THE ARTINIAN CONJECTURE
(FOLLOWING DJAMENT, PUTMAN, SAM, AND SNOWDEN)

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Abstract. This note provides a self-contained exposition of the proof of the artinian conjecture, following closely Djament's Bourbaki lecture. The original proof is due to Putman, Sam, and Snowden.

1. Introduction

This note provides a complete proof of the celebrated artinian conjecture. The proof is due to Putman, Sam, and Snowden [6, 7]. Here, we follow closely the elegant exposition of Djament in [3]. For the origin of the conjecture and its consequences, we refer to those papers and Djament's Bourbaki lecture [4]. In addition, the expository articles by Kuhn, Powell and Schwartz in [5] are recommended.

There are two main result. Fix a locally noetherian Grothendieck abelian category $\mathcal{A}$, for instance, the category of modules over a noetherian ring.

Theorem 1.1. *Let A be a ring whose underlying set is finite. For the category $\mathcal{P}(A)$ of free A-modules of finite rank, the functor category* $\operatorname{Fun}(\mathcal{P}(A)^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

This result amounts to the assertion of the artinian conjecture when A is a finite field and $\mathcal{A}$ is the category of A-modules.

The first theorem is a direct consequence of the following.

Theorem 1.2. *For the category Γ of finite sets, the functor category* $\operatorname{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

The basic idea for the proof is to formulate finiteness conditions on an essentially small category $\mathcal{C}$ such that $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian. This leads to the notion of a Gröbner category. Such finiteness conditions have a 'direction'. For that reason we consider contravariant functors $\mathcal{C} \to \mathcal{A}$, because then the direction is preserved (via Yoneda's lemma) when one passes from $\mathcal{C}$ to $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.

2. Noetherian posets

Let $\mathcal{C}$ be a poset. A subset $\mathcal{D} \subseteq \mathcal{C}$ is a *sieve* if the conditions $x \leq y$ in $\mathcal{C}$ and $y \in \mathcal{D}$ imply $x \in \mathcal{D}$. The sieves in $\mathcal{C}$ are partially ordered by inclusion.

Definition 2.1. A poset $\mathcal{C}$ is called

(1) *noetherian* if every ascending chain of elements in $\mathcal{C}$ stabilises, and
(2) *strongly noetherian* if every ascending chain of sieves in $\mathcal{C}$ stabilises.

The paper is in a final form and no version of it will be submitted for publication elsewhere.

For a poset $\mathcal{C}$ and $x \in \mathcal{C}$, set $\mathcal{C}(x) = \{t \in \mathcal{C} \mid t \leq x\}$. The assignment $x \mapsto \mathcal{C}(x)$ yields an embedding of $\mathcal{C}$ into the poset of sieves in $\mathcal{C}$.

Lemma 2.2. *For a poset $\mathcal{C}$ the following are equivalent:*

(1) *The poset $\mathcal{C}$ is strongly noetherian.*
(2) *For every infinite sequence $(x_i)_{i \in \mathbb{N}}$ of elements in $\mathcal{C}$ there exists $i \in \mathbb{N}$ such that $x_j \leq x_i$ for infinitely many $j \in \mathbb{N}$.*
(3) *For every infinite sequence $(x_i)_{i \in \mathbb{N}}$ of elements in $\mathcal{C}$ there is a map $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\alpha(i) < \alpha(j)$ and $x_{\alpha(j)} \leq x_{\alpha(i)}$.*
(4) *For every infinite sequence $(x_i)_{i \in \mathbb{N}}$ of elements in $\mathcal{C}$ there are $i < j$ in $\mathbb{N}$ such that $x_j \leq x_i$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathcal{C}$ is strongly noetherian and let $(x_i)_{i \in \mathbb{N}}$ be elements in $\mathcal{C}$. For $n \in \mathbb{N}$ set $\mathcal{C}_n = \bigcup_{i \leq n} \mathcal{C}(x_i)$. The chain $(\mathcal{C}_n)_{n \in \mathbb{N}}$ stabilises, say $\mathcal{C}_n = \mathcal{C}_N$ for all $n \geq N$. Thus there exists $i \leq N$ such that $x_j \leq x_i$ for infinitely many $i \in \mathbb{N}$.

