TAKING TILTING MODULES FROM THE POSET OF SUPPORT TILTING MODULES

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Abstract. C. Ingalls and H. Thomas defined support tilting modules for path algebras. From τ-tilting theory introduced by T. Adachi, O. Iyama and I. Reiten, a partial order on the set of basic tilting modules defined by D. Happel and L. Unger is extended as a partial order on the set of support tilting modules. In this report, we study a combinatorial relationship between the poset of basic tilting modules and basic support tilting modules. We will show that the subposet of tilting modules is uniquely determined by the poset structure of the set of support tilting modules.

1. Introduction

Tilting theory first appeared in an article by S. Brenner and M.C.R. Butler [2]. In that article the notion of a tilting module for finite dimensional algebras was introduced. Let T be a tilting module for a finite dimensional algebra Λ and let $B = \mathrm{End}_A(T)$. Then D. Happel showed that the two bounded derived categories $\mathcal{D}^{\mathrm{b}}(A)$ and $\mathcal{D}^{\mathrm{b}}(B)$ are equivalent as triangulated category [3]. Therefore, classifying tilting modules is an important problem.

Tilting mutation introduced by C. Riedtmann and A. Schofield [7] is an approach to this problem. It is an operation which gives a new tilting module from given one by replacing an indecomposable direct summand. They also introduced a tilting quiver whose vertices are (isomorphism classes of) basic tilting modules and arrows correspond to mutations. D. Happel and L. Unger showed that there is a partial order on the set of (isomorphism classes of) basic tilting modules suct that its Hasse quiver coincides to tilting quiver [4, 5]. However, tilting mutation is often impossible. Support τ-tilting modules introduced by T. Adachi,O. Iyama and I. Reiten [1] are generalization of tilting modules. They showed that a mutation (resp. a partial order) on the set of (isomorphism classes of) basic tilting modules is extended as an operation (resp. a partial order) on the set of (isomorphism classes of) support τ-tilting modules and improved behavior of tilting mutation.

In path algebras case, it is known that a support τ-tilting module is a support tilting module introduced by C. Ingalls and H. Thomas [6]. Then the main result of this report is the following.

Theorem 1. *Let Λ be a finite dimensional path algebra. Then the set of basic tilting modules of Λ is determined by poset structure of the set of basic support tilting modules.*

The detailed version of this paper has been submitted for publication elsewhere.

2. Path algebras

Let k be an algebraically closed field and let Q be a finite quiver (=oriented graph). We denote by Q_0 (resp. Q_1) the set of vertices (resp. edges) of Q. For an edge $\alpha : a \to b$, we set $s(\alpha) := a,\ t(\alpha) := b$.

Definition 2. A sequence $w = (\alpha_1|\alpha_2|\cdots|\alpha_l)$ of Q_1 is a path on Q if $t(\alpha_i) = s(\alpha_{i+1})$ holds for any i. Then we call l the length of w and put $s(w) := s(\alpha_1)\ t(w) := \alpha_l$. We regard a vertex $a \in Q_0$ as a path of length 0 with $s(e_a) = a = t(e_a)$ and denote it by e_a.

Then a path algebra $\Lambda = kQ$ is defined as follows:

(1) $\Lambda = \bigoplus_{w:\text{path}} k \cdot w$.

(2) For two paths $w = (\alpha_1|\alpha_2|\cdots|\alpha_l),\ w' = (\beta_1|\beta_2|\cdots|\beta_{l'})$, we define

$$w \cdot w' = \begin{cases} (\alpha_1|\alpha_2|\cdots|\alpha_l \mid \beta_1|\beta_2|\cdots|\beta_{l'}) & \text{if } t(w) = s(w') \\ 0 & \text{if } t(w) \neq s(w'). \end{cases}$$

From now on, we assume that $\Lambda = kQ$ and Q has no oriented cycles ($\Leftrightarrow \dim \Lambda < \infty$).

3. Tilting modules and support tilting modules

In this section, we recall definitions of poset of tilting modules and poset of support tilting modules. For a module $M \in \operatorname{mod}\Lambda$ with indecomposable decomposition

$$M \simeq \oplus_{i=1}^{m} M_i^{r_i}\ (i \neq j \Rightarrow M_i \not\simeq M_j),$$

we put $|M| := m$. M is said to be basic if $r_i = 1$ ($\forall i$).

Definition 3. $T \in \operatorname{mod}\Lambda$ is a tilting module if T satisfies following properties.

(1) $\operatorname{Ext}^1_\Lambda(T,T) = 0$.

(2) $|T| = \# Q_0$.

We denote by tilt Λ the set of (isomorphism classes of) basic tilting modules.

Proposition 4. [4, 5] *The following relation induces a partial order on* tilt Λ.

$$T \geq T^{'} \Leftrightarrow \operatorname{Ext}^1_\Lambda(T,T^{'}) = 0.$$

For a module $M \in \operatorname{mod}\Lambda$, we put $\mathrm{supp(M) := \{a \in Q_0 \mid \dim Me_a > 0\}}$ and denote by $Q(M)$ the full subquiver of Q with $Q(M)_0 = \mathrm{supp(M)}$.

