# ON A GENERALIZATION OF COMPLEXES AND THEIR DERIVED CATEGORIES.

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ABSTRACT. When we want to understand the reason why the equation $d^2 = 0$ has the beautiful consequences, one way is to consider generalizations of it and research how its properties vary. One natural candidate of a generalization is the notion of $N$-complex, that is, gradeds object equipped with a morphism $d$ of degree 1 such that $d^N = 0$. This was introduced by Kapranov [5] and Sarkaria [7] independently. Nowadays there is a vast collection of literatures on the subject.

For an $N$-complex $X$, there are several cohomology functors. More precisely, for $1 \le r \le N-1$, we define a cohomorogy functor to be

$$\mathrm{H}^i_{(r)}(X) := \frac{\mathrm{Ker}[d^r : X^i \to X^{i+r}]}{\mathrm{Im}[d^{N-r} : X^{i-N+r} \to X^i]}.$$

As a new feature, it is observed that there are several relations between these cohomology functors [5, 1].

On the other hands, Iyama-Kato-Miyachi [4] construct and study the homotopy category $\mathsf{K}_N(R)$, the derived category $\mathsf{D}_N(R)$ of $N$-complexes. They showed that the derived category $\mathsf{D}_N(R)$ is equivalent as triangulated categories to the derived category (in the ordinary sense) $\mathsf{D}(R \otimes_{\mathbf{k}} \mathbf{k}\overrightarrow{A}_{N-1})$. Inspired by their results, we introduce the notion of $A$-complexes for a graded self-injective algebra $A$. We construct and study the homotopy category, the derived category of and the cohomology functors. As a consequence, we see that the relations between various cohomology functors of $N$-complexes comes from representation theory of the graded algebra $\mathbf{k}[\delta]/(\delta^N)$ with $deg \mathbf{k} = 0, \deg \delta = 1$.

## 1. $N$-COMPLEXES (KAPRANOV, SARKARIA, G. KATO, DUBOIS-VIOLETTE, HIRAMATSU-G. KATO, IYAMA-K. KATO-MIYACHI ...)

### 1.1. $N$-complexes.

Our setup is the followings:

- $N \ge 2$ is an integer greater than 1.
- $R$ is an algebra over a field $\mathbf{k}$.

For simplicity, in this note $N$-($A$-)complexes are that of $R$-modules.

**Definition 1.** An *$N$-complex* $X$ ( of $R$-modules ) is a graded $R$-module $\bigoplus_{i\in\mathbb{Z}} X^i$ equipped with an endomorphism $d_X$ of degree 1 (the differential of $X$) such that $d_X^N = 0$.

$$d_X^N = d_X \circ d_X \circ \cdots d_X \qquad (N \text{ times }).$$

$$\cdots \to X^{i-1} \xrightarrow{d_X} X^i \xrightarrow{d_X} X^{i+1} \to \cdots$$

The detailed version of this paper will be submitted for publication elsewhere.

A morphism $f : X \to Y$ of $N$-complexes is a morphism of graded $R$-modules which is compatible with the differentials $d_X$ and $d_Y$.

$$\begin{array}{ccccccc} \xrightarrow{d_X} & X^{i-1} & \xrightarrow{d_X} & X^i & \xrightarrow{d_X} & X^{i+1} & \xrightarrow{d_X} \\ & {\scriptstyle f^{i-1}}\downarrow & & {\scriptstyle f^{i}}\downarrow & & {\scriptstyle f^{i+1}}\downarrow & \\ \xrightarrow{d_Y} & Y^{i-1} & \xrightarrow{d_Y} & Y^i & \xrightarrow{d_Y} & Y^{i+1} & \xrightarrow{d_Y} \end{array}$$

The category $\mathsf{C}_N(R)$ of $N$-complexes is abelian.

The notion of $N$-complexes is so natural that it have been studied by many researchers from various point of views.

### 1.2. Cohomology group $\mathrm{H}^i_{(n)}(X)$ of $N$-complexe $X$.

**Definition 2.** For $i \in \mathbb{Z}$ and $0 < n < N$, we define the cohomology group $\mathrm{H}^i_{(n)}(X)$ of $N$-complexe $X$ which has $i$-th degree and $n$-th position to be

$$\mathrm{H}^i_{(n)}(X) := \frac{\mathrm{Ker}[d_X^n : X^i \to X^{i+n}]}{\mathrm{Im}[d_X^{N-n} : X^{i-N+n} \to X^i]}.$$

For $N$-complexes we have cohomology long exact sequences.

