_{l=0}^{2n} Be_1 \otimes e_1 B \oplus \coprod_{i=2}^{s} Be_{a(i)} \otimes e_{a(i)} B \oplus \coprod_{j=2}^{t} Be_{b(j)} \otimes e_{b(j)} B \oplus Be_2 \otimes e_{a(s+1-u)} B,$$

$$P_{2n+1} = \coprod_{l=1}^{2n} Be_1 \otimes e_1 B \oplus \coprod_{i=1}^{s} Be_{a(i+1)} \otimes e_{a(i)} B \oplus \coprod_{j=1}^{t} Be_{b(j+1)} \otimes e_{b(j)} B \oplus Be_2 \otimes e_1 B.$$

The generators $e_1 \otimes e_1$, $e_{a(i)} \otimes e_{a(i)}$, $e_{b(j)} \otimes e_{b(j)}$, $e_2 \otimes e_{a(s+1-u)}$ and $e_2 \otimes e_2$ of P_{2n} are labeled ε_l^{2n} for $0 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n}$ for $2 \leq i \leq s$, and $\varepsilon_{b(j)}^{2n}$ for $2 \leq j \leq t$, $\varepsilon^{2n\prime}$ and $\varepsilon^{0\prime}$ respectively. Similarly, we denote the generators $e_1 \otimes e_1$, $e_{a(i+1)} \otimes e_{a(i)}$, $e_{b(j+1)} \otimes e_{b(j)}$ and $e_2 \otimes e_1$ of P_{2n+1} by ε_l^{2n+1} for $1 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n+1}$ for $1 \leq i \leq s$, $\varepsilon_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ and $\varepsilon^{2n+1\prime}$ respectively.

Then we have the B^e-homomorphisms $\sigma_n\colon \mathcal{G}(L_{n-1}) \to \mathcal{F}(Q_{n-1})$ as follows:

$$\sigma_0(\varepsilon^{0\prime}\gamma) = \gamma,$$
$$\sigma_{2n-1}(\varepsilon^{2n-1\prime}) = \gamma\varepsilon^{2n-2}_{2n-2},$$
$$\sigma_{2n}(\varepsilon^{2n\prime}) = \sum_{l=0}^{v-1}\sum_{l'=0}^{s-1}\gamma X^{sl+l'}\varepsilon^{2n-1}_{a(s-l')}X^{s(v-l)+u-1-l'} + \sum_{l'=0}^{u-1}\gamma X^{sv+l'}\varepsilon^{2n-1}_{a(s-l')}X^{u-1-l'}.$$

So we have the minimal projective bimodule resolution of B as follows.

Theorem 3. *The following sequence $\mathbb{P}$ is a minimal projective resolution of the left B^e-module B:*

$$\mathbb{P}: \cdots \to P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} P_{2n-1} \to \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0.$$

where $d_0\colon P_0 \to B$ is the multiplication map, and left B^e-homomorphisms d_{2n} and d_{2n+1} are defined by

$$d_1:$$
$$\begin{cases} \varepsilon^1_{b(j)} \mapsto \varepsilon^0_{b(j+1)}Y - Y\varepsilon^0_{b(j)} & \textit{for } 1 \le j \le t, \\ \varepsilon^1_{a(i)} \mapsto \varepsilon^0_{a(i+1)}X - X\varepsilon^0_{a(i)} & \textit{for } 1 \le i \le s, \\ \varepsilon^{1\prime} \mapsto \gamma\varepsilon^0_0 - \varepsilon^{0\prime}\gamma, \end{cases}$$

$$d_{2n}:$$
$$\begin{cases} \varepsilon^{2n}_0 \mapsto \sum_{l=0}^{b-1}\sum_{l'=0}^{t-1} Y^{tl+l'}\varepsilon^{2n-1}_{b(t-l')}Y^{t(b-1-l)+t-l'-1}, \\ \varepsilon^{2n}_{b(j)} \mapsto \sum_{l=0}^{b-1}\sum_{l'=0}^{t-1} Y^{tl+l'}\varepsilon^{2n-1}_{b(j-1-l')}Y^{t(b-1-l)+t-l'-1} \textit{ for } 2 \le j \le t, \\ \varepsilon^{2n}_1 \mapsto \\ qY^t\varepsilon^{2n-1}_1 - \varepsilon^{2n-1}_1 Y^t - \sum_{j=1}^{t} X^sY^{t-j}\varepsilon^{2n-1}_{b(j)}Y^{j-1} + q^{bn+1}\sum_{j=1}^{t} Y^{t-j}\varepsilon^{2n-1}_{b(j)}Y^{j-1}X^s, \\ \varepsilon^{2n}_{l''} \mapsto \\ \begin{cases} \sum_{j=0}^{b-1} q^{alj}Y^{tj}\varepsilon^{2n-1}_{2l}Y^{t(b-1-j)} + \sum_{i=0}^{a-1} q^{ib(n-l+1)}X^{s(a-1-i)}\varepsilon^{2n-1}_{2l-1}X^{si} \\ \qquad \textit{if } l'' = 2l \textit{ for } 1 \le l \le n-1, \\ q^{al'+1}Y^t\varepsilon^{2n-1}_{2l'+1} - \varepsilon^{2n-1}_{2l'+1}Y^t - X^s\varepsilon^{2n-1}_{2l'} + q^{b(n-l')+1}\varepsilon^{2n-1}_{2l'}X^s \\ \qquad \textit{if } l'' = 2l'+1 \textit{ for } 1 \le l' \le n-2, \end{cases} \\ \varepsilon^{2n}_{2n-1} \mapsto \\ q^{an+1}\sum_{i=1}^{s} Y^tX^{s-i}\varepsilon^{2n-1}_{a(i)}X^{i-1} - \sum_{i=1}^{s} X^{s-i}\varepsilon^{2n-1}_{a(i)}X^{i-1}Y^t - X^s\varepsilon^{2n-1}_{2n} + q\varepsilon^{2n-1}_{2n}X^s, \\ \varepsilon^{2n}_{a(i)} \mapsto \sum_{l=0}^{a-1}\sum_{l'=0}^{s-1} X^{sl+l'}\varepsilon^{2n-1}_{a(i-1-l')}X^{s(a-1-l)+s-l'-1} \textit{ for } 2 \le i \le s, \\ \varepsilon^{2n}_{2n} \mapsto \sum_{l=0}^{a-1}\sum_{l'=0}^{s-1} X^{sl+l'}\varepsilon^{2n-1}_{a(s-l')}X^{s(a-1-l)+s-l'-1}, \\ \varepsilon^{2n\prime} \mapsto \sum_{l=0}^{v-1}\sum_{l'=0}^{s-1} \gamma X^{sl+l'}\varepsilon^{2n-1}_{a(s-l')}X^{s(v-l)+u-1-l'} \\ \qquad + \sum_{l'=0}^{u-1}\gamma X^{sv+l'}\varepsilon^{2n-1}_{a(s-l')}X^{u-1-l'} - \varepsilon^{2n-1\prime}X^{sv+u}, \end{cases}$$

d_{2n+1} :

$$\begin{cases} \varepsilon^{2n+1}_{b(j)} \mapsto \varepsilon^{2n}_{b(j+1)}Y - Y\varepsilon^{2n}_{b(j)} \ \mathit{for}\ 1 \leq j \leq t, \\ \varepsilon^{2n+1}_{l''} \mapsto \\ \begin{cases} -\sum_{j=0}^{b-1} q^{j(al+1)}Y^{tj}\varepsilon^{2n}_{2l+1}Y^{t(b-1-j)} - X^s\varepsilon^{2n}_{2l} + q^{b(n-l)}\varepsilon^{2n}_{2l}X^s \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathit{if}\ l'' = 2l+1 \ \mathit{for}\ 0 \leq l \leq n-1, \\ -q^{al'}Y^t\varepsilon^{2n}_{2l'} + \varepsilon^{2n}_{2l'}Y^t + \sum_{i=0}^{a-1} q^{i(b(n-l')+1)}X^{s(a-1-i)}\varepsilon^{2n}_{2l'-1}X^{si} \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathit{if}\ l'' = 2l' \ \mathit{for}\ 1 \leq l' \leq n, \end{cases} \\ \varepsilon^{2n+1}_{a(i)} \mapsto \varepsilon^{2n}_{a(i+1)}X - X\varepsilon^{2n}_{a(i)} \ \mathit{for}\ 1 \leq i \leq s, \\ \varepsilon^{2n+1\prime} \mapsto \gamma\varepsilon^{2n}_{2n} - \varepsilon^{2n\prime}X^{s(a-1-v)+s-u}, \end{cases}$$

for $n \geq 1$, *where in the case* $n = 1$, $\varepsilon^2_{l''}$ *vanish, and the image of* ε^2_1 *by* d_2 *is*

$$-\sum_{j=1}^{t} X^sY^{t-j}\varepsilon^1_{b(j)}Y^{j-1} + q\sum_{j=1}^{t} Y^{t-j}\varepsilon^1_{b(j)}Y^{j-1}X^s + q\sum_{i=1}^{s} Y^tX^{s-i}\varepsilon^1_{a(i)}X^{i-1} - \sum_{i=1}^{s} X^{s-i}\varepsilon^1_{a(i)}X^{i-1}Y^t.$$

2. The Hochschild cohomology ring of B modulo nilpotence

In this section, we consider the case that q is a root of unity. We set $\bar{z}$ is the remainder when we divide z by r for any integer z. Then we have $0 \leq \bar{z} \leq r-1$. Using the projective resolution in Theorem 3, we have a basis and Yoneda product of the Hochschild cohomology ring $\mathrm{HH}^*(B)$ of B. Then we determine the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(B)/\mathcal{N}$ as follows.