Remark 5. We can regard M as $kQ(M)$-module.

Definition 6. $T \in \operatorname{mod}\Lambda$ is a support tilting module if T satisfies following properties.

(1) $\operatorname{Ext}^1_\Lambda(T,T) = 0$.

(2) $|T| = \#\mathrm{supp(T)}$.

We denote by stiltΛ the set of (isomorphism classes of) support tilting module.

We note that T is support tilting if and only if $\Lambda(T)$ is tilting as $kQ(T)$-module.

Proposition 7. [1, 6] *The following relation induces a partial order on* stilt Λ.

$$T \geq T' \Leftrightarrow \operatorname{Ext}^1_\Lambda(T,T') = 0\ \&\ \mathrm{supp(T') \subset supp(T)}.$$

Example 8. Let $Q = 1 \to 2$. Then stiltΛ is given by the following.

$$\begin{tikzcd}
& P(1) \oplus P(2) \arrow[dl] \arrow[dr] & \\
P(1) \oplus I(1) \arrow[d] & & P(2) \arrow[d] \\
I(1) \arrow[rr] & & 0
\end{tikzcd}$$

4. Outline of a proof

By definition of support tilting modules, we have

$$T \in \mathrm{stilt}\Lambda \text{ is a tilting module} \Leftrightarrow T \geq I_\Lambda = \oplus_{a \in Q_0} I(a).$$

For a non negative integer i, we define a subset V_i of Q_0 as follows.

- $V_0 = \emptyset$.
- $V_i = V_{i-1} \cup \{a \in Q_0 \mid a \text{ is a source of } Q \setminus V_{i-1}\}$.

We set $I_i := \oplus_{a \in V_i} I(a)$ $(I_0 = 0)$. Then we note that $I_i \in \mathrm{stilt}\Lambda$.

Lemma 9. *Let $i \geq 0$. Then I_{i+1} is a minimum element of*

$$\bigcap_{X \in \mathrm{idp(I_i)}} \{T \in \mathrm{stilt}\Lambda \mid T \geq X\},$$

where $\mathrm{idp(I_i)}$ *I_i is the set of injective direct predecessors of I_i.*

Lemma 1 shows that it is sufficient to determine $\mathrm{idp(I_i)}$ by poset structure of stiltΛ.

4.1. **Deleting non injective direct predecessors of I_i.** Non injective direct predecessor T satisfies one of the following.

(1) $\#\mathrm{supp(T)} = \#\mathrm{supp(I_i)} + 1$.
(2) $\#\mathrm{supp(T)} = \#\mathrm{supp(I_i)}$.

We denote by $\mathcal{N}_i(p)$ $(p = 1, 2)$ the set of non injective direct predecessors of I_i which satisfies (p).

Lemma 10. *Let a, $b \in Q_0$. Then There is an edge $a \to b$ in Q if and only if there are $X \in \mathrm{dp(S(a))}$, $Y \in \mathrm{dp(S(b))}$ such that $X < Y$.*

Since $S(a)$ is injective if and only if $a \in Q_0$ is a source, we can determine $\mathrm{idp}(I_0)$ by poset structure of stilt Λ.

Lemma 11. *Let $T \in \mathcal{N}_i(1)$. Then there are $T' \in \mathrm{dp(I_i)}$, $X \in \mathrm{dp(T)}$, $Y \in \mathrm{dp(T')}$ such that $X > Y$.*

Lemma 12. *Let $T \in \mathrm{idp(I_i)}$. Then for any $T' \in \mathrm{dp(I_i)}$, $X \in \mathrm{dp(T)}$, $Y \in \mathrm{dp(T')}$, we have $X \not> Y$.*

Lemma 3 and Lemma 4 implies that we can delete $\mathcal{N}_i(1)$. For $T \in \mathrm{dp(I_i)}$ and $r \in \mathbb{Z}_{\geq 1}$, we set

$$\mathcal{F}(i, T, r) := \{((X_k)_{k\in\{0,\cdots,r\}}, (T_k)_{k\in\{0\cdots,r-1\}}, (Y_k)_{k\in\{1,\cdots,r-1\}}) \mid (\star)\}$$

where the condition $(\star)$ is as follows: $(\star) := \begin{cases} \bullet \;\; X_0 = I_i,\ T_0 = T \\ \bullet \;\; X_1 \in \mathrm{ds(I_i)},\ \mathrm{X_{k+1} \in ds(X_k)} \\ \bullet \;\; T_k \in \mathrm{dp(X_k) \setminus \{X_{k-1}\}} \\ \bullet \;\; Y_k \in \mathrm{dp(T_k)} \\ \bullet \;\; Y_1 \geq T,\ Y_{k+1} \geq T_k \end{cases}$

