SINGULARITY CATEGORIES OF STABLE RESOLVING SUBCATEGORIES

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Abstract. In this article[1] we study resolving subcategories $\mathcal{X}$ of an abelian category from the structure of their associated triangulated categories. More precisely, we investigate the singularity categories

$$\mathsf{D_{sg}}(\underline{\mathcal{X}}) = \mathsf{D^b}(\mathsf{mod}\,\underline{\mathcal{X}})/\mathsf{K^b}(\mathsf{proj}(\mathsf{mod}\,\underline{\mathcal{X}}))$$

of the stable categories $\underline{\mathcal{X}}$ of $\mathcal{X}$. We consider when the stable categories of two resolving subcategories have triangle equivalent singularity categories. Applying this to resolving subcategories of modules over Gorenstein rings, we characterize simple hypersurface singularities of type (A_1) as complete intersections over which the stable categories of resolving subcategories have trivial singularity categories.

1. Introduction

Let R be a noetherian ring. The *singularity category* of R is by definition the Verdier quotient

$$\mathsf{D_{sg}}(R) = \mathsf{D^b}(\mathsf{mod}\,R)/\mathsf{K^b}(\mathsf{proj}(\mathsf{mod}\,R)),$$

where $\mathsf{mod}\,R$ denotes the category of finitely generated R-modules, $\mathsf{D^b}(\mathsf{mod}\,R)$ the bounded derived category and $\mathsf{K^b}(\mathsf{proj}(\mathsf{mod}\,R))$ the bounded homotopy category. The singularity category $\mathsf{D_{sg}}(R)$ is a triangulated category, which has been introduced by Buchweitz [4] by the name of *stable derived category* and connected to the Homological Mirror Symmetry Conjecture by Orlov [10]. A lot of studies on singularity categories have been done in recent years; see [5, 8, 11, 15] for instance.

In this article, we consider the singularity category of a stable resolving category. Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$, and $\underline{\mathcal{X}}$ its stable category. Then the category $\mathsf{mod}\,\underline{\mathcal{X}}$ of finitely presented right $\underline{\mathcal{X}}$-modules is an abelian category with enough projective objects [1]. We take the Verdier quotient of

$$\mathsf{D_{sg}}(\underline{\mathcal{X}}) := \mathsf{D^b}(\mathsf{mod}\,\underline{\mathcal{X}})/\mathsf{K^b}(\mathsf{proj}(\mathsf{mod}\,\underline{\mathcal{X}})),$$

and call this the *singularity category* of $\underline{\mathcal{X}}$. For two resolving subcategories X, Y we say that $\underline{\mathcal{X}}$, $\underline{\mathcal{Y}}$ are *singulary equivalent* if there is a triangle equivalence $\mathsf{D_{sg}}(\underline{\mathcal{X}}) \cong \mathsf{D_{sg}}(\underline{\mathcal{Y}})$.

The main purpose of this article is to study the following question.

Question 1. Let $\mathcal{A}$ be an abelian category with enough projective objects. Let X, Y be resolving subcategories of $\mathcal{A}$. When are the stable categories $\underline{\mathcal{X}}$, $\underline{\mathcal{Y}}$ singularly equivalent?

[1]The detailed version of this article will be submitted for publication elsewhere.

We give a sufficient condition for two stable resolving subcategories to be singularly equivalent. We also apply it to resolving subcategories of module categories of commutative Gorenstein rings, and characterize the simple hypersurface singularities of type (A_1) in terms of singular equivalence classes.

2. Preliminaries

In this section, we introduce the several notions. Throughout this article, let $\mathcal{A}$ be an abelian category with enough projective objects, and denote by $\mathsf{proj}\,\mathcal{A}$ the full subcategory of projective objects of $\mathcal{A}$.

Definition 2. An object M of $\mathcal{A}$ is said to be *Cohen-Macaulay* if there is an exact sequence

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} \cdots$$

of projectives whose dual by any projective is also exact, such that M is isomorphic to the image of d_0. Denote by $\mathsf{CM}(\mathcal{A})$ the subcategory of $\mathcal{A}$ consisting of Cohen-Macaulay objects and by $\mathsf{CM}_n(\mathcal{A})$ the subcategory of $\mathcal{A}$ consisting objects whose n-th syzygies are Cohen-Macaulay.