Theorem 4. *If* $s, t \geq 2$ *and* q *is an* r*-th root of unity then*

$$\mathrm{HH}^*(B)/\mathcal{N} \cong \begin{cases} k \oplus k[x^{2r}, y^{2r}]x^{2r} & \mathit{if}\ \bar{a} \neq 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^{2r}]x^2 & \mathit{if}\ \bar{a} \neq 0, \bar{b} = 0, \\ k \oplus k[x^{2r}, y^2]x^{2r} & \mathit{if}\ \bar{a} = 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^2]x^2 & \mathit{if}\ \bar{a} = \bar{b} = 0. \end{cases}$$

where $x^l = e_{1,0} + \sum_{j=2}^{t} e_{b(j)}$ *in* $\mathrm{HH}^l(B)$ *and* $x^my^n = e_{1,n}$ *in* $\mathrm{HH}^{m+n}(B)$ *for* $l > 0$, $m, n > 0$.

Theorem 5. *If* $s \geq 2$, $t = 1$ *and* q *is an* r*-th root of unity then*

$$\mathrm{HH}^*(B)/\mathcal{N} \cong$$

$$\begin{cases} \begin{cases} k \oplus k[x^r, y^{2r}]x^r & \mathit{if}\ \bar{a} \neq 0, \bar{b} \neq 0, \\ k \oplus k[x^r, y^2]x^r & \mathit{if}\ \bar{a} = 0, \bar{b} \neq 0, \\ k \oplus k[x, y^2]x & \mathit{if}\ \bar{a} = \bar{b} = 0, \end{cases} & \mathit{if}\ \mathrm{char}\, k = 2, b = 2, r\ \mathit{is\ odd}, \\ \begin{cases} k \oplus k[x^{2r}, y^{2r}]x^{2r} & \mathit{if}\ \bar{a} \neq 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^{2r}]x^2 & \mathit{if}\ \bar{a} \neq 0, \bar{b} = 0, \\ k \oplus k[x^{2r}, y^2]x^{2r} & \mathit{if}\ \bar{a} = 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^2]x^2 & \mathit{if}\ \bar{a} = \bar{b} = 0, \end{cases} & \mathit{others}. \end{cases}$$

where $x^my^n = e_{1,n}$ *in* $\mathrm{HH}^{m+n}(B)$ *for* $m > 0$ *and* $n \geq 0$.

Theorem 6. *If $s = 1$, $t \geq 2$ and q is an r-th root of unity then*

$$\mathrm{HH}^*(B)/\mathcal{N} \cong$$
$$\begin{cases} \begin{cases} k \oplus k[x^{2r}, y^r]x^{2r} & \text{if } \bar{a} \neq 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^r]x^2 & \text{if } \bar{a} \neq 0, \bar{b} = 0, \\ k \oplus k[x^2, y]x^2 & \text{if } \bar{a} = \bar{b} = 0, \end{cases} & \text{if } \operatorname{char} k = 2, a = 2, r \text{ is odd}, \\ \\ \begin{cases} k \oplus k[x^{2r}, y^{2r}]x^{2r} & \text{if } \bar{a} \neq 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^{2r}]x^2 & \text{if } \bar{a} \neq 0, \bar{b} = 0, \\ k \oplus k[x^{2r}, y^2]x^{2r} & \text{if } \bar{a} = 0, \bar{b} \neq 0, \\ k \oplus k[x^2, y^2]x^2 & \text{if } \bar{a} = \bar{b} = 0, \end{cases} & \text{others.} \end{cases}$$

where $x^l = e_{1,0} + \sum_{j=2}^{t} e_{b(j)}$ in $\mathrm{HH}^l(B)$ and $x^m y^n = e_{1,n}$ in $\mathrm{HH}^{m+n}(B)$ for $l > 0$, $m, n > 0$.

It follows from Theorem 4, 5 and 6 that if q is a root of unity then $\mathrm{HH}^*(B)/\mathcal{N}$ is not finitely generated as an algebra. So B negates Snashall-Solberg's conjecture.

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SOURCE ALGEBRAS AND COHOMOLOGY OF BLOCK IDEALS OF FINTIE GROUP ALGEBRAS

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Abstract. Let B a block ideal of the group algebra of a finite group G over a field k with a defect group P. We shall give a criterion for a (kP, kP)-bimodule defined by a (P, P)-double coset to be isomorphic to a direct summand of the source algebra of the block B viewed as a (kP, kP)-bimodule.

Key Words: block ideal, defect group, source algebra, cohomology

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1. Block ideals, source algebras and cohomology rings

Throughout this note we let G denote a finite group and k an algebraically closed field of characteristic p dividing the order of G.

Let B be a block ideal of the group algebra kG; let P be a defect group of B. Let X be a source module of B, which is an indecomposable direct summand of B as $k[G\times P^{\mathrm{op}}]$-module having ΔP as vertex; X has a trivial source. The source module X can be written as $X = kGi$ with a source idempotent i. Let (P, b_P) be the Sylow B-subpair such that the Brauer construction $X(P)$ belongs to b_P; let $\mathcal{F}_{(P,b_P)}(B, X) = \{ (R, b_R) \mid (R, b_R) \subseteq (P, b_P) \}$ be the Brauer category associated with (P, b_P). Then the cohomology ring $H^*(G, B, X)$ of the block B with respect to X is defined to be the subring of the cohomology ring $H^*(P, k)$ of the defect group P consisting of $\mathcal{F}_{(P,b_P)}(B, X)$-stable elements (Linckelmann [4]).

The cohomology ring $H^*(G, B, X)$ is so tightly related to the source algebra $ikGi = X^* \otimes_B X$. Namely

Theorem 1 ([4, Theorem 5.1], [7, Theorem 1]). *Under the notation above an element $\zeta \in H^*(P, k)$ belongs to the cohomology ring $H^*(G, B, X)$ if and only if the diagonal embedding $\delta_P\zeta \in HH^*(kP)$ is $ikGi$-stable, where $ikGi$ is viewed as a (kP, kP)-bimodule.*

Upon this fact the author proposed in [7] a conjecture that the transfer map defined by the source algebra would describe the block cohomology. To be more precise we let $t_{ikGi} : HH^*(kP) \to HH^*(kP)$ be the transfer map defined by $ikGi$ as a (kP, kP)-bimodule. Then we can define a map $t : H^*(P, k) \to H^*(P, k)$ giving rise to the following

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commutative diagram

$$\begin{array}{ccc} H^*(P,k) & \xrightarrow{\delta_P} & HH^*(kP) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t_{ikGi}} \\ H^*(P,k) & \xrightarrow[\delta_P]{} & HH^*(kP) \end{array}.$$

Conjecture. *Under the notation above it would follow that*

$$H^*(G,B,X) = t\,H^*(P,k).$$

If we let

$$ikGi \simeq \bigoplus_{PxP} k[PxP]$$

be a direct sum decomposition of indecomposable (kP,kP)-bimodules, then the map t is described as follows:

$$t : H^*(P,k) \to H^*(P,k); \zeta \mapsto \sum_{PxP} \mathrm{tr}^P \,\mathrm{res}_{P\cap {}^xP}\, {}^x\zeta.$$

However we have had few knowledge for indecomposable direct summands of $ikGi$; we have for an element $x \in G$ outside the inertia group $N_G(P,b_P)$ of the Sylow subpair (P,b_P) almost no information for $k[PxP]$ to be isomorphic to a direct summand of $ikGi$, whereas the direct summands isomorphic to $k[Px]$ for $x \in N_G(P,b_P)$ is so well understood, as we can see in [8, Theorem 44.3].