Lemma 13. *Let $T \in \mathcal{N}_i(2)$. Then there are $r \in \mathbb{Z}_{\geq 1}$ and $((X_k), (T_k), (Y_k)) \in \mathcal{F}(i, T, r)$ such that for any $T_r \in \mathrm{dp(X_r) \setminus \{X_{r-1}\}}$ and $Y_r \in \mathrm{dp(T_r)}$, we have $Y_r \ngeq T_{r-1}$.*

Lemma 14. *Let $T \in \mathrm{idp(I_i)}$. Then for any $r \in \mathbb{Z}_{\geq 1}$ and $((X_k), (T_k), (Y_k)) \in \mathcal{F}(i, T, r)$, there are $T_r \in \mathrm{dp(X_r) \setminus \{X_{r-1}\}}$ and $Y_r \in \mathrm{dp(T_r)}$ such that $Y_r \geq T_{r-1}$.*

Thus we can also delete $\mathcal{N}_i(2)$.

Corollary 15. *Let Λ and Γ be two path algebras, ρ be a poset isomorphism*

$$\rho : \mathrm{stilt}\Lambda \simeq \mathrm{stilt}\Gamma.$$

Then the restriction of ρ to $\mathrm{tilt}\Lambda$ *induces a poset isomorphism*

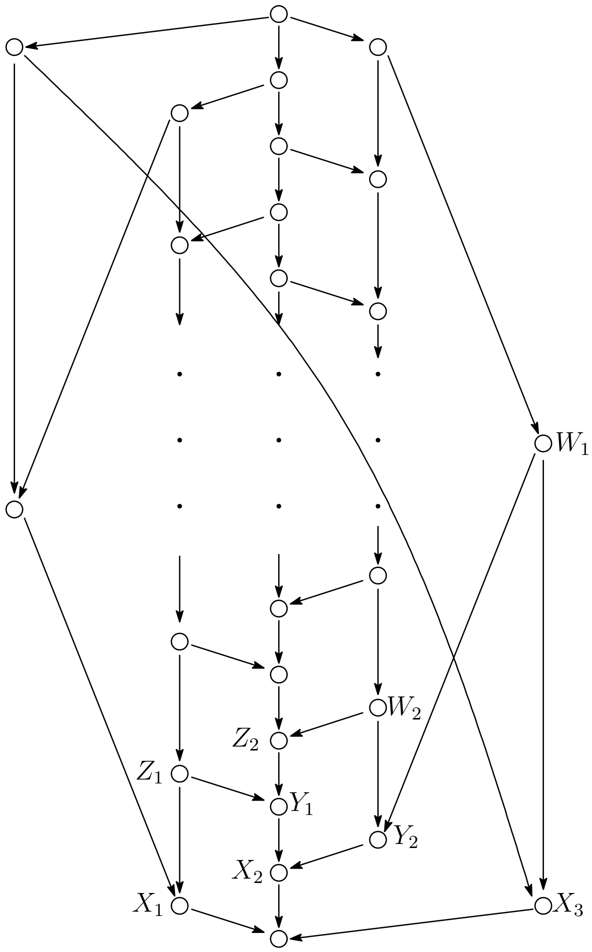
$$\rho|_{\mathrm{tilt}\Lambda} : \mathrm{tilt}\Lambda \simeq \mathrm{tilt}\Gamma.$$

5. Example

We consider the following quiver Q.

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$

Then stilt Λ is given by the following.

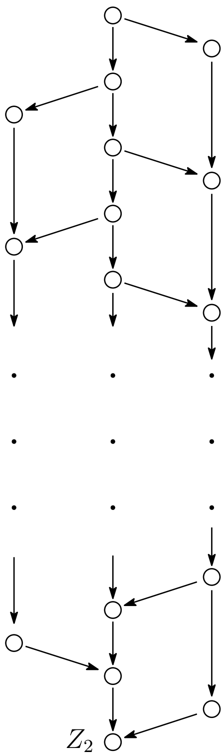


step 1 By applying Lemma 3 and Lemma 4 to $\{0,\ X_1,\ X_2,\ Y_1,\ Z_1\}$, we can see that X_1 is not injective. Similarly we have X_3 is not injective. Therefore X_2 is injective.

step 2 By applying Lemma 5 and Lemma 6 to $\{X_2,\ Y_1,\ Y_2,\ Z_2,\ W_2\}$, we have Y_2 is not injective. Hence Y_1 is injective.

step 3 We consider $\mathcal{F}(1, Z_1, Y_1) \ni ((Y_1, X_2), (Z_1), \emptyset)$. Then Y_2 is a unique direct predecessor of X_2 and $\{W_1, W_2\}$ is the set of direct predecessors of Y_2. Since $W_p \not\geq Z_1$ ($p = 1, 2$), Lemma 5 implies that Z_1 is not injective. Therefore we have $I_\Lambda = Z_2$.

In particular, tiltΛ is given by the following.



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