In [7], a Cohen-Macaulay object is called a Gorenstein projective object. The category consisting of Cohen-Macaulay objects is a Frobenius category, hence its stable category is a triangulated category.

Next, we recall the definition of the category of finitely presented modules over an additive category.

Definition 3. Let $\mathcal{C}$ be an additive category. Denote by $\mathsf{Mod}\,\mathcal{C}$ the *functor category* of $\mathcal{C}$, that is, the objects are additive contravariant functors from $\mathcal{C}$ to the category $\mathbf{Ab}$ of abelian groups, and the morphisms are natural transformations. An object and a morphism of $\mathsf{Mod}\,\mathcal{C}$ are called a *(right) $\mathcal{C}$-module* and a *$\mathcal{C}$-homomorphism*, respectively. A $\mathcal{C}$-module F is said to be *finitely presented* if there is an exact sequence

$$\mathsf{Hom}_{\mathcal{C}}(-, X) \to \mathsf{Hom}_{\mathcal{C}}(-, Y) \to F \to 0$$

in the abelian category $\mathsf{Mod}\,\mathcal{C}$ with $X, Y \in \mathcal{C}$. The full subcategory of $\mathsf{Mod}\,\mathcal{C}$ consisting of finitely presented $\mathcal{C}$-modules is denoted by $\mathsf{mod}\,\mathcal{C}$.

Definition 4. An additive category $\mathcal{C}$ is called *Gorenstein* of dimension at most n if $\Omega^n(\mathsf{mod}\,\mathcal{C}) = \mathsf{CM}(\mathsf{mod}\,\mathcal{C})$.

Example 5. Let Λ be a Gorenstein ring of selfinjective dimension at most n, and denote by $\mathsf{proj}\,\Lambda$ the category of finitely generated Λ-modules. Then $\mathsf{proj}\,\Lambda$ is Gorenstein of dimension at most n.

We introduce the main target in this article.

Definition 6. Let $\mathcal{C}$ be an additive category. The *singularity category* $\mathcal{C}$ is defined as follows:

$$\mathsf{D}_{\mathsf{sg}}(\mathcal{C}) = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{C})/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\,\mathcal{C})).$$

Definition 7. Additive categories $\mathcal{C}, \mathcal{C}'$ are *singularly equivalent* if there is a triangule equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{C}) \cong \mathsf{D}_{\mathsf{sg}}(\mathcal{C}')$, and then denote this by $\mathcal{C} \overset{sg}{\sim} \mathcal{C}'$.

Let us give the definition of a resolving subcategory, which is mainly studied in this article.

Definition 8. A full subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is *resolving* if:

(1) $\mathcal{X}$ contains all projective objects of $\mathcal{A}$.
(2) $\mathcal{X}$ is closed under direct summands, extensions and syzygies.

Here we recall the definition of a stable category.

Definition 9. Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ containing $\mathsf{proj}\,\mathcal{A}$. Then the quotient category

$$\underline{\mathcal{X}} := \mathcal{X}/\,\mathsf{proj}\,\mathcal{A}$$

is called the *stable category* of $\mathcal{X}$; the objects of $\underline{\mathcal{X}}$ are the same as those of $\mathcal{X}$, and the hom-set $\mathsf{Hom}_{\underline{\mathcal{X}}}(M,N)$ of $M,N \in \underline{\mathcal{X}}$ is defined as follows:

$$\mathsf{Hom}_{\underline{\mathcal{X}}}(M,N) := \mathsf{Hom}_{\mathcal{A}}(M,N)/\mathsf{P}_{\mathcal{A}}(M,N),$$

where $\mathsf{P}_{\mathcal{A}}(M,N)$ consists of all morphisms from M to N that factor through objects in $\mathsf{proj}\,\mathcal{A}$.

Finally, we recall a structure result due to Auslander and Reiten on finitely presented modules over the stable category of a resolving subcategory.

Theorem 10. [1] *If $\mathcal{X}$ is a resolving subcategory of $\mathcal{A}$, then the category $\mathsf{mod}\,\underline{\mathcal{X}}$ of finitely presented right $\underline{\mathcal{X}}$-modules is an abelian category with enough projectives.*

3. Singularity categories and singularly equivalent

In this section, we give a sufficient condition for two resolving subcategories to be singularly equivalent. In particular, there is a natural asking when a resolving subcategory is singularly equivalent to $\mathbf{0}$. We give an answer to this question.

