HALF EXACT FUNCTORS ASSOCIATED WITH GENERAL HEARTS ON EXACT CATEGORIES

YU LIU

Abstract. We construct a half exact functor from the exact category to the heart of a cotorsion pair. This is analog of the construction of Abe and Nakaoka for triangulated categories. When the cotorsion pair comes from a cluster tilting subcategory, our half exact functor coincides with the canonical quotient functor from the exact category to the quotient category of it by this cluster tilting subcategory. We will also use this half exact functor to find out the relationship between different hearts.

Key Words: exact category, abelian category, cotorsion pair, heart, half exact functor.

1. Introduction

Cotorsion pairs play an important role in representation theory (see [2] and see [3] for more examples). In [4], we define hearts $\underline{\mathcal{H}}$ of cotorsion pairs $(\mathcal{U},\mathcal{V})$ on exact categories $\mathcal{B}$ and proved that they are abelian. This is similar as Nakaoka's result on triangulated categories [5]. It is natural to ask whether we can find any relationship between the hearts and the original exact categories. Abe and Nakaoka have already given an answer by constructing a cohomological functor in the case of triangulated categories [1]. In this paper we will construct an associated half exact functor H from the exact category $\mathcal{B}$ to the heart $\underline{\mathcal{H}}$, which is similar as the construction of Abe and Nakaoka.

Let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the full subcategory of projectives (resp. injectives) of $\mathcal{B}$.

We recall the definition of a cotorsion pair on $\mathcal{B}$ [4, Definition 2.3]:

Definition 1. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{B}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V})=0$.

(b) For any object $B\in\mathcal{B}$, there exits two short exact sequences

$$V_B \rightarrowtail U_B \twoheadrightarrow B, \quad B \rightarrowtail V^B \twoheadrightarrow U^B$$

satisfying $U_B, U^B \in \mathcal{U}$ and $V_B, V^B \in \mathcal{V}$.

For any cotorsion pairs $(\mathcal{U},\mathcal{V})$, let $\mathcal{W} := \mathcal{U}\cap\mathcal{V}$. We denote the quotient of $\mathcal{B}$ by $\mathcal{W}$ as $\underline{\mathcal{B}} := \mathcal{B}/\mathcal{W}$. For any morphism $f\in\mathrm{Hom}_{\mathcal{B}}(X,Y)$, we denote its image in $\mathrm{Hom}_{\underline{\mathcal{B}}}(X,Y)$ by $\underline{f}$. For any subcategory $\mathcal{C}\supseteq\mathcal{W}$ of $\mathcal{B}$, we denote by $\underline{\mathcal{C}}$ the full subcategory of $\underline{\mathcal{B}}$ consisting of the same objects as $\mathcal{C}$. Let

$$\mathcal{B}^+ := \{B\in\mathcal{B} \mid U_B\in\mathcal{W}\}, \quad \mathcal{B}^- := \{B\in\mathcal{B} \mid V^B\in\mathcal{W}\}.$$

The detailed version of this paper will be submitted for publication elsewhere.

Let

$$\mathcal{H} := \mathcal{B}^+ \cap \mathcal{B}^-.$$

Since $\mathcal{H} \supseteq \mathcal{W}$, we have an additive subcategory $\underline{\mathcal{H}}$ which we call the *heart* of cotorsion pair $(\mathcal{U}, \mathcal{V})$.

Definition 2. A covariant functor F from $\mathcal{B}$ to an abelian category $\mathcal{A}$ is called *half exact* if for any short exact sequence

$$A \rightarrowtail^{f} B \overset{g}{\twoheadrightarrow} C$$

in $\mathcal{B}$, the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact in $\mathcal{A}$.

We will prove the following theorem.

Theorem 3. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$, there exists an associated half exact functor*

$$H : \mathcal{B} \to \underline{\mathcal{H}}.$$

The half exact functor we construct gives us a way to find out the relationship between different hearts. Let $k \in \{1, 2\}$, $(\mathcal{U}_k, V_k)$ be a cotorsion pair on $\mathcal{B}$ and $\mathcal{W}_k = \mathcal{U}_k \cap \mathcal{V}_k$. Let $\mathcal{H}_k/\mathcal{W}_k$ be the heart of $(\mathcal{U}_k, V_k)$ and H_k be the associated half exact functor. For $i, j \in \{1, 2\}$ and $i \neq j$, if $H_i(\mathcal{W}_j) = 0$, then H_i induces a functor $\beta_{ji} : \mathcal{H}_j/\mathcal{W}_j \to \mathcal{H}_i/\mathcal{W}_i$. Moreover, we have the following theorem.