**Theorem 3** (Dubois-Violette)**.** *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of $N$-complexes. Then we have the following exact sequence:*

$$\cdots \to \mathrm{H}^i_{(n)}(X) \to \mathrm{H}^i_{(n)}(Y) \to \mathrm{H}^i_{(n)}(Z) \to$$

$$\to \mathrm{H}^{i+n}_{(N-n)}(X) \to \mathrm{H}^{i+n}_{(N-n)}(Y) \to \mathrm{H}^{i+n}_{(N-n)}(Z) \to$$

$$\to \mathrm{H}^{i+N}_{(n)}(X) \to \mathrm{H}^{i+N}_{(n)}(Y) \to \mathrm{H}^{i+N}_{(n)}(Z) \to$$

$$\to \mathrm{H}^{i+n+N}_{(N-n)}(X) \to \mathrm{H}^{i+n+N}_{(N-n)}(Y) \to \mathrm{H}^{i+n+N}_{(N-n)}(Z) \to \cdots$$

Note that this sequence is 6-periodic up to degree shift.

There is another long exact sequence for cohomology groups of $N$-complexes.

**Theorem 4** ( Second long exact sequence (Dubois-Violett) )**.** *Let $n, m > 0$ be natural numbers such that $n + m < N$. Then, for an $N$-complex $X$, we have*

$$\cdots \to \mathrm{H}^i_{(n)}(X) \to \mathrm{H}^i_{(n+m)}(X) \to \mathrm{H}^{i+n}_{(m)}(X) \to$$

$$\to \mathrm{H}^{i+n}_{(N-n)}(X) \to \mathrm{H}^{i+n+m}_{(N-n-m)}(X) \to \mathrm{H}^{i+n+m}_{(N-m)}(X) \to$$

$$\to \mathrm{H}^{i+N}_{(n)}(X) \to \mathrm{H}^{i+N}_{(n+m)}(X) \to \mathrm{H}^{i+n+N}_{(m)}(X) \to$$

$$\to \mathrm{H}^{i+n+N}_{(N-n)}(X) \to \mathrm{H}^{i+n+m+N}_{(N-n-m)}(X) to \mathrm{H}^{i+n+m+N}_{(N-m)}(X) \to \cdots$$

We remark that for the ordinary complexes (i.e., the case where $N = 2$) the condition for $n$ and $m$ is empty. We note that this sequence is also 6-periodic up to degree shift.

1.3. **Results of Iyama-Kato-Miyachi.** Iyama-Kato-Miyachi showed that $\mathsf{C}_N(R)$ has a Frobenious structure. Then they defined the homotopy category$\mathsf{K}_N(R)$ to be the stable category $\mathsf{K}_N(R) := \underline{\mathsf{C}_N(R)}$ and of $\mathsf{C}_N(R)$ with respect to this Frobenious structure, and the derived category $\mathsf{D}_N(R)$ to be the Verdier quotient of $\mathsf{K}_N(R)$ by the thick subcategories consisting of acyclic $N$-complexes $\mathsf{D}_N(R) := \dfrac{\mathsf{K}_N(R)}{(\text{Acyclic } N\text{-complexes})}$.

I heard that one of their mtivation to define a derived category of $N$-complexes is to get a triangulated category of new kind. But they showed that derived category of $N$-complexes is no new. It turns out to be equivalent to an ordinary derived category. More precisely we have the following equivalence of triangulated categories:

**Theorem 5** (Iyama-Kato-Miyachi)**.**

$$\mathsf{D}_N(R) \simeq \mathsf{D}(\mathbf{k}\overrightarrow{\mathrm{A}}_{N-1} \otimes R)$$

The right hand side is the ordinary derived category of the algebra $\mathbf{k}\overrightarrow{\mathrm{A}}_{N-1} \otimes R$ where $\mathbf{k}\overrightarrow{\mathrm{A}}_{N-1}$ is the path algebra of $A_{N-1}$-quiver.

Since there are interesting results on $N$-complexes, now we would like to ask why $d^N = 0$? For this purpose, we try to find a further generalization of $N$-complexes.

## 2. $A$-complexes

2.1. **An observation on $N$-complexes.** We observe that the notion of $N$-complexes and related things can be reformulated in terms of a graded algebra and its modules.

We define a graded algebra $B_N$ to be $B_N := \mathbf{k}[\delta]/\delta^N$ with $\deg \delta = 1$. A point is that an $N$-complex $X$ is nothing but a graded module over the graded algebra $B_N \otimes R$ and

$$\mathsf{C}_N(R) = (B_N \otimes R)\,\mathrm{GRMod}$$

where we consider $\deg R = 0$.