The aim in this note is to give a criterion for a (kP,kP)-bimodule $k[PgP]$ to be isomorphic to a direct summand of $ikGi$. Here we fix a notation; for a double coset PgP we let

$$t_{PgP} : H^*(P,k) \to H^*(P,k); \zeta \mapsto \mathrm{tr}^P \,\mathrm{res}_{P\cap {}^gP}\, {}^g\zeta.$$

Theorem 2. *Let $(R,b_R),(S,b_S) \subseteq (P,b_P)$; assume that $C_P(R)$ is a defect group of b_R or $C_P(S)$ is a defect group of b_S. For $g \in G$ with ${}^g(R,b_R) = (S,b_S)$ if the map*

$$t_g : H^*(P,k) \to H^*(P,k); \zeta \mapsto \mathrm{tr}^P \,\mathrm{res}_S\, {}^g\zeta$$

does not vanish, then the following hold:

(1) *$S = P \cap {}^gP$; hence $t_g = t_{PgP}$,*
(2) *the (kP,kP)-bimodule $k[PgP]$ is isomorphic to a direct summand of $ikGi$,*
(3) *a (kP,kP)-bimodule $k[Pg'P]$ is isomorphic to $k[PgP]$ if and only if $Pg'P = PcgP$ for some $c \in C_G(S)$.*

Note in the above that the blocks b_R and b_S are considered as blocks in $kC_G(R)$ and $kC_G(S)$, respectively. We prove the theorem above in Section 2.

In Kawai–Sasaki [1] we calculated cohomology rings of 2-blocks of tame representation type and of blocks with defect groups isomorphic to wreathed 2-groups of rank 2. There we constructed transfer maps on the cohomology rings of defect groups; the images of these maps are just the cohomology rings of the blocks. In Section 3 we shall apply Theorem 2 to show that our transfer maps are defined by direct summands of the source algebras of block ideals of tame representation type.

Proof of Theorem 2. We first show in this section the following proposition.

Proposition 3. *Let $P \leqslant G$ be an arbitrary p-subgroup. The (kP, kP)-bimodules $k[PxP]$ and $k[PyP]$, where $x, y \in G$, are isomorphic if and only if $PyP = PcxP$ for some $c \in C_G(P \cap {}^xP)$ with the property that $P \cap {}^xP = P \cap {}^{cx}P$. In this case $P \cap {}^xP$ and $P \cap {}^yP$ are conjucate in P and the transfer maps t_{PxP} and t_{DyD} coincide.*

Proof. Proof. The (kP, kP)-bimodule $k[PxP]$ as a $k[P \times P^{\mathrm{op}}]$-module has ${}^{(x,1)}\Delta({}^{x^{-1}}P \cap P)$ as vertex; and we see that

$$k[PxP] = k[P \times P^{\mathrm{op}}] \otimes_{k[{}^{(x,1)}\Delta({}^{x^{-1}}P \cap P)]} k.$$

Hence we have that

$$\begin{aligned} k[PxP] \simeq k[PyP] \Longleftrightarrow \exists\, (a,b) \in P \times P^{\mathrm{op}}\, &\text{s.t.} \\ {}^{(x,1)}\Delta({}^{x^{-1}}P \cap P) &= {}^{(a,b)}\Big({}^{(y,1)}\Delta({}^{y^{-1}}P \cap P)\Big). \end{aligned}$$

The last equation is equivalent to the following equation:

$$\{\, ({}^xs, s^{-1}) \mid s \in {}^{x^{-1}}P \cap P \,\} = \{\, ({}^{ay}t, b{\cdot}t^{-1}{\cdot}b^{-1}) \mid t \in {}^{y^{-1}}P \cap P \,\}.$$

Here, the multiplication in right component of pairs is in the oposite group P^{op} so that, rewriting it by using the multiplication in P, we obtain that $b{\cdot}t^{-1}{\cdot}b^{-1} = b^{-1}t^{-1}b$. Namely we see for an arbitrary $s \in {}^{x^{-1}}P \cap P$ that there exists a unique element $t \in {}^{y^{-1}}P \cap P$ such that

$${}^xs = {}^{ay}t, \quad s^{-1} = b^{-1}t^{-1}b.$$

The second equation above implies that $t = {}^bs$. Substitute this to the first one to obtain

$${}^xs = {}^{ayb}s.$$

This equation holds for an arbitrary $s \in {}^{x^{-1}}P \cap P$; hence there exists an element $c \in C_G(P \cap {}^xP)$ such that

$$ayb = cx.$$

Note that $P \cap {}^xP = {}^{xb^{-1}}({}^{y^{-1}}P \cap P)$, since $s = {}^{b^{-1}}t$. Then we have that

$$\begin{aligned} P \cap {}^{cx}P = P \cap {}^{ayb}P &= P \cap {}^{ay}P && (\because b \in P) \\ &= {}^a(P \cap {}^yP) && (\because a \in P) \\ &= {}^{cxb^{-1}y^{-1}}(P \cap {}^yP) = {}^{cxb^{-1}}({}^{y^{-1}}P \cap P) = {}^c(P \cap {}^xP) \\ &= P \cap {}^xP. && (\because c \in C_G(P \cap {}^xP)) \end{aligned}$$

Suppose conversely for an element $c \in C_G(P \cap {}^xP)$ that $PyP = PcxP$ with $P \cap {}^xP = P \cap {}^{cx}P$. Then we have

$$\begin{aligned} {}^{(cx)^{-1}}P \cap P = {}^{(cx)^{-1}}(P \cap {}^{cx}P) &= {}^{(cx)^{-1}}(P \cap {}^xP) = {}^{x^{-1}c^{-1}x}({}^{x^{-1}}P \cap P) \\ &= {}^{x^{-1}}P \cap P. \qquad (\because x^{-1}c^{-1}x \in C_G({}^{x^{-1}}P \cap P)) \end{aligned}$$

Then the indecomposable $k[P \times P^{\mathrm{op}}]$-module $k[PcxP]$ has vertex

$$\begin{aligned} {}^{(cx,1)}\Delta({}^{(cx)^{-1}}P \cap P) &= {}^{(cx,1)}\Delta({}^{x^{-1}}P \cap P) = \{\, ({}^{cx}s, s^{-1}) \mid s \in {}^{x^{-1}}P \cap P \,\} \\ &= \{\, ({}^{x}s, s^{-1}) \mid s \in {}^{x^{-1}}P \cap P \,\} = {}^{(x,1)}\Delta({}^{x^{-1}}P \cap P). \end{aligned}$$

Hence we see that $k[PcxP] \simeq k[PxP]$, as desired.

Also wee see, under the condition above, since we can write $cx = ayb$ with suitable a, $b \in P$, that

$$P \cap {}^{x}P = P \cap {}^{cx}P = P \cap {}^{ayb}P = P \cap {}^{ay}P = {}^{a}(P \cap {}^{y}P),$$

hence clearly the last assertion holds. □

Proof of Theorem 2. Because $S \leqslant P \cap {}^{g}P$ we see for $\zeta \in H^*(P,k)$ that

$$\begin{aligned} \mathrm{tr}^P \mathrm{res}_S {}^{g}\zeta &= \mathrm{tr}^P \mathrm{tr}^{P\cap {}^{g}P} \mathrm{res}_S \mathrm{res}_{P\cap {}^{g}P} {}^{g}\zeta \\ &= |\, P \cap {}^{g}P : S \,|\, \mathrm{tr}^P \mathrm{res}_{P\cap {}^{g}P} {}^{g}\zeta. \end{aligned}$$

Hence if $P \cap {}^{g}P > S$, then the map t_g vanishes. Thus we have that $S = P \cap {}^{g}P$.

If $C_P(R)$ is a defect group of b_R, then we see by [3, Lemma 3.3 (iv)] that the (kS, kR)-bimodule $k[gR] = k[Sg]$ is isomorphic with a direct summand of $ikGi$. If on the other hand $C_P(S)$ is a defect group of b_S, then an argument similar to that in the proof of [3, Lemma 3.3 (iv)] tells us that the (kS, kR)-bimodule $k[Sg]$ is isomorphic to a direct summand of $ikGi$. Namely in both cases $k[Sg]$ is isomorphic to a direct summand of $ikGi$ as (kS, kR)-bimodule.

