ON ISOMORPHISMS OF GENERALIZED MULTIFOLD EXTENSIONS OF ALGEBRAS WITHOUT NONZERO ORIENTED CYCLES

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Abstract. We show that an algebra of the form $\hat{A}/\langle\phi\rangle$ where A is an algebra and ϕ is an automorphism of $\hat{A}$ such that $\phi(A^{[0]}) = A^{[n]}$ for some integer n is isomorphic to an algebra of the form $\hat{A}/\langle\hat{\phi_0}\nu_{\hat{A}}^n\rangle$ where $\hat{\phi_0}$ is an automorphism of $\hat{A}$ induced by ϕ and $\nu_{\hat{A}}$ is the Nakayama automorphism of $\hat{A}$ if A has no nonzero oriented cycles. Throughout this paper we do not assume that the action of groups (or automorphisms of $\hat{A}$) are free. Therefore this result give us applying a derived equivalence classification in [1] and [3] to $n = 0$.

1. Introduction

Throughout this paper $\Bbbk$ is an algebraically closed field, algebras are basic finite-dimensional $\Bbbk$-algebras and categories are $\Bbbk$-categories.

We say that an algebra is a generalize multifold extension of algebra A if it has the form $\hat{A}/\langle\phi\rangle$ where $\hat{A}$ is the repetitive category of $\hat{A}$ and ϕ is an automorphism of $\hat{A}$ with jump n for some integer n (see Definition 1 and Proposition 2). In [3], we gave a derived equivalence classification of generalized multifold extensions of algebras which are piecewise hereditary of tree type (i.e., algebras are derived equivalent to some hereditary algebra whose ordinary quiver is oriented tree) if automorphisms act on algebras have positive jump. To give a classification, we showed that for a positive integer $n \in \mathbb{Z}$, a generalized n-fold extension $\hat{A}/\langle\phi\rangle$ is derived equivalent to $T^n_{\phi_0}(A) := \hat{A}/\langle\hat{\phi_0}\nu_{\hat{A}}^n\rangle$ where $\hat{\phi_0}$ is the automorphism of $\hat{A}$ naturally induced from automorphism $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu_{\hat{A}}^{-n}\phi\mathbb{1}^{[0]}$ of A and $\nu_{\hat{A}}$ is the Nakayama automorphism of $\hat{A}$. Also, we posed a following question

Problem. If A is piecewise hereditary of tree type, when are the algebras $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ isomorphic?

In this paper we will give the answer to this question.

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The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

For a category R we denote by R_0 and R_1 the class of objects and morphisms of R, respectively. A category R is said to be *locally bounded* if it satisfies the following:

- Distinct objects of R are not isomorphic;
- $R(x,x)$ is a local algebra for all $x \in R_0$;
- $R(x,y)$ is finite-dimensional for all $x, y \in R_0$; and
- The set $\{y \in R_0 \mid R(x,y) \neq 0 \text{ or } R(y,x) \neq 0\}$ is finite for all $x \in R_0$.

A category is called *finite* if it has only a finite number of objects.

A pair (A, E) of an algebra A and a complete set $E := \{e_1, \dots, e_n\}$ of orthogonal primitive idempotents of A can be identified with a locally bounded and finite category R by the following correspondences. Such a pair (A, E) defines a category $R_{(A,E)} := R$ as follows: $R_0 := E$, $R(x,y) := yAx$ for all $x, y \in E$, and the composition of R is defined by the multiplication of A. Then the category R is locally bounded and finite. Conversely, a locally bounded and finite category R defines such a pair (A_R, E_R) as follows: $A_R := \bigoplus_{x,y \in R_0} R(x,y)$ with the usual matrix multiplication (regard each element of A as a matrix indexed by R_0), and $E_R := \{(\mathbb{1}_x \delta_{(i,j),(x,x)})_{i,j \in R_0} \mid x \in R_0\}$. We always regard an algebra A as a locally bounded and finite category by fixing a complete set A_0 of orthogonal primitive idempotents of A.