Theorem 4. *If $H_j(\mathcal{U}_i) = H_j(\mathcal{V}_i) = 0$ and $H_i(\mathcal{U}_j) = H_i(\mathcal{V}_j) = 0$, then we have an equivalence $\mathcal{H}_i/\mathcal{W}_i \simeq \mathcal{H}_j/\mathcal{W}_j$ between two hearts. More precisely, we have natural isomorphisms $\beta_{ij}\beta_{ji} \simeq \mathrm{id}_{\mathcal{H}_j/\mathcal{W}_j}$ and $\beta_{ji}\beta_{ij} \simeq \mathrm{id}_{\mathcal{H}_i/\mathcal{W}_i}$ of functors.*

2. Notations

For briefly review of the important properties of exact categories, we refer to [4, §2].

Throughout this paper, let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the full subcategory of projectives (resp. injectives) of $\mathcal{B}$.

Definition 5. For any $B \in \mathcal{B}$, we define B^+ as follows:
Take two short exact sequences:

$$V_B \rightarrowtail U_B \overset{u_B}{\twoheadrightarrow} B\ , \quad U_B \overset{w'}{\rightarrowtail} W^0 \twoheadrightarrow U^0$$

where $U_B, U^0 \in \mathcal{U}$, W^0,$\mathcal{V}_B \in \mathcal{V}$. In fact, $W^0 \in \mathcal{W}$ since $\mathcal{U}$ is closed under extension. We get the following commutative diagram

$$\begin{tikzcd}
V_B \arrow[r, tail] \arrow[d, equal] & U_B \arrow[r, two heads, "u_B"] \arrow[d, tail, "w'"'] & B \arrow[d, tail, "\alpha_B"] \\
V_B \arrow[r, tail] & W^0 \arrow[r, two heads, "w"'] \arrow[d, two heads] & B^+ \arrow[d, two heads] \\
& U^0 \arrow[r, equal] & U^0
\end{tikzcd}$$

where the upper-right square is both a push-out and a pull-back.

By [4, Lemma 3.2], $B^+ \in \mathcal{B}^+$, and if $B \in \mathcal{B}^-$, then $B^+ \in \mathcal{H}$.

Proposition 6. [4, Proposition 3.3] *For any $B \in \mathcal{B}$ and $Y \in \mathcal{B}^+$,* $\mathrm{Hom}_{\mathcal{B}}(\alpha_B, Y) : \mathrm{Hom}_{\mathcal{B}}(B^+, Y) \to \mathrm{Hom}_{\mathcal{B}}(B, Y)$ *is surjective and* $\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{\alpha_B}, Y) : \mathrm{Hom}_{\underline{\mathcal{B}}}(B^+, Y) \to \mathrm{Hom}_{\underline{\mathcal{B}}}(B, Y)$ *is bijective.*

We define a functor σ^+ from $\underline{\mathcal{B}}$ to $\underline{\mathcal{B}}^+$ as follows:
For any object $B \in \mathcal{B}$, since all the $B^{+'}s$ are isomorphic to each other in $\underline{\mathcal{B}}$ by Proposition 6, we fix a B^+ for B. Let

$$\sigma^+ : \underline{\mathcal{B}} \to \underline{\mathcal{B}}^+$$
$$B \mapsto B^+$$

and for any morphism $\underline{f} : B \to C$, we define $\sigma^+(\underline{f})$ as the unique morphism given by Proposition 6

$$\begin{tikzcd}
B \arrow[r, "\underline{f}"] \arrow[d, "\underline{\alpha_B}"'] & C \arrow[d, "\underline{\alpha_C}"] \\
B^+ \arrow[r, dotted, "\sigma^+(\underline{f})"'] & C^+.
\end{tikzcd}$$

Dually, we can define σ^-.
Let $\pi : \mathcal{B} \to \underline{\mathcal{B}}$ be the canonical functor. We denote $\sigma^- \circ \sigma^+ \circ \pi$ by

$$H : \mathcal{B} \to \underline{\mathcal{H}}.$$

3. Main results

Proposition 7. *The functor H has the following properties:*

(a) *For any objects A and B in $\mathcal{B}$, $H(A \oplus B) \simeq H(A) \oplus H(B)$ in $\underline{\mathcal{H}}$.*
(b) $H|_{\mathcal{H}} = \pi|_{\mathcal{H}}$.
(c) $H(\mathcal{U}) = 0$ *and* $H(\mathcal{V}) = 0$. *In particular,* $H(\mathcal{P}) = 0$ *and* $H(\mathcal{I}) = 0$.

Theorem 8. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{B}$, the functor*

$$H : \mathcal{B} \to \underline{\mathcal{H}}$$

is half exact. We call H the associated half exact functor to $(\mathcal{U}, \mathcal{V})$.

We have the following general property of half exact functors which H satisfies.