As in the proof of [7, Theorem 1] we can take an indecomposable direct summand $k[PxP]$ of $ikGi$ such that

$$k[Sg] \,\big|\, k[PxP]$$

as (kS, kR)-bimodules; then we can write $g = caxb$ using suitable elements $a, b \in P$ and an element $c \in C_G(S)$. Since $S = P \cap {}^{caxb}P \leqslant {}^{cax}P$, we have that ${}^{a^{-1}}S = {}^{a^{-1}c^{-1}}S \leqslant {}^{x}P$ so that ${}^{a^{-1}}S \leqslant P \cap {}^{x}P$. Therefore it follows that

$$\begin{aligned} \mathrm{res}_S {}^{g}\zeta &= \mathrm{res}_S {}^{caxb}\zeta = \mathrm{res}_S {}^{cax}\zeta \\ &= {}^{c}\mathrm{res}_S {}^{ax}\zeta = \mathrm{res}_S {}^{ax}\zeta && (\because c \in C_G(S)) \\ &= {}^{a}\mathrm{res}_{{}^{a^{-1}}S} {}^{x}\zeta \\ &= {}^{a}\mathrm{res}_{{}^{a^{-1}}S} \mathrm{res}_{P\cap {}^{x}P} {}^{x}\zeta && (\because {}^{a^{-1}}S \leqslant P \cap {}^{x}P) \end{aligned}$$

so that

$$\begin{aligned} \mathrm{tr}^P \mathrm{res}_S {}^{g}\zeta &= \mathrm{tr}^P {}^{a}\mathrm{res}_{{}^{a^{-1}}S} \mathrm{res}_{P\cap {}^{x}P} {}^{x}\zeta \\ &= \mathrm{tr}^P \mathrm{tr}^{P\cap {}^{x}P} \mathrm{res}_{{}^{a^{-1}}S} \mathrm{res}_{P\cap {}^{x}P} {}^{x}\zeta && (\because a \in P) \\ &= \mathrm{tr}^P \,|\, P \cap {}^{x}P : {}^{a^{-1}}S \,|\, \mathrm{res}_{P\cap {}^{x}P} {}^{x}\zeta \\ &= |\, P \cap {}^{x}P : {}^{a^{-1}}S \,|\, \mathrm{tr}^P \mathrm{res}_{P\cap {}^{x}P} {}^{x}\zeta. \end{aligned}$$

This implies that

$${}^{a^{-1}}S = P \cap {}^{x}P, \quad \mathrm{tr}^P \mathrm{res}_S {}^{g}\zeta = \mathrm{tr}^P \mathrm{res}_{P\cap {}^{x}P} {}^{x}\zeta.$$

Thus we see that

$$S = {}^{a}(P \cap {}^{x}P) = P \cap {}^{ax}P = P \cap {}^{c^{-1}g}P.$$

Because $c \in C_G(S)$, we can apply Proposition 3 to $k[PgP]$ and $k[Pc^{-1}gP]$ to conclude that $k[Pc^{-1}gP] \simeq k[PgP]$. Moreover, since $Pc^{-1}gP = PaxbP = PxP$, we have that $k[PgP] \bigm| ikGi$.

Finally we see for $c \in C_G(S)$ that ${}^{cg}(R, b_R) = (S, b_S)$ and that

$$\mathrm{tr}^P \mathrm{res}_S {}^{cg}\zeta = \mathrm{tr}^P {}^{c}\mathrm{res}_S {}^{g}\zeta = \mathrm{tr}^P \mathrm{res}_S {}^{g}\zeta \quad \text{for } \zeta \in H^*(P, k).$$

Hence we have that $P \cap {}^{cg}P = S$. Again Proposition 3 says that $k[PcgP] \simeq k[PgP]$. The "only if" part of our assertion (3) is obvious by Proposition 3. □

3. Tame 2-blocks

Linckelmann [5] says that the family

$$\mathcal{F} = \{\, (S, b_S) \subseteq (P, b_P) \mid (S, b_S) \text{ is extremal and essential} \,\} \cup \{\, (P, b_P) \,\}$$

is a conjugation family.

For a subpair $(S, b_S) \subseteq (P, b_P)$ we consider the following stability condition:

$$\mathrm{S}(S, b_S) \qquad\qquad \mathrm{res}_S {}^{g}\zeta = \mathrm{res}_S \zeta \quad \forall\, g \in N_G(S, b_S).$$

Then we see

$$H^*(G, B, X) = \{\, \zeta \in H^*(P, k) \mid \zeta \text{ satisfies } \mathrm{S}(S, b_S) \text{ for an arbitrary } (S, b_S) \in \mathcal{F} \,\}.$$

In the rest of the note we let $p = 2$ and assume that the block B is of tame representation type; the defect group P is one of the followings:

(1) dihedral 2-group

$$D_n = \langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{-1} \,\rangle,\ n \geqslant 3;$$

(2) generalized quaternion 2-group

$$Q_n = \langle\, x, y \mid x^{2^{n-2}} = y^2 = z, z^2 = 1, yxy^{-1} = x^{-1} \,\rangle,\ n \geqslant 3;$$

(3) semidihedral 2-group

$$SD_n = \langle\, x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{-1+2^{n-2}} \,\rangle,\ n \geqslant 4.$$

The following would be well known.

Proposition 4. (1) *If $P = D_n$ $(n \geqslant 3)$, then*

$$\{\, (E, b_E) \subseteq (P, b_P) \mid E \simeq \textit{a four-group},\ N_G(E, b_E)/C_G(E) \simeq \mathrm{GL}(2, 2) \,\} \cup \{\, (P, b_P) \,\}$$

is a conjugation family.

(2) *If $P = SD_n$ $(n \geqslant 4)$, then*

$$\begin{aligned} &\{\, (E, b_E) \subseteq (P, b_P) \mid E \simeq \textit{a four-group},\ N_G(E, b_E)/C_G(E) \simeq \mathrm{GL}(2, 2) \,\} \\ &\cup \{\, (V, b_V) \subseteq (P, b_P) \mid V \simeq \textit{a quaternion group},\ N_G(V, b_V)/VC_G(V) \simeq \mathrm{GL}(2, 2) \,\} \\ &\cup \{\, (P, b_P) \,\} \end{aligned}$$

is a conjugation family.

(3) *If* $P = Q_n$ $(n \geqslant 4)$, *then*

$\{ (V, b_V) \subseteq (P, b_P) \mid V \simeq$ *a quaternion group*, $N_G(V, b_V)/VC_G(V) \simeq \mathrm{GL}(2,2) \}$
$\cup \{ (P, b_P) \}$

is a conjugation family.

3.1. **Blocks with semidihedral defect groups.** In this subsection we let $P = SD_n$ $(n \geqslant 4)$; let

$$E = \langle x^{2^{n-2}}, y \rangle, \quad V = \langle x^{2^{n-3}}, xy \rangle.$$

Here we state the cohomology ring $H^*(P, k)$ of $P = SD_n = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{-1+2^{n-2}} \rangle$, $n \geqslant 4$. Let $\xi = x^*$, $\eta = y^* \in H^1(P, k)$. Let $\alpha \in H^2(\langle x \rangle, k)$ be the standard element. Let $\nu = \mathrm{norm}^P \alpha \in H^4(P, k)$. Choose an element $\theta \in H^3(P, k)$ appropriately; we can describe as follows:

$$H^*(P, k) = k[\xi, \eta, \theta, \nu]/(\xi^2 - \xi\eta, \xi^3, \xi\theta, \theta^2 - \eta^6 - \eta^2\nu - \xi^2\nu).$$

We constructed in Kawai–Sasaki [1] a transfer map from $H^*(P, k)$ to $H^*(G, B, X)$.

From now on we assume that $N_G(E, b_E)/C_G(E) \simeq \mathrm{GL}(2,2)$ and $N_G(V, b_V)/VC_G(V) \simeq \mathrm{GL}(2,2)$. Let ω and ω' are automorphisms of E and V of order three, respectively. Then the cohomology ring of the block is described as follows:

$$H^*(G, B, X) = \{ \zeta \in H^*(P, k) \mid \mathrm{res}_E \zeta = \mathrm{res}_E {}^{\omega}\zeta, \mathrm{res}_V \zeta = \mathrm{res}_V {}^{\omega'}\zeta \}.$$

We let $g_0 \in N_G(E, b_E)$ and $g_1 \in N_G(V, b_V)$ induce the automorphisms $\omega \in \mathrm{Aut}\, E$ and $\omega' \in \mathrm{Aut}\, V$, respectively:

(1) $\langle x^{2^{n-3}}, g_0 \rangle C_G(E)/C_G(E) = N_G(E, b_E)/C_G(E) \simeq \mathrm{GL}(2,2)$,
(2) $\langle x^{2^{n-4}}, g_1 \rangle VC_G(V)/VC_G(V) = N_G(V, b_V)/VC_G(V) \simeq \mathrm{GL}(2,2)$.