Definition 1. Let A be a locally bounded category.

(1) The *repetitive category* $\hat{A}$ of A is a $\Bbbk$-category defined as follows ($\hat{A}$ turns out to be locally bounded again):

- $\hat{A}_0 := A_0 \times \mathbb{Z} = \{x^{[i]} := (x, i) \mid x \in A_0, i \in \mathbb{Z}\}$.
- $\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} \mid f \in A(x,y)\} & \text{if } j = i, \\ \{\phi^{[i]} \mid \phi \in DA(y,x)\} & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$ for all $x^{[i]}, y^{[j]} \in \hat{A}_0$.
- For each $x^{[i]}, y^{[j]}, z^{[k]} \in \hat{A}_0$ the composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is given as follows.
  - (i) If $i = j, j = k$, then this is the composition of A $A(y,z) \times A(x,y) \to A(x,z)$.
  - (ii) If $i = j, j+1 = k$, then this is given by the right A-module structure of DA: $DA(z,y) \times A(x,y) \to DA(z,x)$.
  - (iii) If $i+1 = j, j = k$, then this is given by the left A-module structure of DA: $A(y,z) \times DA(y,x) \to DA(z,x)$.
  - (iv) Otherwise, the composition is zero.

(2) We define an automorphism ν_A of $\hat{A}$, called the *Nakayama automorphism* of $\hat{A}$, by $\nu_A(x^{[i]}) := x^{[i+1]}$, $\nu_A(f^{[i]}) := f^{[i+1]}$, $\nu_A(\phi^{[i]}) := \phi^{[i+1]}$ for all $i \in \mathbb{Z}, x \in A_0, f \in A_1, \phi \in \bigcup_{x,y \in A_0} DA(y,x)$.

(3) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $\hat{A}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow \hat{A}, x \mapsto x^{[n]}$, the embedding functor.

We cite the following [3, Proposition 1.6.].

Proposition 2. *Let A be an algebra, n an integer, and ϕ an automorphism of $\hat{A}$. Then the following are equivalent:*

(1) ϕ *is an automorphism with jump* n*;*
(2) $\phi(A^{[i]}) = A^{[i+n]}$ *for some integer* i*;*
(3) $\phi(A^{[j]}) = A^{[j+n]}$ *for all integers* j*; and*
(4) $\phi = \phi_L \nu_A^n$ *for some automorphism* ϕ_L *of* $\hat{A}$ *with jump* 0*.*
(5) $\phi = \nu_A^n \phi_R$ *for some automorphism* ϕ_R *of* $\hat{A}$ *with jump* 0*.*

We cite the following from [1, Lemma 2.3].

Lemma 3. *Let* $\psi\colon A \to B$ *be an isomorphism of locally bounded categories. Denote by* $\psi_x^y\colon A(y,x) \to B(\psi y, \psi x)$ *the isomorphism defined by* ψ *for all* $x, y \in A$*. Define* $\hat{\psi}\colon \hat{A} \to \hat{B}$ *as follows.*

- *For each* $x^{[i]} \in \hat{A}, \hat{\psi}(x^{[i]}) := (\psi x)^{[i]}$*;*
- *For each* $f^{[i]} \in \hat{A}(x^{[i]}, y^{[i]}), \hat{\psi}(f^{[i]}) := (\psi f)^{[i]}$*; and*
- *For each* $\phi^{[i]} \in \hat{A}(x^{[i]}, y^{[i+1]}), \hat{\psi}(\phi^{[i]}) := (D((\psi_x^y)^{-1})(\phi))^{[i]} = (\phi \circ (\psi_x^y)^{-1})^{[i]}$*.*