2.2. **$A$-complexes and their cohomologies.** We define a notion of $A$-complex by replacing the graded algebra $B_N$ with a graded algebra $A$ satisfying some conditions, which allow us to develop general theory.

Let $A := \bigoplus_{i\in\mathbb{Z}} A^i$ be a finite dimensional graded Frobenius algebra having Gorenstein parameter $\ell \in \mathbb{Z}$, i.e., $\mathrm{Hom}_k(A, \mathbf{k}) \cong A(\ell)$ for some $\ell \in \mathbb{Z}$.

**Definition 6.** An $A$-complex is a graded $A \otimes R$-module. We set the category $\mathsf{C}_A(R)$ of $A$-complexes to be the category of graded $A \otimes R$-modules.

$$\mathsf{C}_A(R) := (A \otimes R)\,\mathrm{GRMod}\,.$$

*Remark* 7. The above definition and the following results can be generalized to the case where $A$ is a self-injective $\mathbf{k}$-linear category with a Serre functor satisfying some conditions.

For $A$-complex $X$ we have a notion of cohomology groups $\mathrm{H}^t(X)$. The indexes $t$ are not integers any more.

**Definition 8.** Let $t$ be a graded $A$-module. We define $t$-th cohomology group of an $A$-complexes $X$ to be

$$\mathrm{H}^t(X) := \mathrm{Ext}^1_{A\,\mathrm{GRMod}}(t, X)$$

The cohomology group $\mathrm{H}^t(X)$ is functorial in $X$ and hence gives a functor

$$\mathrm{H}^t(-) : \mathsf{C}_A(R) \to RMod, X \mapsto \mathrm{H}^t(X).$$

**Theorem 9** (Cohomology long exact sequences for $A$-complexes)**.**
*Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of $A$-complexes. Then we have the following exact sequence*

$$\to \mathrm{H}^{\Omega^{-1}(t)}(X) \to \mathrm{H}^{\Omega^{-1}(t)}(Y) \to \mathrm{H}^{\Omega^{-1}(t)}(Z) \to \cdots$$
$$\to \mathrm{H}^t(X) \to \mathrm{H}^t(Y) \to \mathrm{H}^t(Z) \to$$
$$\to \mathrm{H}^{\Omega(t)}(X) \to \mathrm{H}^{\Omega(t)}(Y) \to \mathrm{H}^{\Omega(t)}(Z) \to \cdots$$

*where $\Omega$ and $\Omega^{-1}$ denote the syzygy functor and co-syzygy functor.*

**Theorem 10** (Cohomology long exact sequence for indexes)**.**
*Let $0 \to s \to t \to u \to 0$ be an exact sequence of graded $A$-modules. Then, for an $A$-complex $X$, we have the following long exact sequence*

$$\to \mathrm{H}^{\Omega^{-1}(u)}(X) \to \mathrm{H}^{\Omega^{-1}(t)}(X) \to \mathrm{H}^{\Omega^{-1}(s)}(X) \to$$
$$\to \mathrm{H}^u(X) \to \mathrm{H}^t(X) \to \mathrm{H}^s(X) \to$$
$$\to \mathrm{H}^{\Omega(u)}(X) \to \mathrm{H}^{\Omega(t)}(X) \to \mathrm{H}^{\Omega(s)}(X) \to$$

Now we discuss a Frobenius Structure in $\mathsf{C}_A(R)$.

**Lemma 11.** *Let $\mathcal{E}$ be the class of exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathsf{C}_A(R)$ which become a split exact sequence when they are considered as graded $R$-modules. Then $\mathcal{E}$ gives a Frobenius structure in $\mathsf{C}_A(R)$.*

**Definition 12.** We define the homotopy category $\mathsf{K}_A(R)$ of $A$-complexes to be the stable category of $\mathsf{C}_A(R)$ with respect to the above Frobenious structure.

$$\mathsf{K}_A(R) := \underline{\mathsf{C}_A(R)}$$

*Remark* 13. There exists a notion of *homotopy equivalence* for a morphism $f : X \to Y$ of $A$-complexes. It can be proved that the homotopy category $\mathsf{K}_A(R)$ is isomorphic to the residue category of $\mathsf{C}_A(R)$ modulo homotopy equivalences.

The cohomology functor $\mathrm{H}^t(X)$ descend to

$$\mathrm{H}^t(-) : \mathsf{K}_A(R) \to RMod, X \mapsto \mathrm{H}^t(X).$$

An $A$-complex $X$ is said to be *acyclic* if $\mathrm{H}^t(X) = 0$ for all $A$-module $t$.