Definition 5. We let

$$\mathrm{Tr}_P^B : H^*(P, k) \to H^*(P, k); \; \zeta \mapsto \zeta + \mathrm{tr}^P \mathrm{res}_E {}^{g_0}\zeta + \mathrm{tr}^P \mathrm{res}_V {}^{g_1}\zeta.$$

Theorem 6. *The image of* Tr_P^B *above coincides with the cohomology ring* $H^*(G, B, X)$.

Since (E, b_E), $(V, b_V) \subseteq (P, b_P)$ are extremal, we see that $C_P(E)$ and $C_P(V)$ are defect groups of b_E and b_V, respectively. Theorem 2 together with the facts that the maps $[\zeta \mapsto \mathrm{tr}^P \mathrm{res}_E {}^{g_0}\zeta]$ and $[\zeta \mapsto \mathrm{tr}^P \mathrm{res}_V {}^{g_1}\zeta]$ do not vanish implies that both of (kP, kP)-bimodules $k[Pg_0P]$ and $k[Pg_1P]$ are isomorphic to direct summands of $ikGi$; we obtain the following theorem.

Theorem 7. *Let* $M = kP \oplus k[Pg_0P] \oplus k[Pg_1P]$. *Then*

(1) $M \bigm| ikGi$;
(2) *the map* Tr_P^B *is induced by the transfer map* $t_M : HH(kP) \to HH(kP)$;
(3) *an element* $\zeta \in H^*(P, k)$ *belongs to* $H^*(G, B, X)$ *if and only if* $\delta_P\zeta \in HH^*(kP)$ *is* M*-stable.*

In the other cases of the iniertia quotients, we have similar results.

Suppose that the (kP, kP)-bimodules $k[PgP]$ is isomorphic to a direct summand of (kP, kP)-bimodule $ikGi$.

Let $R = {}^{g^{-1}}P \cap P$ and $S = P \cap {}^{g}P$ and let (R, b_R), $(S, b_S) \subseteq (P, b_P)$. Then Külshammer–Okuyama–Watanabe [2, Proposition 5] says that

$$^{g}(R, b_R) = (S, b_S) \subseteq (P, b_P).$$

The Brauer category $\mathcal{F}_{(P,b_P)}(B, X)$ is well understood so that the posibilities of fusions above are completely described and the transfer maps t_{PgP} are also determined.

Hence we obtain the following.

Theorem 8. *The source algebra ikGi induces, as a (kP, kP)-bimodule, the transfer map $t_{ikGi} : HH^*(kP) \to HH^*(kP)$ whose restriction to the cohomology ring $H^*(P, k)$ maps $\zeta \in H^*(P, k)$ as follows:*

$$\zeta \mapsto \zeta + l_0 \operatorname{tr}^P \operatorname{res}_E {}^{g_0}\zeta + l_1 \operatorname{tr}^P \operatorname{res}_V {}^{g_1}\zeta.$$

Here $l_0, l_1 \in \mathbf{Z}$.

3.2. **Blocks with dihedral or quaternion defect groups.** In the case of $P = D_n$ ($n \geqslant 3$), let us take four-groups

$$E_0 = \langle\, x^{2^{n-2}}, y \,\rangle, \quad E_1 = \langle\, x^{2^{n-2}}, xy \,\rangle.$$

In the case of $P = Q_n$ ($n \geqslant 4$, let us take quaternion groups

$$V_0 = \langle\, x^{2^{n-3}}, y \,\rangle, \quad V_1 = \langle\, x^{2^{n-3}}, xy \,\rangle.$$

Then we can construct the transfer maps $\operatorname{Tr}_P^B : H^(P, k) \to H^(P, k)$ whose images are $H^*(G, B, X)$s, by similar constructions in semidihedral case.

We have also results corresponding to Theorems 7 and 8.

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COMPLEMENTS AND CLOSED SUBMODULES RELATIVE TO TORSION THEORIES

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ABSTRACT. A submodule of a module M is called to be closed if it has no proper essential extensions in M. A submodule X of M is called to be a complement if it is maximal with respect to $X \cap Y = 0$, for some submodule Y of M. It is well known that closed and complement submodule are the same. A module M is called to be extending (M has condition (C_1)) if any submodule of M is essential in a summand of M. It is known that quasi-injective module is extending. In this note we generalize this by using hereditary torsion theories and state related results.

1. INTRODUCTION

Throughout this paper R is a ring with a unit element, every right R-module is unital and Mod-R is the category of right R-modules. A subfunctor of the identity functor of Mod-R is called a preradical. For preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R | \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R | \sigma(M) = 0\}$ is the class of σ-torsion free right R-modules. A preradical t is called to be idempotent(a radical) if $t(t(M)) = t(M)(t(M/t(M)) = 0)$. Let $\mathcal{C}$ be a subclass of Mod-R. A torsion theory for $\mathcal{C}$ is a pair of $(\mathcal{T}$,$\mathcal{F})$ of classes of objects of $\mathcal{C}$ such that (i) $\text{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$. (ii) If $\text{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$. (iii) If $\text{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$. It is well known that $(\mathcal{T}_t, \mathcal{F}_t)$ is a torsion theory for an idempotent radical t. A preradical t is called to be left exact if $t(N) = N \cap t(M)$ holds for any module M and its submodule N. For a preradical σ and a module M and its submodule N, N is called to be σ-dense submodule of M if $M/N \in \mathcal{T}_\sigma$. If N is an essential and σ-dense submodule of M, then N is called to be a σ-essential submodule of M(M is a σ-essential extension of N). If N is essential in M, we denote $N \subseteq^e M$. If N is σ-essential in M, we denote $N \subseteq^{\sigma e} M$. For an idempotent radical σ a module M is called to be σ-injective if the functor $\text{Hom}_R(\_, M)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$. We denote $E(M)$ the injective hull of a module M. For an idempotent radical σ, $E_\sigma(M)$ is called the σ-injective hull of a module M, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$. Then even if σ is not left exact, $E_\sigma(M)$ is σ-injective and a σ-essential extension of M, is a maximal σ-essential extension of M and is a minimal σ-injective extension of M. If N is σ-essential in M, then it holds that $E_\sigma(N) = E_\sigma(M)$. Let B be a submodule of a module M. We call B is σ-essentially closed in M if B has no proper σ-essential extension in M.

The final version of this paper will be submitted for publication elsewhere.

2. COMPLEMENT AND CLOSED SUBMODULE

First we state σ-essentially closed submodules and complement submodules relative to torsion theories. Following proposition generalize Proposition 1.4 in [2].

Proposition 1. *Let σ be a left exact radical and B be a submodule of a module M. We denote $\overline{B}/B := \sigma(M/B)$. Then the following conditions from* (1) *to* (9) *are equivalent.*

(1) *B is essentially closed in $\overline{B}$.*
(2) *B is σ-essentially closed in M.*
(3) *B is a complement of a submodule in $\overline{B}$.*
(4) *If X is a complement of B in $\overline{B}$, then B is a complement of X in $\overline{B}$.*
(5) *It holds that $B = E_\sigma(B) \cap M$.*
(6) *If $B \subseteq X \subseteq^e \overline{B}$, then $X/B \subseteq^e \overline{B}/B$.*
(7) *It holds that $B = E(B) \cap \overline{B}$.*
(8) *There exists submodules M_1 and K of M such that $K \subseteq M_1$, $M/M_1 \in \mathcal{F}_\sigma$ and B is a complement of K in M_1.*
(9) *If $B \subseteq X \subseteq^{\sigma e} M$, then $X/B \subseteq^{\sigma e} M/B$.*

Proof. (2)→(1): Let B be σ-essentially closed in M. Let H a module such that $B \subseteq^e H \subseteq \overline{B}$. Since $H/B \subseteq \overline{B}/B = \sigma(M/B) \in \mathcal{T}_\sigma$, $H/B \in \mathcal{T}_\sigma$. Thus $B \subseteq^{\sigma e} H \subseteq M$, and so $H = B$ by (2).

(1)→(2): Let B be essentially closed in $\overline{B}$. Let N be a module such that $B \subseteq^{\sigma e} N \subseteq M$, and so $B \subseteq^e N$ and $N/B \in \mathcal{T}_\sigma$. Then $N/B \subseteq \sigma(M/B) = \overline{B}/B$. Thus it holds that $B \subseteq^e N \subseteq \overline{B}$. By (1), $B = N$.