Then

(1) $\hat{\psi}$ *is an isomorphism.*
(2) *Given an isomorphism* $\rho\colon \hat{A} \to \hat{B}$*, the following are equivalent.*
 (a) $\rho = \hat{\psi}$*;*
 (b) ρ *satisfies the following.*
  (i) $\rho\nu_A = \nu_B\rho$*;*
  (ii) $\rho(A^{[0]}) = A^{[0]}$*;*
  (iii) *The diagram*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\mathbf{1}^{[0]}"'] & B \arrow[d, "\mathbf{1}^{[0]}"] \\ A^{[0]} \arrow[r, "\rho"'] & B^{[0]} \end{tikzcd}$$

  is commutative; and
  (iv) $\rho(\phi^{[0]}) = (\phi \circ (\psi_x^y)^{-1})^{[0]}$ *for all* $x, y \in A$ *and all* $\phi \in DA(y,x)$*.*

3. Automorphisms of repetitive category with jump 0

Throughout this section A is an algebra. We set $\mathrm{Aut}^0(\hat{A})$ to be the group of all automorphisms of $\hat{A}$ with jump 0.

Lemma 4. *Let* $\phi \in \mathrm{Aut}^0(\hat{A})$*. Then* ϕ *gives a family of* $\Bbbk$*-linear maps* $(\phi_i, f_i)_{i\in\mathbb{Z}}$*, where* ϕ_i *is an automorphism of* A *and* $f_i : A \to A$ *is a bijective* ϕ_i*-*ϕ_{i+1}*-semilinear map for all* $i \in \mathbb{Z}$*.*

Proof. Let $i \in \mathbb{Z}$. Then by definition, we have $\phi(A^{[i]}) = A^{[i]}$. We set $\phi_i := (\mathbb{1}_A^{[i]})^{-1}\phi\mathbb{1}_A^{[i]} : A \to A$, then ϕ_i is an automorphism of A. On the other hand, also by definition, we have $\phi(DA^{[i]}) = DA^{[i]}$. Hence we get a bijective $\Bbbk$-linear map $D(f_i^{-1}) := D(\mathbb{1}_{A^{[i]}})\phi(D(\mathbb{1}_{A^{[i]}}))^{-1} : A \to A$. For morphisms $a, b \in A_1$ and $\mu^* \in DA_1$, $b^{[i+1]}\mu^{*[i]}a^{[i]} = (a\mu^*b)^{[i]} \in DA^{[i]}$ and

$$\phi(b^{[i+1]}\mu^{*[i]}a^{[i]}) = \phi(b^{[i+1]})\phi(\mu^{*[i]})\phi(a^{[i]}).$$

Since
$$LHS = (D(f_i^{-1})(a\mu^* b))^{[i]} = ((a\mu^* b)f_i^{-1})^{[i]}$$
and
$$RHS = \phi_{i+1}(b)^{[i+1]}(D(f_i^{-1})(\mu^*))^{[i]}\phi_i(a)^{[i]} = \phi_{i+1}(b)^{[i+1]}(\mu^* f_i^{-1})^{[i]}\phi_i(a)^{[i]},$$
we have $f_i(a\alpha b) = \phi_i(a)f_i(\alpha)\phi_{i+1}(b)$ for each $\alpha \in A_1$, which shows that f_i is ϕ_i-ϕ_{i+1}-semilinear. □

We identify ϕ with $(\phi_i, f_i)_{i\in\mathbb{Z}}$ and write $\phi = (\phi_i, f_i)_{i\in\mathbb{Z}}$.

For $\psi \in \mathrm{Aut}(\hat{A})$ with jump $n \in \mathbb{Z}$, we also get a family of $\Bbbk$-linear maps by following way. By Proposition 2, there exists an automorphism $\psi_R = (\psi_{Ri}, f_i)_{i\in\mathbb{Z}}$ of $\hat{A}$ with jump 0 such that $\psi = \nu_A^n \psi_R$. We can define $(\psi_i, g_i)_{i\in\mathbb{Z}}$ by $\psi_i := \psi_{Ri}$, $g_i := f_i$ for all $i \in \mathbb{Z}$.