Proposition 9. *Let $\mathcal{A}$ be an abelian category and $F : \mathcal{B} \to \mathcal{A}$ be a half exact functor satisfying $F(\mathcal{P}) = 0$ and $F(\mathcal{I}) = 0$. Then for any short exact sequence*

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

in $\mathcal{B}$, there exist morphisms $h : C \to \Omega^- A$ and $h' : \Omega C \to A$ such that the following sequence

$$\begin{aligned} \cdots \xrightarrow{F(\Omega h')} F(\Omega A) \xrightarrow{F(\Omega f)} F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(h')} F(A) \xrightarrow{F(f)} F(B) \\ \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Omega^- A) \xrightarrow{F(\Omega^- f)} F(\Omega^- B) \xrightarrow{F(\Omega^- g)} F(\Omega^- C) \xrightarrow{F(\Omega^- h)} \cdots \end{aligned}$$

is exact in $\mathcal{A}$.

Let $i \in \{1, 2\}$. Let $(\mathcal{U}_i, \mathcal{V}_i)$ be a cotorsion pair on $\mathcal{B}$ and $\mathcal{W}_i = \mathcal{U}_i \cap \mathcal{V}_i$.

(a) $\mathcal{B}_i^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$V_B \rightarrowtail U_B \twoheadrightarrow B$$

where $U_B \in \mathcal{W}_i$ and $V_B \in \mathcal{V}_i$.

(b) $\mathcal{B}_i^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$B \rightarrowtail V^B \twoheadrightarrow U^B$$

where $V^B \in \mathcal{W}_i$ and $U^B \in \mathcal{U}_i$.

Denote

$$\mathcal{H}_i := \mathcal{B}_i^+ \cap \mathcal{B}_i^-.$$

Then $\mathcal{H}_i/\mathcal{W}_i$ is the heart of $(\mathcal{U}_i, \mathcal{V}_i)$. Let $\pi_i : \mathcal{B} \to \mathcal{B}/\mathcal{W}_i$ be the canonical functor and $\iota_i : \mathcal{H}_i/\mathcal{W}_i \hookrightarrow \mathcal{B}/\mathcal{W}_i$ be the inclusion functor.

If $H_2(\mathcal{W}_1) = 0$, then there exists a functor $h_{12} : \mathcal{B}/\mathcal{W}_1 \to \mathcal{H}_2/\mathcal{W}_2$ such that $H_2 = h_{12}\pi_1$.

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\pi_1"] \arrow[dr, "H_2"'] & & \mathcal{B}/\mathcal{W}_1 \arrow[dl, dotted, "h_{12}"] \\ & \mathcal{H}_2/\mathcal{W}_2 & \end{tikzcd}$$

Hence we get a functor $\beta_{12} := h_{12}\iota_1 : \mathcal{H}_1/\mathcal{W}_1 \to \mathcal{H}_2/\mathcal{W}_2$.

Proposition 10. *Let $(\mathcal{U}_1, \mathcal{V}_1)$, $(\mathcal{U}_2, \mathcal{V}_2)$ be cotorsion pairs on $\mathcal{B}$. If $H_2(\mathcal{W}_1) = 0$ and $H_1(\mathcal{U}_2) = 0 = H_1(\mathcal{V}_2)$, then we have a natural isomorphism $\beta_{21}\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$ of functors.*

Moreover, we have the following theorem.

Theorem 11. *If $H_1(\mathcal{U}_2) = H_1(\mathcal{V}_2) = 0$ and $H_2(\mathcal{U}_1) = H_2(\mathcal{V}_1) = 0$, then we have an equivalence $\mathcal{H}_1/\mathcal{W}_1 \simeq \mathcal{H}_2/\mathcal{W}_2$ between two hearts. More precisely, we have natural isomorphisms $\beta_{12}\beta_{21} \simeq \mathrm{id}_{\mathcal{H}_2/\mathcal{W}_2}$ and $\beta_{21}\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$ of functors.*

References

[1] N. Abe, H. Nakaoka. General heart construction on a triangulated category (II): Associated cohomological functor. Appl. Categ. Structures **20(2)** (2012), 162–174.
[2] M. Auslander, I. Reiten. Application of contravariantly finite subcategories. Adv. Math. **86**, 111–152(1991).
[3] Z. Huang, O. Iyama. Auslander-type conditions and cotorsion pairs. J. Algebra. **318** (2007), 93–110.
[4] Y. Liu. Hearts of twin cotorsion pairs on exact categories. J. Algebra. **394** (2013), 245–284.
[5] H. Nakaoka. General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories. Appl. Categ. Structures **19(6)** (2011), 879–899.

Graduate School of Mathematics
Nagoya University
Furo-cho, Nagoya 464-8602 JAPAN
E-mail address: `d11005m@math.nagoya-u.ac.jp`