(2) $\Rightarrow$ (3): Define $\alpha\colon \mathbb{N} \to \mathbb{N}$ recursively by taking for $\alpha(0)$ the smallest $i \in \mathbb{N}$ such that $x_j \leq x_i$ for infinitely many $j \in \mathbb{N}$. For $n > 0$ set

$$\alpha(n) = \min\{i > \alpha(n-1) \mid x_j \leq x_i \leq x_{\alpha(n-1)} \text{ for infinitely many } j \in \mathbb{N}\}.$$

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (1): Suppose there is a properly ascending chain $(\mathcal{C}_n)_{n \in \mathbb{N}}$ of sieves in $\mathcal{C}$. Choose $x_n \in \mathcal{C}_{n+1} \setminus \mathcal{C}_n$ for each $n \in \mathbb{N}$. There are $i < j$ in $\mathbb{N}$ such that $x_j \leq x_i$. This implies $x_j \in \mathcal{C}_{i+1} \subseteq \mathcal{C}_j$ which is a contradiction. $\square$

3. Functor categories

Let $\mathcal{C}$ be an essentially small category and $\mathcal{A}$ a Grothendieck abelian category. We denote by $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. The morphisms between two functors are the natural transformations. Note that $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is a Grothendieck abelian category.

Given an object $x \in \mathcal{C}$, the evaluation functor

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}) \longrightarrow \mathcal{A}, \quad F \mapsto F(x)$$

admits a left adjoint

$$\mathcal{A} \longrightarrow \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}), \quad M \mapsto M[\mathcal{C}(-,x)]$$

where for any set X we denote by $M[X]$ a coproduct of copies of M indexed by the elements of X. Thus we have a natural isomorphism

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})(M[\mathcal{C}(-,x)], F) \cong \mathcal{A}(M, F(x)). \tag{3.1}$$

Lemma 3.1. *If $(M_i)_{i \in I}$ is a set of generators of $\mathcal{A}$, then the functors $M_i[\mathcal{C}(-,x)]$ with $i \in I$ and $x \in \mathcal{C}$ generate $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.*

Proof. Use the adjointness isomorphism (3.1). $\square$

A Grothendieck abelian category $\mathcal{A}$ is *locally noetherian* if $\mathcal{A}$ has a generating set of noetherian objects. In that case an object $M \in \mathcal{A}$ is noetherian iff M is *finitely presented* (that is, the representable functor $\mathcal{A}(M, -)$ preserves filtered colimits); see [8, Chap. V] for details.

Lemma 3.2. *Let $\mathcal{A}$ be locally noetherian. Then* $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian iff* $M[\mathcal{C}(-,x)]$ *is noetherian for every noetherian* $M \in \mathcal{A}$ *and* $x \in \mathcal{C}$.

Proof. First observe that $M[\mathcal{C}(-,x)]$ is finitely presented if M is finitely presented. This follows from the isomorphism (3.1) since evaluation at $x \in \mathcal{C}$ preserves colimits. Now the assertion of the lemma is an immediate consequence of Lemma 3.1. □

4. Noetherian functors

Let $\mathcal{C}$ be an essentially small category and fix an object $x \in \mathcal{C}$. Set

$$\mathcal{C}(x) = \coprod_{t \in \mathcal{C}} \mathcal{C}(t,x).$$

Given $f, g \in \mathcal{C}(x)$, let $\langle f \rangle$ denote the set of morphisms in $\mathcal{C}(x)$ that factor through f, and set $f \leq_x g$ if $\langle f \rangle \subseteq \langle g \rangle$. We identify f and g when $\langle f \rangle = \langle g \rangle$. This yields a poset which we denote by $\bar{\mathcal{C}}(x)$.

A functor is *noetherian* if every ascending chain of subfunctors stabilises.

Lemma 4.1. *The functor* $\mathcal{C}(-,x)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *is noetherian iff the poset* $\bar{\mathcal{C}}(x)$ *is strongly noetherian.*

Proof. Sending $F \subseteq \mathcal{C}(-,x)$ to $\bigcup_{t \in \mathcal{C}} F(t)$ induces an inclusion preserving bijection between the subfunctors of $\mathcal{C}(-,x)$ and the sieves in $\bar{\mathcal{C}}(x)$. □

For a poset $\mathcal{T}$ let $\mathrm{Set} \wr \mathcal{T}$ denote the category consisting of pairs (X, ξ) such that X is a set and $\xi\colon X \to \mathcal{T}$ is a map. A morphism $(X,\xi) \to (X', \xi')$ is a map $f\colon X \to X'$ such that $\xi(a) \leq \xi' f(a)$ for all $a \in X$.

A functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \wr \mathcal{T}$ is given by a pair (F, ϕ) consisting of a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and a map $\phi\colon \coprod_{t \in \mathcal{C}} F(t) \to \mathcal{T}$ such that $\phi(a) \leq \phi(F(f)(a))$ for every $a \in F(t)$ and $f\colon t' \to t$ in $\mathcal{C}$.

Lemma 4.2. *Let $\mathcal{T}$ be a noetherian poset. If $\mathcal{C}(-,x)$ is noetherian, then any functor* $(\mathcal{C}(-,x), \phi)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \wr \mathcal{T}$ *is noetherian.*

Proof. Let $(F_n, \phi_n)_{n \in \mathbb{N}}$ be a strictly ascending chain of subfunctors of (F, ϕ). The chain $(F_n)_{n \in \mathbb{N}}$ stabilises since $\mathcal{C}(-,x)$ is noetherian. Thus we may assume that $F_n = F$ for all $n \in \mathbb{N}$, and we find $f_n \in \coprod_{t \in \mathcal{C}} F(t)$ such that $\phi_n(f_n) < \phi_{n+1}(f_n)$. The poset $\bar{\mathcal{C}}(x)$ is strongly noetherian by Lemma 4.1. It follows from Lemma 2.2 that there is a map $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\alpha(i) < \alpha(j)$ and $f_{\alpha(j)} \leq_x f_{\alpha(i)}$. Thus

$$\phi_{\alpha(n)}(f_{\alpha(n)}) < \phi_{\alpha(n)+1}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n+1)}).$$

This yields a strictly ascending chain in $\mathcal{T}$, contradicting the assumption on $\mathcal{T}$. □

Definition 4.3. A partial order $\leq$ on $\mathcal{C}(x)$ is *admissible* if the following holds:

(1) The order $\leq$ restricted to $\mathcal{C}(t,x)$ is total and noetherian for every $t \in \mathcal{C}$.
(2) For $f, f' \in \mathcal{C}(t,x)$ and $e \in \mathcal{C}(s,t)$, the condition $f < f'$ implies $fe < f'e$.

Fix an admissible partial order $\leq$ on $\mathcal{C}(x)$ and an object M in a Grothendieck abelian category $\mathcal{A}$. Let $\mathrm{Sub}(M)$ denote the poset of subobjects of M and consider the functor

$$\mathcal{C}(-,x) \wr M\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \mathrm{Sub}(M), \quad t \mapsto \big(\mathcal{C}(t,x), (M)_{f \in \mathcal{C}(t,x)}\big).$$

For a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ define a subfunctor $\tilde{F} \subseteq \mathcal{C}(-,x) \wr M$ as follows:

$$\tilde{F} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \mathrm{Sub}(M), \quad t \mapsto \Big(\mathcal{C}(t,x), \big(\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t))\big)_{f \in \mathcal{C}(t,x)}\Big)$$

where $\mathcal{C}(t,x)_f = \{g \in \mathcal{C}(t,x) \mid f \leq g\}$ and $\pi_f \colon M[\mathcal{C}(t,x)_f] \to M$ is the projection onto the factor corresponding to f. For a morphism $e \colon t' \to t$ in $\mathcal{C}$, the morphism $\tilde{F}(e)$ is induced by precomposition with e. Note that

$$\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t)) \subseteq \pi_{fe}(M[\mathcal{C}(t',x)_{fe}] \cap F(t'))$$

since $\leq$ is compatible with the composition in $\mathcal{C}$.

Lemma 4.4. *Suppose there is an admissible partial order on $\mathcal{C}(x)$. Then the assignment which sends a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ to $\tilde{F}$ preserves proper inclusions. Therefore $M[\mathcal{C}(-,x)]$ is noetherian provided that $\mathcal{C}(-,x) \wr M$ is noetherian.*