Remark 5. We can define a group homomorphism $\Psi : \mathrm{Aut}^0(\hat{A}) \to \mathrm{Aut}(A)$ by $\Psi(\phi) := \phi_0$ for all $\phi \in \mathrm{Aut}^0(\hat{A})$. Then we have $\hat{\sigma} \in \mathrm{Aut}^0(\hat{A})$ and $\Psi(\hat{\sigma}) = \sigma$ for all $\sigma \in \mathrm{Aut}(A)$ by lemma 3. Thus Ψ is an epimorphism, in particular split epimorhism.

Clearly an automorphism ϕ in the kernel of Ψ is whose ϕ_0 is the identity of A. Therefore to see the kernel of Ψ more particularly, we are interested to construct an automorphism of $\hat{A}$ from the identity of A.

Definition 6. We define a map $\xi : (\Bbbk^\times)^{A_0} \to \mathrm{Aut}(A)$ by
$$\xi(\lambda)(e) := e$$
and
$$\xi(\lambda)(a) := \lambda(t(a))^{-1}\lambda(s(a))a$$
for all $\lambda = (\lambda(x))_{x\in A_0} \in (\Bbbk^\times)^{A_0}$, all objects e and morphisms a in A.

Then ξ is a group homomorphism.

Lemma 7. *Let $\lambda = (\lambda_i)_{i\in\mathbb{Z}} \in (\Bbbk^\times)^{\hat{A}_0}$ (We regard $(\Bbbk^\times)^{\hat{A}_0} = ((\Bbbk^\times)^{A_0})^{\mathbb{Z}}$ by the canonical isomorphism $(\Bbbk^\times)^{\hat{A}_0} = (\Bbbk^\times)^{A_0\times\mathbb{Z}} \cong ((\Bbbk^\times)^{A_0})^{\mathbb{Z}}$). Then a family $(\phi_i, f_i)_{i\in\mathbb{Z}}$ of maps where*
$$\phi_i := \begin{cases} \xi(\lambda_i\lambda_{i+1}\cdots\lambda_{-1}) & \text{if } i < 0 \\ \mathbb{1}_A & \text{if } i = 0 \\ \xi(\lambda_0\lambda_1\cdots\lambda_{i-1}) & \text{if } i > 0 \end{cases}$$
and $f_i : A \to A$ is defined by $f_i(a) := \lambda_i(s(a))\phi_i(a) (= \lambda_i(t(a))\phi_{i+1}(a))$ for $a \in A_1$, gives an automorphism of $\hat{A}$ with jump 0.

We assume the following property which is necessary for our purpose.

Definition 8. If $eAe \cong \Bbbk$ for all primitive idempotents of A, then A is said to have no nonzero oriented cycles.

Let $A := \Bbbk Q/I$ where Q is a quiver and I is an admissible ideal of $\Bbbk Q$. The definition 8 means that I contains all cycles in Q.

Proposition 9. *Assume that A has no nonzero oriented cycles. Then there is an exact sequence of groups*
$$1 \to (\Bbbk^\times)^{\hat{A}_0} \xrightarrow{\Phi} \mathrm{Aut}^0(\hat{A}) \xrightarrow{\Psi} \mathrm{Aut}(A) \to 1.$$

Proof. For $\lambda \in (\Bbbk^{\times})^{\hat{A}_0}$, we define $\Phi(\lambda)$ the automorphism constructed by Lemma 7. Since ξ is group homomorphism, clearly Φ is a group homomorphism. First, we show that Φ is injective. If $\Phi(\lambda) = \mathbb{1}_{\hat{A}}$, then $\phi_i = \mathbb{1}_A$ and $D(f_i^{-1}) = \mathbb{1}_{DA}$ for all $i \in \mathbb{Z}$. By induction, the former implies that $\xi(\lambda_i) = \mathbb{1}_A$ for all $i \in \mathbb{Z}$. Hence for each $i \in \mathbb{Z}$, we get an element $k_i \in \Bbbk^{\times}$ such that $f_i = k_i \mathbb{1}_A$ for all $x \in A_0$. Therefore $D(f_i^{-1}) = \lambda_i^{-1}\mathbb{1}_{DA} = \mathbb{1}_{DA}$, so that $k_i = 1$ for all $i \in \mathbb{Z}$. This shows that Φ is injective.

