CLASSIFICATION OF CATEGORICAL SUBSPACES OF LOCALLY NOETHERIAN SCHEMES

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Abstract. This paper is an announcement of our results in [2]. We classify the prelocalizing subcategories of the category of quasi-coherent sheaves on a locally noetherian scheme. In order to give the classification, we introduce the notion of a local filter of subobjects of the structure sheaf. We also classify the localizing subcategories and the closed subcategories in terms of filters.

Key Words: Locally noetherian scheme, Prelocalizing subcategory, Localizing subcategory, Closed subcategory, Local filter.

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1. Gabriel's results

Let $\mathcal{A}$ be a Grothendieck category. For example, the category $\operatorname{Mod}\Lambda$ of right modules over a ring Λ and the category $\operatorname{QCoh}X$ of quasi-coherent sheaves on a scheme X are Grothendieck categories. In this paper, we deal with the following classes of subcategories.

Definition 1. Let $\mathcal{Y}$ be a full subcategory of $\mathcal{A}$.

(1) $\mathcal{Y}$ is called a *prelocalizing subcategory* (or a *weakly closed subcategory*) if $\mathcal{Y}$ is closed under subobjects, quotient objects, and arbitrary direct sums.
(2) $\mathcal{Y}$ is called a *closed subcategory* if $\mathcal{Y}$ is a prelocalizing subcategory closed under arbitrary direct products.
(3) $\mathcal{Y}$ is called a *localizing subcategory* if $\mathcal{Y}$ is a prelocalizing subcategory closed under extensions.

For a ring Λ, Gabriel [1] classified the prelocalizing subcategories and the localizing subcategories of $\operatorname{Mod}\Lambda$ by using the notion of filters. We define filters for objects in Grothendieck categories.

Definition 2. Let M be an object in $\mathcal{A}$. A *filter* (of subobjects) of M in $\mathcal{A}$ is a set $\mathcal{F}$ of subobjects of M satisfying the following conditions.

(1) $M \in \mathcal{F}$.
(2) If $L \subset L'$ are subobjects of M with $L \in \mathcal{F}$, then $L' \in \mathcal{F}$.
(3) If $L_1, L_2 \in \mathcal{F}$, then $L_1 \cap L_2 \in \mathcal{F}$.

For each subobject L of M, denote by $\mathcal{F}(L)$ the filter consisting of all subobjects L' of M with $L \subset L'$. A filter of the form $\mathcal{F}(L)$ is called a *principal filter*.

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Remark 3. The principal filter $\mathcal{F}(L)$ is closed under arbitrary intersection. Conversely, if a filter $\mathcal{F}$ of M is closed under arbitrary intersection, then $\mathcal{F} = \mathcal{F}(L)$, where L is the smallest element of $\mathcal{F}$.

Definition 4. For a ring Λ, we say that a filter $\mathcal{F}$ (of right ideals) of Λ in $\operatorname{Mod}\Lambda$ is *prelocalizing* if for each $L \in \mathcal{F}$ and $a \in \Lambda$, the right ideal

$$a^{-1}L = \{\, b \in \Lambda \mid ab \in L \,\}$$

of Λ belongs to $\mathcal{F}$.

Note that every filter $\mathcal{F}$ of a commutative ring R is prelocalizing.

The following theorem is the motivating result of our study.

Theorem 5 ([1, Lemma V.2.1]). *Let Λ be a ring. Then the map*

$$\{\, \textit{prelocalizing subcategories of } \operatorname{Mod}\Lambda \,\} \to \{\, \textit{prelocalizing filters of } \Lambda \textit{ in } \operatorname{Mod}\Lambda \,\}$$

given by

$$\mathcal{Y} \mapsto \left\{ L \subset \Lambda \textit{ in } \operatorname{Mod}\Lambda \;\middle|\; \frac{\Lambda}{L} \in \mathcal{Y} \right\}$$

is bijective. The inverse map is given by

$$\begin{aligned} \mathcal{F} &\mapsto \{\, M \in \operatorname{Mod}\Lambda \mid \operatorname{Ann}_\Lambda(x) \in \mathcal{F} \textit{ for every } x \in M \,\} \\ &= \left\langle \frac{\Lambda}{L} \in \operatorname{Mod}\Lambda \;\middle|\; L \in \mathcal{F} \right\rangle_{\text{preloc}}, \end{aligned}$$

where $\langle \mathcal{S} \rangle_{\text{preloc}}$ is the smallest prelocalizing subcategory containing the set $\mathcal{S}$ of objects.

By considering the principal filters, we can recover the classification of the closed subcategories of $\operatorname{Mod}\Lambda$ due to Rosenberg [3].

Theorem 6 (Gabriel [1, Lemma V.2.1] and Rosenberg [3, Proposition III.6.4.1]). *Let Λ be a ring. Then there exist bijections between the following sets.*

1. *The set of closed subcategories of* $\operatorname{Mod}\Lambda$.
2. *The set of principal prelocalizing filters of right ideals of Λ.*
3. *The set of two-sided ideals of Λ.*

The bijection between (1) and (2) is induced by the bijection in Theorem 5.