The following result is the key to study singular equivalence.

Theorem 11. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$ such that $\Omega^{-1}\Omega^{n}\mathcal{X} \subset \Omega^{n}\mathcal{X} \subset \mathsf{CM}(\mathcal{A})$. Then:*

(1) *$\underline{\mathcal{X}}$ is Gorenstein of dimension at most $3n$.*
(2) *There is a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}})$.*

This theorem gives some characterizations of a singularity category.

Corollary 12. *For each $n \geq 0$ there is a triangle equivalence*

$$\mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{CM}}_n(\mathcal{A})) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathsf{CM}}_n(\mathcal{A})).$$

Corollary 13. *Let R be a local complete intersection. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$. Then there is a triangle equivalence*

$$\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}}).$$

Let $n = 0$ in Theorem 11. Then the following result holds, whose assertion is nothing but [14].

Corollary 14. *Let $\mathcal{X}$ be a resolving subcateory of $\mathcal{A}$ contained in $\mathsf{CM}(\mathcal{A})$ and closed under cosyzygies. Then $\mathsf{mod}\,\underline{\mathcal{X}} = \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}})$, and hence $\mathsf{mod}\,\underline{\mathcal{X}}$ is a Frobenius category.*

Taking advantage of Theorem 11, we obtain a sufficient condition for singular equivalence.

Theorem 15. *Let $\mathcal{X}, \mathcal{Y}$ be resolving subcategories of $\mathcal{A}$ such that $\Omega^n \mathcal{X} \cup \Omega^{-1}\mathcal{Y} \subseteq \mathcal{Y} \subseteq \mathcal{X} \cap \mathsf{CM}(\mathcal{A})$ for some $n \geq 0$. Then there are triangle equivalences*

$$\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{Y}}) \cong \mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{Y}}).$$

Hence $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ are singularly equivalent.

Sketch of proof. The restriction $F \mapsto F|_{\underline{\mathcal{Y}}}$ makes a covariant exact functor

$$\Phi : \mathsf{Mod}\,\underline{\mathcal{X}} \to \mathsf{Mod}\,\underline{\mathcal{Y}}$$

of abelian categories. This induces an equivalent functor

$$\phi : \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}}) \to \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{Y}}).$$

of triangulated categories. ■

Corollary 16. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$ with $\Omega^n \mathcal{X} \subseteq \mathsf{CM}(\mathcal{A}) \subseteq \mathcal{X}$ for some $n \geq 0$. Then $\underline{\mathcal{X}}$ and $\underline{\mathsf{CM}}(\mathcal{A})$ are singularly equivalent. In particular, $\underline{\mathsf{CM}}_p(\mathcal{A})$ and $\underline{\mathsf{CM}}_q(\mathcal{A})$ are singularly equivalent for all $p, q \geq 0$.*

Remark 17. A singular equivalence between $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ does not necessarily imply that $\mathcal{X}$, $\mathcal{Y}$ have an inclusion relation. Indeed, let $(R, \mathfrak{m})$ be a Gorenstein local domain of dimension at least 2. Set

$$\mathcal{X} = \{M \in \mathsf{mod}\,R \mid \mathfrak{m} \notin \mathsf{Ass}M\},$$
$$\mathcal{Y} = \{M \in \mathsf{mod}\,R \mid \mathsf{Ass}M \subseteq \{0, \mathfrak{m}\}\}.$$

These are resolving subcategories of $\mathsf{mod}\,R$ containing $\mathsf{CM}(R)$. Hence $\underline{\mathcal{X}} \overset{sg}{\approx} \underline{\mathsf{CM}}(R) \overset{sg}{\approx} \underline{\mathcal{Y}}$. However, $\mathcal{X}$ and $\mathcal{Y}$ have no inclusion relation.

In the proof of our last theorem, the following two lemmas are necessary.