Next we show that $\operatorname{Im}\Phi = \operatorname{Ker}\Psi$. We easily have $\Psi\Phi = 1$ by definition, hence it is enough to show $\operatorname{Im}\Phi \supseteq \operatorname{Ker}\Psi$. Let $\psi = (\psi_i, g_i)_{i\in\mathbb{Z}} \in \operatorname{Ker}\Psi$. Since g_i is a ψ_i-ψ_{i+1}-semilinear bijection, the equality $\psi_0 = \mathbb{1}_A$ imply that $g_i(x) = g_i(x^3) = \psi_i(x)g_i(x)\psi_{i+1}(x) = xg_i(x)x$ for all $x \in A_0$. Hence $g_i(x) \in xAx = \Bbbk x$, because A have no nonzero oriented cycles. Therefore we get $\lambda_i(x) \in \Bbbk^{\times}$ such that $g_i(x) = \lambda_i(x)x$ for each $i \in \mathbb{Z}$ and $x \in A_0$. We claim that $\psi = \Phi(\lambda)$. Set $\Phi(\lambda) = (\phi_i, f_i)_{i\in\mathbb{Z}}$. To see that, we take an arbitrary morphism $a \in A(x,y)$. If $\phi_i = \psi_i$ for all $i \in \mathbb{Z}$, then

$$\begin{aligned} f_i(a) &= \lambda_i(x)\phi_i(a) \\ &= \lambda_i(x)\psi_i(a) \\ &= \lambda_i(x)\psi_i(ax) \\ &= \psi_i(a)\lambda_i(x)\psi_i(x) \\ &= \psi_i(a)\lambda_i(x)x \\ &= \psi_i(a)g_i(x) \\ &= g_i(a). \end{aligned}$$

Hence we check that $\psi_i = \xi(\lambda_0\lambda_1\cdots\lambda_{i-1})$ for all $0 \le i \in \mathbb{Z}$ and $\psi_i = \xi(\lambda_i\lambda_{i+1}\cdots\lambda_{-1})^{-1}$ for all $0 > i \in \mathbb{Z}$. We prove by induction on $0 \le i \in \mathbb{Z}$ the first equality, the other one following in a similar way. Since $\psi_0 = \mathbb{1}_A = \phi_0$, it is enough to show it in the case that $1 \le i$. For any morphism $a \in A(x,y)$,

$$\begin{aligned} \phi_i(a) &= \xi(\lambda_0\lambda_1\cdots\lambda_{i-1})(a) \\ &= \lambda_0\cdots\lambda_{i-1}(x)(\lambda_0\cdots\lambda_{i-1}(y))^{-1}a \\ &= \lambda_{i-1}(x)(\lambda_{i-1}(y))^{-1}\psi_{i-1}(a) \\ &= \lambda_{i-1}(x)(\lambda_{i-1}(y))^{-1}\psi_{i-1}(a)x \\ &= (\lambda_{i-1}(y))^{-1}\psi_{i-1}(a)g_{i-1}(x) \\ &= (\lambda_{i-1}(y))^{-1}g_{i-1}(a) \\ &= (\lambda_{i-1}(y))^{-1}g_{i-1}(ya) \\ &= (\lambda_{i-1}(y))^{-1}g_{i-1}(y)\psi_i(a) \\ &= y\psi_i(a) \\ &= \psi_i(a) \end{aligned}$$

as desired. □

Remark 10.