(2)→(6): Suppose that B is σ-essentially closed in M and X an essential submodule of $\overline{B}$ containing B. Let Y/B be a submodule of $\overline{B}/B$ such that $X/B \cap Y/B = \overline{0}$. Then $X \cap Y = B$. Since X is essential in $\overline{B}$, $B = Y \cap X$ is essential in $Y \cap \overline{B} = Y$. Since $Y/B = Y/(Y \cap X) \cong (Y + X)/X \subseteq \overline{B}/X \twoheadleftarrow \overline{B}/B \in \mathcal{T}_\sigma$, B is σ-essential in Y. As B is σ-essentially closed in M, it follows that $Y = B$. Thus X/B is essential in $\overline{B}/B$.

(6)→(4): Let X be a complement of B in $\overline{B}$. Let B' be a complement of X in $\overline{B}$ containing B. Then $(X \oplus B) \cap B' = (X \cap B') \oplus B = B$. Thus $((X \oplus B)/B) \cap (B'/B) = \overline{0}$. Since $X \oplus B$ is essential in $\overline{B}$, it holds that $(X \oplus B)/B$ is essential in $\overline{B}/B$ by (6) . Since $((X \oplus B)/B) \cap (B'/B) = \overline{0}$, then $B' = B$, as desired.

(4)→(3): Since there exists a complement of B in $\overline{B}$, it is obvious.

(3)→(2): Let B be a complement of a submodule K of $\overline{B}$. Then B is essentially closed in $\overline{B}$. We show that B is σ-essentially closed in M. Let B' be a submodule of M such that B' is a σ-essential extension of B. Then $B \cap \overline{B} = B$ is essential in $B' \cap \overline{B}$. Since B is essentially closed in $\overline{B}$, $B = B' \cap \overline{B}$. Since $\mathcal{T}_\sigma \ni B'/B = B'/(B' \cap \overline{B}) \cong (B' + \overline{B})/\overline{B} \subseteq M/\overline{B} \cong (M/B)/\sigma(M/B) \in \mathcal{F}_\sigma$, it follows that $B' = B$, as desired.

(2)→(5): It is easily verified that $E_\sigma(B) \cap M$ is σ-essential extension of B in M. By (2), it follows that $E_\sigma(B) \cap M = B$.

(5)→(2): Let X be a module such that $B \subseteq X \subseteq M$ and B is σ-essential in X. Then $E_\sigma(B) = E_\sigma(X)$. By (5), $B = E_\sigma(B) \cap M$. Since $B \subseteq X \subseteq E_\sigma(X) \cap M = E_\sigma(B) \cap M = B$, it follows that $X = B$, as desired.

(1)→(7): Since $E(B) \cap \overline{B}$ is essential extension of B in $\overline{B}$, it holds that $B = E(B) \cap \overline{B}$.

(7)$\to$(1): Let X be a module such that $B \subseteq X \subseteq \overline{B}$ such that X is σ-essential extension of B. Then it follows that $E(X) = E(B)$. Since $B \subseteq X \subseteq E(X) \cap \overline{B} = E(B) \cap \overline{B} = B$, it concludes that $B = X$.

(2)$\to$(8): Let B be σ-essentially closed in M. Then $M/\overline{B} \in \mathcal{F}_\sigma$. We take a complement K of B in $\overline{B}$. Then $B \oplus K$ is essential in $\overline{B}$ and $(B \oplus K)/K$ is essential in $\overline{B}/K$. We take a complement L of K containing B in $\overline{B}$. Since $(B \oplus K)/K$ is σ-essential in $\overline{B}/K$, $(B \oplus K)/K$ is σ-essential in $(L \oplus K)/K$. Thus L is σ-essential extension of B. Thus by (2) $B = L$, and so B is a complement of K in $\overline{B}$.

(8)$\to$(2): Suppose that there exists submodules M_1 and K of M such that $K \subseteq M_1$, $M/M_1 \in \mathcal{F}_\sigma$ and B is a complement of K in M_1. Then B is essentially closed in M_1. We show that B is σ-essentially closed in M. Let B_1 be a submodule of M such that B is σ-essential in B_1. Then $B = B \cap M_1$ is essential in $B_1 \cap M_1 (\subseteq M_1)$. Since B is essentially closed in M_1, $B = B_1 \cap M_1$. Since $\mathcal{T}_\sigma \ni B_1/B = B_1/(B_1 \cap M_1) \cong (B_1 + M_1)/M_1 \subseteq M/M_1 \in \mathcal{F}_\sigma$, it follows that $B_1 = B$.

(2)$\to$(9): Suppose that B is σ-essentially closed in M. Let X be a submodule of M such that $B \subseteq X \subseteq^{\sigma e} M$. Let Q be a submodule of M containing B such that $(X/B) \cap (Q/B) = 0$. Then $B = Q \cap X \subseteq^{e} Q \cap M = Q$. Since $Q/B = Q/(Q \cap X) \cong (Q + X)/X \subseteq M/X \in \mathcal{T}_\sigma$, it holds that $B \subseteq^{\sigma e} Q \subseteq M$. Since B is σ-essentially closed in M, $B = Q$, and so $(Q/B) = 0$. Thus X/B is σ-essential in M/B.

(9)$\to$(2): Suppose that $B \subseteq^{\sigma e} X \subseteq M$. Let B' be a complement of B in M. Then $B \oplus B' \subseteq^{\sigma e} M$ and hence by (9) $(B \oplus B')/B \subseteq^{\sigma e} M/B$. Since $B \cap (B' \cap X) = 0$, $B' \cap X = 0$. Since $((B \oplus B')/B) \cap (X/B) = [(B \oplus B') \cap X]/B = [B \oplus (B' \cap X)]/B = 0$, $(X/B) = 0$, as desired. □

3. σ-QUASI-INJECTIVE MODULE

We call A σ-M-injective if $\mathrm{Hom}_R(-, A)$ preserves the exactness for any exact sequence $0 \to N \to M \to M/N \to 0$, where $M/N \in \mathcal{T}_\sigma$. The following proposition is a generalization of Theorem 15 in [1].

Proposition 2. *Let σ be a left exact radical. Then A is σ-M-injective if and only if $f(M) \subseteq A$ for any $f \in Hom_R(E_\sigma(M), E_\sigma(A))$.*

Proof. ($\leftarrow$): Let σ be an idempotent radical and N be a submodule of M such that $M/N \in \mathcal{T}_\sigma$. Since $E_\sigma(M)/M \in \mathcal{T}_\sigma$ and $\mathcal{T}_\sigma$ is closed under taking extensions, it follows that $E_\sigma(M)/N \in \mathcal{T}_\sigma$. Consider the following diagram.

$$\begin{array}{l} 0 \to N \to E_\sigma(M) \to E_\sigma(M)/N \to 0 \\ \qquad \downarrow_f \qquad \downarrow_g \\ 0 \to A \longrightarrow E_\sigma(A) \end{array}$$

For any $f \in \mathrm{Hom}_R(N, A)$, f is extended to $g \in \mathrm{Hom}_R(E_\sigma(M), E_\sigma(A))$. By the assumption it follows that $g(M) \subseteq A$, and so f is extended to $g|_M \in \mathrm{Hom}_R(M, A)$, as desired.

($\to$): Let σ be a left exact radical and $f \in \mathrm{Hom}_R(E_\sigma(M), E_\sigma(A))$. Then $f|_{M \cap f^{-1}A} \in \mathrm{Hom}_R(M \cap f^{-1}(A), A)$. Since $M/(M \cap f^{-1}(A)) \simeq (M + f^{-1}(A))/f^{-1}(A) \simeq (f(M) + A)/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$, $M/(M \cap f^{-1}(A)) \in \mathcal{T}_\sigma$. Consider the following diagram.

$$\begin{array}{ccccccc} 0 \to & M \cap f^{-1}(A) & \to & M & \to & M/(M \cap f^{-1}(A)) & \to 0 \\ & {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & & & & \\ & A & & & & & \end{array}$$

Thus by the assumption $f|_{M\cap f^{-1}(A)}$ is extended to $g \in \mathrm{Hom}_R(M, A)$, and so $(g-f)(M\cap f^{-1}(A)) = 0$. Hence we obtain $ker(g - f) \supseteq M \cap f^{-1}(A)$. If $x \in (g - f)^{-1}(A)$, then there exists an $a \in A$ such that $g(x) - f(x) = a$, and then $f(x) = g(x) - a \in A$ and so $x \in f^{-1}(A)$. It follows that $(g - f)^{-1}(A) \subseteq f^{-1}(A)$, and so $M \cap (g - f)^{-1}(A) \subseteq M \cap f^{-1}(A) \subseteq \ker(g - f)$. If $a = (g - f)(m) \in (g - f)M \cap A$ for $a \in A$ and $m \in M$, then $m \in (g - f)^{-1}a \subseteq M \cap (g - f)^{-1}A \subseteq \ker(g - f)$, and so $0 = (g - f)(m) = a$. Thus it follows that $(g - f)M \cap A = 0$. Since A is essential in $E_\sigma(A)$, $(g - f)M = 0$, and so we obtain that $f(M) = g(M) \subseteq A$, as desired. □

We obtain the following corollary as a torsion theoretic generalization of the Johnson Wong theorem [4] by putting $M = A$ in Proposition 2. We call a module A σ-quasi-injective if A is σ-A-injective.