Lemma 18. *Let R be a Gorenstein complete local ring. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{CM}(R)$ and closed under cosyzygies. Assume that there exists a nonsplit exact sequence*

$$\sigma : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of R-modules with $X, Y, Z \in \mathcal{X}$ such that X, Z are indecomposable. If $\underline{\mathcal{X}}$ is singularly equivalent to $\mathbf{0}$, then Y is free, and X is isomorphic to ΩZ.

Lemma 19. *Let R and S be Gorenstein complete local rings. Let $\Phi : \underline{\mathsf{CM}}(R) \to \underline{\mathsf{CM}}(S)$ be a triangle equivalence. If f is an irreducible homomorphism of nonfree indecomposable MCM R-modules and g is a homomorphism of S-modules such that $\Phi(\underline{f}) = \underline{g}$, then g is an irreducible homomorphism of nonfree indecomposable MCM S-modules.*

Let R be a local ring. Recall that M is said to have *complexity* c, denoted by $\mathsf{cx}_R\,M = c$, if c is the least nonnegative integer n such that there exists a real number r satisfying the inequality $\beta_i^R(M) \leq r i^{n-1}$ for all $i \gg 0$. It is known that if R is a complete intersection, then the codimension of R is the maximum of the complexities of R-modules. For details on the complexity of a module, we refer the reader to [2, §4.2].

Let R be a d-dimensional Gorenstein local ring with algebraically closed residue field k of characteristic zero. Then R contains a field isomorphic to k, and it is known that R has finite CM-representation type if and only if R is a *simple (hypersurface) singularity* [13, §8], namely, R is isomorphic to a hypersurface

$$k[[x_0, \dots, x_d]]/(f),$$

where f is one of the following.

$$\begin{array}{ll} (\mathsf{A}_n) & x_0^2 + x_1^{n+1} + x_2^2 + \cdots + x_d^2, \\ (\mathsf{D}_n) & x_0^2 x_1 + x_1^{n-1} + x_2^2 + \cdots + x_d^2, \\ (\mathsf{E}_6) & x_0^3 + x_1^4 + x_2^2 + \cdots + x_d^2, \\ (\mathsf{E}_7) & x_0^3 + x_0 x_1^3 + x_2^2 + \cdots + x_d^2, \\ (\mathsf{E}_8) & x_0^3 + x_1^5 + x_2^2 + \cdots + x_d^2. \end{array}$$

For each $\mathsf{T} \in \{\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8\}$, a simple hypersurface singularity of type (T) is shortly called a (T)-*singularity.*

We give a characterization of the (A_1)-singularities in terms of singular equivalence.

Theorem 20. *Let R be a d-dimensional nonregular complete local ring with algebraically closed residue field k of characteristic 0. Then the following conditions are equivalent;*

(1) *R is a Gorenstein ring, and $\underline{\mathsf{CM}}(R)$ is singularly equivalent to* $\mathbf{0}$.
(2) *R is a complete intersection, and $\underline{\mathcal{X}}$ is singularly equivalent to $\mathbf{0}$ for every resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$.*
(3) *R is a complete intersection, and $\underline{\mathcal{X}}$ is singularly equivalent to $\mathbf{0}$ for some resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ that containing a module of maximal complexity.*
(4) *R is an (A_1)-singularity.*

Sketch of proof. (1) $\Rightarrow$ (4): Using Lemma 18, we can show that R has finite CM representation type. By [13, Corollary 8.16] R is a simple singularity. The classification of the Auslander-Reiten quivers of the MCM modules over simple singularities [13, Chapters 8–12] together with Lemma 19 implies that the only simple singularities R where $\mathsf{CM}(R)$ possesses such an Auslander-Reiten quiver are (A_1)-singularities. ■

Let R be a simple hypersurface singularity. Theorem 20 especially says that $\underline{\mathsf{CM}}(R)$ is not singularly equivalent to $\mathbf{0}$ unless R is an (A_1)-singularity. One can actually confirm this for a 1-dimensional (A_2)- singularity by direct calculation.

Example 21. Let k be an algebraically closed field of characteristic 0. Let R be an (A_2)-singularity of dimension 1 over k. Then there is a triangle equivalence

$$\mathsf{D_{sg}}(\underline{\mathsf{CM}}(R)) \cong \mathsf{D_{sg}}(k[t]/(t^2)).$$

In particular, $\underline{\mathsf{CM}}(R)$ is not singularly equivalent to $\mathbf{0}$.

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