(1) By Remark 5, the exact sequence in Proposition 9 splits. Therefore an automorphism of $\hat{A}$ with jump is characterized by an automorphism of A and a map

from $\hat{A}_0$ to $\Bbbk^\times$. Let $\phi = (\phi_i, f_i)_{i\in\mathbb{Z}}$ be an automorphism of $\hat{A}$ with jump. For all morphism $a \in A(x, y)$,

$$f_i(a) = \phi_i(a) f_i(x) = f_i(y)\phi_{i+1}(a)$$

and

$$f_i(x) = f_i(x^3) = \phi_i(x) f_i(x)\phi_{i+1}(x).$$

By Proposition 2, $\phi_i(x) = \phi_{i+1}(x)$ therefore $f_i(x) \in \phi_i(x)A\phi_i(x) = \Bbbk\phi_i(x)$. Hence we get $\lambda_i(x) \in \Bbbk$ such that $f_i(x) = \lambda_i(x)\phi_i(x)$ and

$$\phi_{i+1}(a) = \lambda_i(x)(\lambda_i(y))^{-1}\phi_i(a).$$

(2) In [5, section 3] automorphisms of repetitive category with jump 0 is characterized in general case i.e., it does not assume that algebras have no nonzero oriented cycles, and automorphisms are "algebra automorphisms". In their results, the left term of exact sequence is given by $\mathrm{U}(A)^{\mathbb{Z}}$ where $\mathrm{U}(A)$ is the set of all units in A.

4. Orbit categories

Throughout this section G is a group. A pair $(\mathcal{C}, A)$ of a category and a group homomorphism $A : G \to \mathrm{Aut}(\mathcal{C})$ (we write $A_\alpha := A(\alpha)$) is called a category with G-action.

We cite the following definition and lemma from [2, Section 4].

Definition 11. Let $(\mathcal{C}, A)$, $(\mathcal{C}', A')$ be categories with G-actions and $F : \mathcal{C} \to \mathcal{C}'$ a functor. Then an *equivariance adjuster* of F is a family $\eta = (\eta_\alpha)_{\alpha\in G}$ of natural isomorphisms $\eta_\alpha : A'_\alpha F \Rightarrow FA_\alpha$ ($\alpha \in G$) such that the following diagram commutes for each $\alpha, \beta \in G$

$$\begin{tikzcd}
A'_{\beta\alpha}F = A'_\beta A'_\alpha F \arrow[r, "A'_\beta\eta_\alpha"] \arrow[dr, "\eta_{\beta\alpha}"'] & A'_\beta F A_\alpha \arrow[d, "\eta_\beta A_\alpha"] \\
& FA_{\beta\alpha} = FA_\beta A_\alpha
\end{tikzcd}$$

and a pair (F, η) is called a G-*equivariant* functor.

Lemma 12. *Let $(\mathcal{C}, A)$, $(\mathcal{C}', A')$ be categories with G-actions, and $(F, \eta) : \mathcal{C} \to \mathcal{C}'$ a G-equivariant equivalence. Then $\mathcal{C}/G$ and $\mathcal{C}'/G$ are equivalent.*

Proposition 13. *Let R be a locally bounded category, and g, h automorphisms of R. If there exists a map $\rho \colon R_0 \to \Bbbk^\times$ such that $\rho(y)g(f) = h(f)\rho(x)$ for all morphisms $f \in R(x, y)$, then $R/\langle g\rangle \cong R/\langle h\rangle$.*

Remark 14. Proposition 13 does not assume free actions. Therefore we extend a derived equivalence classification in [3] to "0-fold" case.

5. Main results

Throughout this section we assume that A is an algebra without nonzero oriented cycles unless we note.

Lemma 15. *Let ϕ and ψ be automorphisms of $\hat{A}$ with jump $n \in \mathbb{Z}$. If there exists a map $\rho_0 \colon A_0 \to \Bbbk^\times$ such that $\rho_0(y)\phi_0(a) = \psi_0(a)\rho_0(x)$ for all morphisms $a \in A(x, y)$, then $\hat{A}/\langle\phi\rangle$ and $\hat{A}/\langle\psi\rangle$ are isomorphic.*