**Definition 14.** We define the derived category $\mathsf{D}_A(R)$ of $A$-complexes to be the Verdier quotient of $\mathsf{K}_A(R)$ by the acyclic $A$-complexes.

$$\mathsf{D}_A(R) := \frac{\mathsf{K}_A(R)}{(\text{Acyclic } A\text{-complexes})}$$

An $A$-complex $X$ is said to be $\mathsf{K}$-*projective* if we have $\mathrm{Hom}_{\mathsf{K}_A(R)}(X, Y) = 0$ for any acyclic $A$-complex $Y$. We denote by $\mathsf{K}_A{-}\mathsf{Proj}$ the full subcategory of $\mathsf{K}_A(R)$ consisting of $\mathsf{K}$-projective $A$-complexes.

**Proposition 15.** *(1) There is a semi-orthogonal decomposition*

$$\mathsf{K}_A(R) = \langle \mathsf{K}_A\text{−}\mathsf{Proj}, (\text{Acyclic } A\text{-complexes})\rangle$$

*(2) The canonical functor induces an equivalence*

$$\mathsf{K}_A\text{−}\mathsf{Proj} \to \mathsf{K}_A(R) \to \mathsf{D}_A(R).$$

## 3. Back to $N$-complexes

Let $B_N = \mathbf{k}[\delta]/\delta^N$ with $\deg \delta = 1$. Recall that $\mathsf{C}_N(R) = \mathsf{C}_{B_N}(R)$.

**Definition 16.** For $i \in \mathbb{Z}, 0 < n < N$, we define a graded $B_N$-module $t(i,n)$ to be

$$t(i,n) := (\mathbf{k}[\delta]/\delta^{N-n})(N-n-i)$$

Then we have

$$\mathrm{H}^{t(i,n)}(X) = \mathrm{H}^i_{(n)}(X)$$

where in the left hand side $X$ is considered as a $B_N$-complex and in the right hand side as an $N$-complex Moreover,

$$\Omega(t(i,n)) = t(i+n, N-n).$$

Now it can be easily seen that the cohomology long exact sequence of $N$-complexes (Theorem 3) is nothing but that of $B_N$-complexes (Theorem 9). More precisely, the sequence

$$\to \mathrm{H}^i_{(n)}(X) \to \mathrm{H}^i_{(n)}(Y) \to \mathrm{H}^i_{(n)}(Z) \to \mathrm{H}^{i+n}_{(N-n)}(X) \to \mathrm{H}^{i+n}_{(N-n)}(Y) \to \mathrm{H}^{i+n}_{(N-n)}(Z) \to$$

is equal to the sequence

$$\to \mathrm{H}^{t(i,n)}(X) \to \mathrm{H}^{t(i,n)}(Y) \to \mathrm{H}^{t(i,n)}(Z) \to \mathrm{H}^{\Omega(t(i,n))}(X) \to \mathrm{H}^{\Omega(t(i,n))}(Y) \to \mathrm{H}^{\Omega(t(i,n))}(Z) \to$$

Now we see that the periodicity of the cohomology long exact sequence of $N$-complexes is a consequence of the well-known fact that the syzygy functor $\Omega_{B_N}$ is 2-periodic up to degree $-N$-shift: $\Omega^2_{B_N} \cong (-N)$.

In the same way, we can see that the second cohomology long exact sequence for $N$-complexes (Theorem 4) is nothing bu the cohomology long exact sequence for indexes (Theorem 10), by using the following exact sequence of graded $B_N$-modules:

$$0 \to \frac{\mathbf{k}[\delta]}{\delta^m}(-n) \to \frac{\mathbf{k}[\delta]}{\delta^{n+m}} \to \frac{\mathbf{k}[\delta]}{\delta^n} \to 0.$$

## 4. Iyama-Kato-Miyachi equivalence for $A$-complexes (Ogawa)

The Iyama-Kato-Miyachi equivalence (Theorem 5) is generalized for $A$-complexes by Y. Ogawa.

**Theorem 17** (Ogawa)**.** *We assume that* $\mathbf{k}$ *is an algebraically closed field. Let* $\Lambda$ *be a finite dimensional algebra and* $A := \Lambda \oplus \Lambda^*$ *the trivial extension algebra equipped with the grading that* $\deg \Lambda = 0, \deg \Lambda^* = 1$. *Then there is an equivalence of triangulated categories:*

$$\mathsf{D}_A(R) \simeq \mathsf{D}(\Lambda \otimes R).$$

In a nutshell, this is a relative version of Happel's equivalence ([2]):

$$\underline{\mathrm{grmod}}\, A \simeq \mathsf{D}(\Lambda).$$

## References


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