Proof. Let $F \subseteq G \subseteq M[\mathcal{C}(-,x)]$. Then $\tilde{F} \subseteq \tilde{G}$. Now suppose that $F \neq G$. Thus there exists $t \in \mathcal{C}$ such that $F(t) \neq G(t)$. We have $\mathcal{C}(t,x) = \bigcup_{f \in \mathcal{C}(t,x)} \mathcal{C}(t,x)_f$, and this union is directed since $\leq$ is total. Thus

$$F(t) = \sum_{f \in \mathcal{C}(t,x)} \big(M[\mathcal{C}(t,x)_f] \cap F(t)\big)$$

since filtered colimits in $\mathcal{A}$ are exact. This yields f such that

$$M[\mathcal{C}(t,x)_f] \cap F(t) \neq M[\mathcal{C}(t,x)_f] \cap G(t).$$

Choose $f \in \mathcal{C}(t,x)$ maximal with respect to this property, using that $\leq$ is noetherian. Now observe that the projection π_f induces an exact sequence

$$0 \longrightarrow \sum_{f<g} \big(M[\mathcal{C}(t,x)_g] \cap F(t)\big) \longrightarrow F(t) \longrightarrow \pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \longrightarrow 0$$

since the kernel of π_f equals the directed union $\sum_{f<g} M[\mathcal{C}(t,x)_g]$. For the directedness one uses again that $\leq$ is total. Thus

$$\pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \neq \pi_f\big(M[\mathcal{C}(t,x)_f] \cap G(t)\big)$$

and therefore $\tilde{F} \neq \tilde{G}$. □

Proposition 4.5. *Let $x \in \mathcal{C}$. Suppose that $\mathcal{C}(-,x)$ is noetherian and that $\mathcal{C}(x)$ has an admissible partial order. If $M \in \mathcal{A}$ is noetherian, then $M[\mathcal{C}(-,x)]$ is noetherian.*

Proof. Combine Lemmas 4.2 and 4.4. □

5. Gröbner categories

Definition 5.1. An essentially small category $\mathcal{C}$ is a *Gröbner category* if the following holds:

(1) The functor $\mathcal{C}(-,x)$ is noetherian for every $x \in \mathcal{C}$.
(2) There is an admissible partial order on $\mathcal{C}(x)$ for every $x \in \mathcal{C}$.

Theorem 5.2. *Let $\mathcal{C}$ be a Gröbner category and $\mathcal{A}$ a Grothendieck abelian category. If $\mathcal{A}$ is locally noetherian, then $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

Proof. Combine Lemma 3.1 and Proposition 4.5. □

Example 5.3. (1) A strongly noetherian poset (viewed as a category) is a Gröbner category.

(2) The additive monoid $\mathbb{N}$ of natural numbers (viewed as a category with a single object) is a Gröbner category. Let $\mathcal{A}$ be the module category of a noetherian ring A. Then $\operatorname{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{A})$ equals the module category of the polynomial ring in one variable over A. Thus Theorem 5.2 generalises Hilbert's Basis Theorem.

6. Base change

Given functors $F, G\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, we write $F \rightsquigarrow G$ if there is a finite chain

$$F = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = G$$

of epimorphisms and monomorphisms of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

Definition 6.1. A functor $\phi\colon \mathcal{C} \to \mathcal{D}$ is *contravariantly finite*[1] if the following holds:

(1) Every object $y \in \mathcal{D}$ is isomorphic to $\phi(x)$ for some $x \in \mathcal{C}$.

(2) For every object $y \in \mathcal{D}$ there are objects $x_1, \ldots, x_n$ in $\mathcal{C}$ such that

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) \rightsquigarrow \mathcal{D}(\phi-, y).$$

The functor ϕ is *covariantly finite* if $\phi^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is contravariantly finite.

Note that a composite of contravariantly finite functors is contravariantly finite.

Lemma 6.2. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a Grothendieck abelian category. Fix $M \in \mathcal{A}$ and suppose that $M[\mathcal{C}(-, x)]$ is noetherian for all $x \in \mathcal{C}$. Then $M[\mathcal{D}(-, y)]$ is noetherian for all $y \in \mathcal{D}$.*

Proof. A finite chain

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = \mathcal{D}(\phi-, y)$$

of epimorphisms and monomorphisms induces a chain

$$\prod_{i=1}^{n} M[\mathcal{C}(-, x_i)] = \bar{F}_0 \twoheadrightarrow \bar{F}_1 \leftarrowtail \bar{F}_2 \twoheadrightarrow \cdots \twoheadrightarrow \bar{F}_{n-1} \leftarrowtail \bar{F}_n = M[\mathcal{D}(\phi-, y)]$$

of epimorphisms and monomorphisms in $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. Thus $M[\mathcal{D}(\phi-, y)]$ is noetherian. It follows that $M[\mathcal{D}(-, y)]$ is noetherian, since precomposition with ϕ yields a faithful and exact functor $\operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A}) \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. □

Proposition 6.3. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a locally noetherian Grothendieck abelian category. If the category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian, then $\operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

Proof. Combine Lemmas 3.2 and 6.2. □

[1] The terminology follows that introduced by Auslander and Smalø [1] for an inclusion functor.