The bijection between (1) and (3) is given by

$$\begin{aligned} &(1) \to (3):\ \mathcal{Y} \mapsto \bigcap_{M \in \mathcal{Y}} \operatorname{Ann}_\Lambda(M), \\ &(3) \to (1):\ I \mapsto \{\, M \in \operatorname{Mod}\Lambda \mid MI = 0 \,\} = \left\langle \frac{\Lambda}{I} \right\rangle_{\text{preloc}}. \end{aligned}$$

Gabriel [1] also classified the localizing subcategories of $\operatorname{Mod}\Lambda$. For more details, see [2, section 10].

2. Classification for QCoh X

In this section, let X be a locally noetherian scheme. Its structure sheaf is denoted by $\mathcal{O}_X$. We give classifications of the three classes of subcategories of QCoh X. In order to do that, we need to refine the notion of filters.

Definition 7. Let X be a locally noetherian scheme. We say that a filter $\mathcal{F}$ of subobjects of $\mathcal{O}_X$ in QCoh X is a *local filter* of $\mathcal{O}_X$ if it satisfies the following condition: let I be a subobject of $\mathcal{O}_X$, and assume that for each $x \in X$, there exist an open neighborhood U of x in X and $I' \in \mathcal{F}$ such that $I'|_U \subset I|_U$ as a subobject of $\mathcal{O}_U$. Then we have $I \in \mathcal{F}$.

We can show that every principal filter of $\mathcal{O}_X$ is a local filter. In the case where X is noetherian, every filter of $\mathcal{O}_X$ is a local filter.

The following theorem is our main result.

Theorem 8. *Let X be a locally noetherian scheme.*

(1) *The map*

$$\{\textit{ prelocalizing subcategories of } \operatorname{QCoh} X \} \to \{\textit{ local filters of } \mathcal{O}_X \textit{ in } \operatorname{QCoh} X \}$$

given by

$$\mathcal{Y} \mapsto \left\{ I \subset \mathcal{O}_X \textit{ in } \operatorname{QCoh} X \;\middle|\; \frac{\mathcal{O}_X}{I} \in \mathcal{Y} \right\}$$

is bijective. The inverse map is given by

$$\mathcal{F} \mapsto \left\langle \frac{\mathcal{O}_X}{I} \in \operatorname{QCoh} X \;\middle|\; I \in \mathcal{F} \right\rangle_{\mathrm{preloc}}.$$

(2) *The bijection in (1) induces bijections*

$$\{\textit{ closed subcategories of } \operatorname{QCoh} X \} \to \{\textit{ principal filters of } \mathcal{O}_X \}$$

and

$$\{\textit{ localizing subcategories of } \operatorname{QCoh} X \} \to \{\textit{ local filters of } \mathcal{O}_X \textit{ closed under products} \}.$$

Corollary 9. *There exist bijections between the following sets.*

(1) *The set of closed subcategories of* QCoh X.
(2) *The set of subobjects of* $\mathcal{O}_X$ *in* QCoh X.
(3) *The set of closed subschemes of* X.

The key of the proof is the fact that every prelocalizing subcategory $\mathcal{Y}$ of QCoh X has the description

$$\mathcal{Y} = \{ M \in \operatorname{QCoh} X \mid M_x \in \mathcal{Y}_x \text{ for each } x \in X \}.$$

Example 10. Let k be an algebraically closed field, and consider the projective line $X = \mathbb{P}^1_k$. Denote by $\varPhi$ the set of closed points in X. For each $r \in \prod_{x \in \varPhi}(\mathbb{Z}_{\geq 0} \cup \{\infty\})$, we define the prelocalizing subcategory $\mathcal{Y}_r$ of QCoh X by

$$\mathcal{Y}_r = \{ M \in \operatorname{QCoh} X \mid M_x \mathfrak{m}_x^{r(x)} = 0 \text{ for each } x \in \varPhi \text{ with } r(x) \neq \infty \}.$$

The set of prelocalizing subcategories of $\operatorname{QCoh} X$ is

$$\left\{ \mathcal{Y}_r \;\middle|\; r \in \prod_{x \in \Phi} (\mathbb{Z}_{\geq 0} \cup \{\infty\}) \right\} \cup \{\operatorname{QCoh} X\},$$

the set of localizing subcategories of $\operatorname{QCoh} X$ is

$$\left\{ \mathcal{Y}_r \;\middle|\; r \in \prod_{x \in \Phi} \{0, \infty\} \right\} \cup \{\operatorname{QCoh} X\},$$

and the set of closed subcategories of $\operatorname{QCoh} X$ is

$$\left\{ \mathcal{Y}_r \;\middle|\; r \in \bigoplus_{x \in \Phi} \mathbb{Z}_{\geq 0} \right\} \cup \{\operatorname{QCoh} X\}.$$

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References

[1] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.
[2] R. Kanda, *Classification of categorical subspaces of locally noetherian schemes*, arXiv:1405.4473v1, 47 pp.
[3] A. L. Rosenberg, *Noncommutative algebraic geometry and representations of quantized algebras*, Mathematics and its Applications, 330, Kluwer Academic Publishers Group, Dordrecht, 1995, xii+315 pp.

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