Corollary 3. *Let σ be a left exact radical. Then A is σ-quasi-injective if and only if $f(A) \subseteq A$ for any $f \in Hom_R(E_\sigma(A), E_\sigma(A))$.*

The following lemma generalizes Proposition 2.3 in [3].

Lemma 4. *If A is σ-quasi-injective and $E_\sigma(A) = M \oplus N$, then $A = (M \cap A) \oplus (N \cap A)$.*

Proof. Let $p_M(p_N)$ be a canonical projection from $E_\sigma(A)$ to $M(N)$ respectively. Then by Corollay 3, it follows that $p_M(A) \subseteq A$ and $p_N(A) \subseteq A$. If $A \ni a = m + n \in M + N$ for $m \in M$ and $n \in N$, then $A \ni p_M(a) = p_M(m + n) = m \in M$, and so $m \in A \cap M$, and it is similarly proved that $n \in A \cap N$. Thus $A \subseteq (M \cap A) \oplus (N \cap A)$, as desired. □

4. (σ-C_i) CONDITIONS

Next we consider (C_i) conditions relative to torsion theories. For (C_i) conditions, see [5]. We call a module M σ-quasi-injective if for any σ-dense submodule N of M, $\mathrm{Hom}_R(\_, M)$ preserves the exactness of a short exact sequence $0 \to N \to M \to M/N \to 0$. The following proposition generalize Proposition 2.1 in [5]. We call a module M has (σ-C_1) if every σ-dense submodule of M is essential in a summand of M. We call a module M has (σ-C_2) if a σ-dense submodule A of M is isomorphic to a summand A_1 of M, then A is a summand of M.

From now on we assume that σ is a left exact radical.

Proposition 5. *Any σ-quasi-injective module M has (σ-C_1) and (σ-C_2).*

Proof. (σ-C_1): Let N be a σ-dense submodule of a σ-quasi-injective module M. Consider the exact sequence $0 \to M/N \to E_\sigma(M)/N \to E_\sigma(M)/M \to 0$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, it follows that $E_\sigma(M)/N \in \mathcal{T}_\sigma$. Since $\mathcal{T}_\sigma$ is closed under taking factor modules, it holds that $E_\sigma(M)/E_\sigma(N) \in \mathcal{T}_\sigma$. As $E_\sigma(N)$ is σ-injective, there exists a submodule E of $E_\sigma(M)$ such that $E_\sigma(M) = E_\sigma(N) \oplus E$. Since M is σ-quasi-injective, it follows that $M = (M \cap E_\sigma(N)) \oplus (E \cap M)$ by Lemma 4. Thus N is σ-essential in $M \cap E_\sigma(N)$ which is a summand of M, as desired. (σ-C_2): Since M is σ-quasi-injective,

M is σ-M-injective. As A_1 is a direct summand of M, A_1 is σ-M-injective. Consider the following exact sequence.

$$\begin{array}{lllll} 0 \to A & \overset{g}{\to} & M & \to M/A \to 0 & (\text{with } M/A \in \mathcal{T}_\sigma) \\ \downarrow_h & & \downarrow_f & & \\ A_1 & \subseteq_\oplus & M & & \end{array}$$

, where h is isomorphism from A to A_1 and f is a homomorphism from M to A_1 such that $fg = h$. It is easily verified that A is a summand of M. □

We call a module M has (σ-C_3) if M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$ and $M/(M_1 \oplus M_2) \in \mathcal{T}_\sigma$, then $M_1 \oplus M_2$ is a summand of M. We call a module M has (σ-C_3') if M_1 and M_2 are summands of M such that $M_1, M/M_2 \in \mathcal{T}_\sigma$ and $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M. It is easily verified that (σ-C_3)$\Rightarrow$(σ-C_3'). The following proposition generalize Proposition 2.2 in [5].

Proposition 6. *If a module M has (σ-C_2), then M has (σ-C_3').*

Proof. Let M_1 and M_2 be summands of M such that $M_1, M/M_2 \in \mathcal{T}_\sigma$ and $M_1 \cap M_2 = 0$. Since M_1 is a summand of M, there exists a submodule M_1^* such that $M = M_1 \oplus M_1^*$. Let π be a projection $M = M_1 \oplus M_1^* \to M_1^*$. By modular law, $M_1 \oplus M_2 = M \cap (M_1 \oplus M_2) = (M_1 \oplus M_1^*) \cap (M_1 \oplus M_2) = M_1 \oplus (M_1^* \cap (M_1 \oplus M_2))$. Thus $\pi(M_2) = \pi(M_1 \oplus M_2) = \pi(M_1 \oplus (M_1^* \cap (M_1 \oplus M_2))) = M_1^* \cap (M_1 \oplus M_2)$. Thus $M_1 \oplus M_2 = M_1 \oplus \pi(M_2)$ and $\pi(M_2) \subseteq M_1^*$. Then $\ker \pi|_{M_2} = \ker \pi \cap M_2 = M_1 \cap M_2 = 0$, $\pi|_{M_2} : M_2 \twoheadrightarrow \pi(M_2) (\subseteq M)$ is an isomorphism. Since $M_1^*/\pi(M_2) \simeq M/M_2 \in \mathcal{T}_\sigma$ and $M/M_1^* \simeq M_1 \in \mathcal{T}_\sigma$, the middle term of $0 \to M_1^*/\pi(M_2) \to M/\pi(M_2) \to M/M_1^* \to 0$ is in $\mathcal{T}_\sigma$. Thus $\pi(M_2)$ is σ-dense submodule of M. Thus we get $\pi(M_2) \subseteq^\oplus M$ by (σ-C_2). Thus there exists a module X such that $M = X \oplus \pi(M_2)$. By modular law, $M_1^* = (X \cap M_1^*) \oplus \pi(M_2)$. Thus $M = M_1 \oplus M_1^* = M_1 \oplus (X \cap M_1^*) \oplus \pi(M_2) = (M_1 \oplus \pi(M_2)) \oplus (X \cap M_1^*) = M_1 \oplus M_2 \oplus (X \cap M_1^*)$, and so $M_1 \oplus M_2 \subseteq^\oplus M$. □

We call a module of M σ-continuous if it has (σ-C_1) and (σ-C_2). We call a module M σ-quasi-continuous if it has (σ-C_1) and (σ-C_3'). We have just seen that the following implications hold: σ-injective $\Rightarrow$ σ-quasi-injective $\Rightarrow$ σ-continuous $\Rightarrow$ σ-quasi-continuous $\Rightarrow$ σ-C_1

Proposition 7. *A module M has (σ-C_1) if and only if every essentially closed σ-dense-submodule of M is a summand of M.*

Proof. $\Rightarrow$): Let N be an essentially closed σ-dense submodule of M. Since $M/N \in \mathcal{T}_\sigma$, there exists a decomposition $M = X \oplus Y$ such that $N \subseteq^{\sigma e} X \subseteq M$. As N is essentially closed in M and so $N = X$. Thus $M = N \oplus Y$.