7. Categories of finite sets

Let Γ denote the category of finite sets (a skeleton is given by the sets $\mathbf{n} = \{1, 2, \ldots, n\}$). The subcategory of finite sets with surjective morphisms is denoted by Γ_{sur}. A surjection $f\colon \mathbf{m} \to \mathbf{n}$ is *ordered* if $i < j$ implies $\min f^{-1}(i) < \min f^{-1}(j)$. We write Γ_{os} for the subcategory of finite sets whose morphisms are ordered surjections. Given a surjection $f\colon \mathbf{m} \to \mathbf{n}$, let $f^!\colon \mathbf{n} \to \mathbf{m}$ denote the map given by $f^!(i) = \min f^{-1}(i)$. Note that $ff^! = \mathrm{id}$, and $gf = f^! g^!$ provided that f and g are ordered surjections.

Lemma 7.1. (1) *The inclusion $\Gamma_{\text{sur}} \to \Gamma$ is contravariantly finite.*
(2) *The inclusion $\Gamma_{\text{os}} \to \Gamma_{\text{sur}}$ is contravariantly finite.*

Proof. (1) For each integer $n \geq 0$ there is an isomorphism

$$\coprod_{\mathbf{m} \hookrightarrow \mathbf{n}} \Gamma_{\text{sur}}(-, \mathbf{m}) \xrightarrow{\sim} \Gamma(-, \mathbf{n})$$

which is induced by the injective maps $\mathbf{m} \to \mathbf{n}$.

(2) For each integer $n \geq 0$ there is an isomorphism

$$\Gamma_{\text{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \xrightarrow{\sim} \Gamma_{\text{sur}}(-, \mathbf{n})$$

which sends a pair (f, σ) to σf. The inverse sends a surjective map $g\colon \mathbf{m} \to \mathbf{n}$ to $(\tau^{-1} g, \tau)$ where $\tau \in \mathfrak{S}_n$ is the unique permutation such that $g^! \tau$ is increasing. □

Fix an integer $n \geq 0$. Given $f, g \in \Gamma(\mathbf{n})$ we set $f \leq g$ if there exists an ordered surjection h such that $f = gh$.

Lemma 7.2. *The poset $(\Gamma(\mathbf{n}), \leq)$ is strongly noetherian.*

Proof. We fix some notation for each $f \in \Gamma(\mathbf{m}, \mathbf{n})$. Set $\lambda(f) = m$. If f is not injective, set

$$\mu(f) = m - \max\{i \in \mathbf{m} \mid \text{there exists } j < i \text{ such that } f(i) = f(j)\}$$

and $\pi(f) = f(m - \mu(f))$. Define $\tilde{f} \in \Gamma(\mathbf{m} - \mathbf{1}, \mathbf{n})$ by setting $\tilde{f}(i) = f(i)$ for $i < m - \mu(f)$ and $\tilde{f}(i) = f(i+1)$ otherwise.

Note that $f \leq \tilde{f}$. Moreover, $\mu(f) = \mu(g)$, $\pi(f) = \pi(g)$, and $\tilde{f} \leq \tilde{g}$ imply $f \leq g$.

Suppose that $(\Gamma(\mathbf{n}), \leq)$ is not strongly noetherian. Then there exists an infinite sequence $(f_r)_{r \in \mathbb{N}}$ in $\Gamma(\mathbf{n})$ such that $i < j$ implies $f_j \not\leq f_i$; see Lemma 2.2. Call such a sequence *bad*. Choose the sequence *minimal* in the sense that $\lambda(f_i)$ is minimal for all bad sequences $(g_r)_{r \in \mathbb{N}}$ with $g_j = f_j$ for all $j < i$. There is an infinite subsequence $(f_{\alpha(r)})_{r \in \mathbb{N}}$ (given by some increasing map $\alpha\colon \mathbb{N} \to \mathbb{N}$) such that μ and π agree on all $f_{\alpha(r)}$, since the values of μ and π are bounded by n. Now consider the sequence $f_0, f_1, \ldots, f_{\alpha(0)-1}, \tilde{f}_{\alpha(0)}, \tilde{f}_{\alpha(1)}, \ldots$ and denote this by $(g_r)_{r \in \mathbb{N}}$. This sequence is not bad, since $(f_r)_{r \in \mathbb{N}}$ is minimal. Thus there are $i < j$ in $\mathbb{N}$ with $g_j \leq g_i$. Clearly, $j < \alpha(0)$ is impossible. If $i < \alpha(0)$, then