Theorem 16. *Let ϕ and ψ be automorphisms of $\hat{A}$ with jump $n \in \mathbb{Z}$. If there exist $i, j \in \mathbb{Z}$ and $\rho : A_0 \to \Bbbk^\times$ such that $\rho(y)\phi_i(a) = \psi_j(a)\rho(x)$ for all morphisms $a \in A(x, y)$, then $\hat{A}/\langle\phi\rangle$ and $\hat{A}/\langle\psi\rangle$ are isomorphic.*

Proof. By Remark 10(1), we get each of the elements $((\lambda_k(x))_{x\in A_0})_{k\in\mathbb{Z}}, ((\mu_k(x))_{x\in A_0})_{k\in\mathbb{Z}} \in ((\Bbbk^\times)^{A_0})^{\mathbb{Z}}$ from ϕ and ψ. Define $\rho_0 : A_0 \to \Bbbk^\times$ by

$$\rho_0(x) := \begin{cases} (\lambda_j \cdots \lambda_{-1}(x))^{-1}\mu_i \cdots \mu_{-1}(x)\rho(x) & \text{if } i, j < 0 \\ \lambda_0 \cdots \lambda_j(x)\mu_i \cdots \mu_{-1}(x)\rho(x) & \text{if } i < 0, j > 0 \\ (\lambda_j \cdots \lambda_{-1}(x))^{-1}(\mu_0 \cdots \mu_i(x))^{-1}\rho(x) & \text{if } i > 0, j < 0 \\ \lambda_0 \cdots \lambda_j(x)(\mu_0 \cdots \mu_i(x))^{-1}\rho(x) & \text{if } i, j > 0 \end{cases}$$

for all $x \in A_0$. Then for a morphism $a \in A(x, y)$,

$$\begin{aligned} \rho(y)\phi_i(a) &= \rho(y)((\lambda_{i-1}(x))^{-1}\lambda_{i-1}(y)\phi_{i-1}(a)) \\ &= \rho(y)(\lambda_{i-1}(x))^{-1}\lambda_{i-1}(y)((\lambda_{i-2}(x))^{-1}\lambda_{i-2}(y)\phi_{i-2}(a)) \\ &\vdots \\ &= \rho(y)(\lambda_0 \cdots \lambda_{i-2}\lambda_{i-1}(x))^{-1}\lambda_0 \cdots \lambda_{i-2}\lambda_{i-1}(y)\phi_0(a) \end{aligned}$$

and similarly

$$\psi_j(a)\rho(x) = (\mu_0 \cdots \mu_{j-2}\mu_{j-1}(x))^{-1}\mu_0 \cdots \mu_{j-2}\mu_{j-1}(y)\psi_0(a)\rho(x).$$

Hence we get $\rho_0(y)\phi_0(a) = \psi_0(a)\rho_0(x)$. By Lemma 15, $\hat{A}/\langle\phi\rangle$ and $\hat{A}/\langle\psi\rangle$ are isomorphic. $\square$

Corollary 17. *Let ϕ be an automorphism of $\hat{A}$ with jump $n \in \mathbb{Z}$. Then $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are isomorphic.*

What we want to know is when $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are isomorphic if A is piecewise hereditary algebra of tree type. The following lemma gives us the answer.

Lemma 18. *A piecewise hereditary algebra has no nonzero oriented cycles.*

Proof. If A is a piecewise hereditary algebra, then there is a tilting complex T on a hereditary algebra H such that $A \cong \operatorname{End}(T)$. For all idempotents e in A, eAe is isomorphic to $\operatorname{End}(T_e)$ where T_e is a direct summand of T. By [4, Corollary 5.5], eAe is a piecewise hereditary algebra because T_e is a partial tilting complex. Since piecewise hereditary algebras have finite global dimension and eAe is local, eAe is isomorphic to $\Bbbk$. Hence A have no nonzero oriented cycles if A is a piecewise hereditary algebra. $\square$

Corollary 19. *Let A be a piecewise hereditary algebra and ϕ be an automorphism of $\hat{A}$ with jump $n \in \mathbb{Z}$. Then $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are isomorphic.*

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