$\Leftarrow$): Let N be a σ-dense submodule of M. Let X be a complement of N in M and Y be a complement of X in M containing N . Then Y is essentially closed σ-dense in M. By the assumption Y is a summand of M. We show that N is essential in Y. If N is not essential in Y, there exists a nonzero submodule H of Y such that $N \cap H = 0$. If $N \cap (X \oplus H) \ni n = x + h$, where $n \in N, x \in X$ and $h \in H$. Then $x = n - h \in X \cap Y = 0$. Thus $x = 0$, and so $n = h \in N \cap H = 0$. Therefore $N \cap (X \oplus H) = 0$. By construction of X, $X = X \oplus H$, and so $H = 0$. Thus N is essential in Y. Thus if $M/N \in \mathcal{T}_\sigma$, then there exists a submodule Y of M such that $N \subseteq^e Y$ and Y is a summand of M. □

Proposition 8. *For a submodule A of a module M, if A is σ-essentially closed in a summand of M, then A is σ-essentially closed in M.*

Proof. Let $M = M_1 \oplus M_2$ with A σ-essentially closed in M_1. Let π denote the projection $M_1 \oplus M_2 \twoheadrightarrow M_1$. Assume that $A \subseteq^{\sigma e} B \subseteq M$. It is easy to see that $A = \pi(A) \subseteq^{\sigma e} \pi(B) \subseteq M_1$. Since A is σ-essentially closed in M_1, $\pi(B) = A \subseteq B$, and so $(1-\pi)(B) \subseteq B$. Since $(1-\pi)(B) \cap A = 0$ and $A \subseteq^e B$, $(1-\pi)(B) = 0$. Thus $A \subseteq^{\sigma e} B = \pi(B) \subseteq M_1$. Since A is σ-essentially closed in M_1, it holds that $A = B$. □

Lemma 9. *If $M = A \oplus B$ and $A \subseteq^e K \subseteq M$, then $K = A$.*

Proof. By modular law it follows that $K = A \oplus (K \cap B)$, and so $A \cap (K \cap B) = 0$. Since A is essential in K, $K \cap B = 0$, and so $K = A \oplus (K \cap B) = A$. □

The following proposition generalize Theorem 2.8 in [5].

Proposition 10. *Consider the following conditions.*

It holds that (3) $\Leftrightarrow$ (4) $\rightarrow$ (1) $\rightarrow$ (2). If $\ker f \in \mathcal{T}_\sigma$ for any idempotent $f \in End_R(E_\sigma(M))$, then (2) $\rightarrow$ (3) holds.

(1) *M has (σ-C_1) and (σ-C_3).*
(2) *$M = X \oplus Y$ for σ-dense submodules X, Y of M such that X is a complement of Y in M and Y is a complement of X in M.*
(3) *$f(M) \subseteq M$ for any idempotent f in $End_R(E_\sigma(M))$.*
(4) *If $E_\sigma(M) = \oplus E_i$, then $M = \oplus\ (M \cap E_i)$.*

Proof. (1)$\rightarrow$(2): Let X and Y be σ-dense submodules of M such that X is a complement of Y in M and Y is a complement of X in M. Since X and Y are essentially closed in M, X and Y are direct summands of M by (σ-C_1). Then $X \oplus Y$ is σ-essential in M. By (σ-C_3), $X \oplus Y$ is a direct summand of M, and so $M = X \oplus Y \oplus Z \supseteq^e (X \oplus Y)$. Therefore it follows that $Z = 0$, and so $M = X \oplus Y$.

(2)$\rightarrow$(3): We assume that $\ker f \in \mathcal{T}_\sigma$ for any idempotent $f \in \mathrm{End}_R(E_\sigma(M))$. Let $A_1 = M \cap f(E_\sigma(M))$ and $A_2 = M \cap (1-f)(E_\sigma(M))$. Then $A_1 \cap A_2 = 0$. Since $E_\sigma(M) = f(E_\sigma(M)) \oplus \ker f$ for any idempotent f in $\mathrm{End}_R(E_\sigma(M))$ and $M/A_i \simeq (M + f(E_\sigma(M)))/f(E_\sigma(M)) \subseteq E_\sigma(M)/f(E_\sigma(M)) \simeq \ker f \in \mathcal{T}_\sigma$, $M/A_i \in \mathcal{T}_\sigma$ for $i = 1, 2$. Let B_1 be a complement of A_1 containing A_2 in M and B_2 be a complement of B_1 containing A_2 in M. Then by (2) $M = B_1 \oplus B_2$. Let π be a projection $B_1 \oplus B_2 \twoheadrightarrow B_1$. We claim that $M \cap (f-\pi)(M) = 0$. Let $x, y \in M$ such that $(f-\pi)(x) = y$. Then $f(x) = y + \pi(x) \in M$, and so $f(x) \in A_1$. Moreover $(1-f)(x) \in M$, and so $(1-f)(x) \in A_2$. Therefore $x = f(x) \oplus (1-f)(x) \in A_1 \oplus A_2 \subseteq B_1 \oplus B_2 = M$. $\pi(x) = \pi(f(x)) + \pi(1-f)(x) = f(x) + 0$, and so $y = 0$. Thus $M \cap (f-\pi)(M) = 0$. Since M is essential in $E_\sigma(M)$, $(f-\pi)(M) = 0$, and so $f(M) = \pi(M) \subseteq M$.

(3)$\rightarrow$(4): Let $E_\sigma(M) = \underset{i\in I}{\oplus} E_i$, then it is clear that $M \supseteq \underset{i\in I}{\oplus}(M \cap E_i)$. Let m be an element of $M \subseteq E_\sigma(M) = \underset{i\in I}{\oplus} E_i$. Then there exists a finite index subset F of I such that $m \in \underset{i\in F}{\oplus} E_i$. Write $E_\sigma(M) = (\underset{i\in F}{\oplus} E_i) \oplus (\underset{i\in I-F}{\oplus} E_i)$. Then there exist orthogonal idempotents $f_i \in \mathrm{End}_R(E_\sigma(M))(i \in F)$ such that $E_i = f_i(E_\sigma(M))$. Since $f_i(M) \subseteq M$ by (3), $m = \sum_{i\in F} f_i(m) \in \underset{i\in F}{\oplus}(M \cap E_i)$. Thus $M \subseteq \underset{i\in I}{\oplus}(M \cap E_i)$, and $M = \underset{i\in I}{\oplus}(M \cap E_i)$.

(4)→(1): Let A be a σ-dense submodule of M. Consider the following exact sequence. $0 \to M/A \to E_\sigma(M)/A \to E_\sigma(M)/M \to 0$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, $E_\sigma(M)/A \in \mathcal{T}_\sigma$. As $E_\sigma(M)/A \twoheadrightarrow E_\sigma(M)/E_\sigma(A)$, $E_\sigma(M)/E_\sigma(A) \in \mathcal{T}_\sigma$. Thus $0 \to E_\sigma(A) \to E_\sigma(M) \to E_\sigma(M)/E_\sigma(A) \to 0$ splits. Then $E_\sigma(M) = E_\sigma(A) \oplus E$. By (4) $M = (M \cap E_\sigma(A)) \oplus (M \cap E)$. Since $(M \cap E_\sigma(A))/A \subseteq E_\sigma(A)/A \in \mathcal{T}_\sigma$, A is σ-essential in $M \cap E_\sigma(A)$ which is a direct summand of M. Thus M has (σ-C_1).

Let M_1 and M_2 be direct summands of M such that $M_1 \cap M_2 = 0$ and $M/M_1, M_2 \in \mathcal{T}_\sigma$. Then $M/(M_1 \oplus M_2) \in \mathcal{T}_\sigma$. Consider the following exact sequence. $0 \to M/(M_1 \oplus M_2) \to E_\sigma(M)/(M_1 \oplus M_2) \to E_\sigma(M)/M \to 0$. Thus $E_\sigma(M)/(M_1 \oplus M_2) \in \mathcal{T}_\sigma$. Thus $E_\sigma(M)/(E_\sigma(M_1) \oplus E_\sigma(M_2)) \in \mathcal{T}_\sigma$. Thus $0 \to E_\sigma(M_1) \oplus E_\sigma(M_2) \to E_\sigma(M) \to E_\sigma(M)/(E_\sigma(M_1) \oplus E_\sigma(M_2)) \to 0$ splits. Thus there exists a submodule E of $E_\sigma(M)$ such that $E_\sigma(M) = E_\sigma(M_1) \oplus E_\sigma(M_2) \oplus E$. Then by (4) $M = (M \cap E_\sigma(M_1)) \oplus (M \cap E_\sigma(M_2)) \oplus (M \cap E)$. Since M_i is a summand of M and M_i is essential in $M \cap E_\sigma(M_i)$, $M_i = M \cap E_\sigma(M_i)$ by Lemma 9. Thus $M = M_1 \oplus M_2 \oplus (M \cap E)$, as desired. Thus M has (σ-C_3).

(4)→(3): $\mathrm{End}_R(E_\sigma(M)) \ni f = f^2$, then $E_\sigma(M) = f(E_\sigma(M)) \oplus f^{-1}(0)$. By (4) $M = (M \cap f(E_\sigma(M))) \oplus (M \cap f^{-1}(0))$. For any $m \in M$, there exists $x \in M \cap f(E_\sigma(M))$ and $y \in M \cap f^{-1}(0)$ such that $m = x + y$. Then $f(m) = f(x) + f(y) = x + 0 \in M$, and so $f(M) \subseteq M$. □

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