$$f_{\alpha(j-\alpha(0))} \leq \tilde{f}_{\alpha(j-\alpha(0))} = g_j \leq g_i = f_i,$$

which is a contradiction, since $i < \alpha(0) \leq \alpha(j - \alpha(0))$. If $i \geq \alpha(0)$, then $f_{\alpha(j-\alpha(0))} \leq f_{\alpha(i-\alpha(0))}$; this is a contradiction again. Thus $(\Gamma(\mathbf{n}), \leq)$ is strongly noetherian. □

Proposition 7.3. *The category Γ_{os} is a Gröbner category.*

Proof. Fix an integer $n \geq 0$. The poset $\bar{\Gamma}_{\mathrm{os}}(\mathbf{n})$ is strongly noetherian by Lemma 7.2, and it follows from Lemma 4.1 that the functor $\Gamma_{\mathrm{os}}(-, \mathbf{n})$ is noetherian.

The admissible partial order on $\Gamma_{\mathrm{os}}(\mathbf{n})$ is given by the lexicographic order. Thus for $f, g \in \Gamma_{\mathrm{os}}(\mathbf{m}, \mathbf{n})$, we have $f < g$ if there exists $j \in \mathbf{m}$ with $f(j) < g(j)$ and $f(i) = g(i)$ for all $i < j$. □

Theorem 7.4. *Let $\mathcal{A}$ be a locally noetherian Grothendieck abelian category. Then the category* $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof. The category Γ_{os} is a Gröbner category by Proposition 7.3. It follows from Theorem 5.2 that $\mathrm{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian. The inclusion $\Gamma_{\mathrm{os}} \to \Gamma$ is contravariantly finite by Lemma 7.1. Thus $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 6.3. □

8. The artinian conjecture

Let A be a ring. We denote by $\mathcal{P}(A)$ the category of free A-modules of finite rank. If A is finite, then the functor $\Gamma \to \mathcal{P}(A)$ sending X to $A[X]$ is a left adjoint of the forgetful functor $\mathcal{P}(A) \to \Gamma$.

Lemma 8.1. *Let A be finite. Then the functor $\Gamma \to \mathcal{P}(A)$ is contravariantly finite.*

Proof. The assertion follows from the adjointness isomorphism

$$\mathcal{P}(A)(A[X], P) \cong \Gamma(X, P).$$ □

Theorem 8.2. *Let A be a finite ring and $\mathcal{A}$ a locally noetherian Grothendieck abelian category. Then the category* $\mathrm{Fun}(\mathcal{P}(A)^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof. Combine Theorem 7.4 with Lemma 8.1 and Proposition 6.3. □

9. FI-modules

The proof of the artinian conjecture yields an alternative proof of the following result due to Church, Ellenberg, Farb, and Nagpal.

Let Γ_{inj} denote the category whose objects are finite sets and whose morphisms are injective maps.

Theorem 9.1 ([2, Theorem A])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck abelian category. Then the category* $\mathrm{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ *is locally noetherian.*

Proof. The following argument has been suggested by Kai-Uwe Bux. Consider the functor $\phi\colon \Gamma_{\mathrm{os}} \to (\Gamma_{\mathrm{inj}})^{\mathrm{op}}$ which is the identity on objects and takes a map $f\colon \mathbf{m} \to \mathbf{n}$ to $f^!\colon \mathbf{n} \to \mathbf{m}$ given by $f^!(i) = \min f^{-1}(i)$. This functor is contravariantly finite, since for each integer $n \geq 0$ the morphism

$$\Gamma_{\mathrm{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \longrightarrow \Gamma_{\mathrm{inj}}(\mathbf{n}, \phi -)$$

which sends a pair (f, σ) to $f^!\sigma$ is an epimorphism.

It follows from Proposition 6.3 that the category $\mathrm{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ is locally noetherian, since $\mathrm{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 7.3 and Theorem 5.2. □

Note added in proof

After completing this paper I found that Theorem 5.2 is precisely the statement of Theorem 3.1 in [G. Richter, Noetherian semigroup rings with several objects, in *Group and semigroup rings (Johannesburg, 1985)*, 231–246, North-Holland Math. Stud., 126, North-Holland, Amsterdam, 1986].

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