τ-RIGID-FINITE ALGEBRAS WITH RADICAL SQUARE ZERO

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Abstract. In this note, we study τ-rigid-finite algebras with radical square zero.

Throughout this note, by an algebra we mean a basic connected finite dimensional algebra over an algebraically closed field K. By a module we mean a finite dimensional right module. Let Λ be an algebra. For a Λ-module M with a minimal projective presentation $P^{-1} \xrightarrow{p} P^0 \to M \to 0$, we define a Λ-module τM by an exact sequence

$$0 \to \tau M \to \nu P^{-1} \xrightarrow{\nu p} \nu P^0,$$

where $\nu := \mathrm{Hom}_K(\mathrm{Hom}_\Lambda(-, \Lambda), K)$ is the Nakayama functor.

The following module plays an important role in this note.

Definition 1. A Λ-module M is *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$. We denote by $\tau\text{-rigid}\Lambda$ the set of isomorphism classes of indecomposable τ-rigid Λ-modules.

In 1980's, Auslander-Smalo [4] have already studied τ-rigid modules from the viewpoint of torsion theory. Recently, from the perspective of tilting mutation theory, the authors in [2] introduced the notion of (support) τ-tilting modules as a special class of τ-rigid modules. They correspond bijectively with many important objects in representation theory, *i.e.*, functorially finite torsion classes, two-term silting complexes and cluster-tilting objects in a special cases. By the following proposition, finiteness of these objects is induced by that of $\tau\text{-rigid}\Lambda$.

Proposition 2. [5] *Let Λ be an algebra. The following are equivalent:*

(1) *The set $\tau\text{-rigid}\Lambda$ is finite.*
(2) *There are finitely many isomorphism classes of basic support τ-tilting Λ-modules.*

Definition 3. An algebra Λ is called *τ-rigid-finite* if it satisfies the equivalent conditions in Proposition 2.

Our aim of this note is to study τ-rigid-finite algebras with radical square zero. In the rest of this note, let Λ be an algebra with radical square zero and $Q = (Q_0, Q_1)$ the quiver of Λ, where Q_0 is the vertex set and Q_1 is the arrow set. Namely, $\Lambda = \Lambda_Q$ is the path algebra of a quiver Q modulo the ideal generated by all paths of length 2. In representation theory of algebras with radical square zero, the notion of the separated quiver play a central role. For a quiver $Q = (Q_0, Q_1)$, we define a new quiver $Q^s = (Q_0^s, Q_1^s)$, called the

The detailed version of this paper will be submitted for publication elsewhere.

separated quiver of Q, as follows:

$$Q_0^s := \{i^+, i^- \mid i \in Q_0\}, \quad Q_1^s := \{i^+ \to j^- \mid (i \to j) \in Q_1\}.$$

Note that the separated quiver Q^s is bipartite and not connected even if Q is connected.

$$\begin{tikzcd}[row sep=small] Q: & & 2 \arrow[dl, shift left] \arrow[dd, shift left] \\ & 1 \arrow[ur, shift left] \arrow[dr, shift left] & \\ & & 3 \arrow[ul, shift left] \arrow[uu, shift left] \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] Q^s: & & 2^- & 3^+ \arrow[l] \arrow[dr] & \\ & 1^+ \arrow[ur] \arrow[dr] & & & 1^- \\ & & 3^- & 2^+ \arrow[l] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} Q: 1 \arrow[r, shift left] \arrow[d, shift left] & 4 \arrow[l, shift left] \arrow[d, shift left] \\ 2 \arrow[r, shift left] \arrow[u, shift left] & 3 \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd} \qquad \begin{tikzcd} Q^s: 1^+ \arrow[r] \arrow[d] & 4^- & 1^- & 4^+ \arrow[l] \arrow[d] \\ 2^- & 3^+ \arrow[l] \arrow[u] & 2^+ \arrow[r] \arrow[u] & 3^- \end{tikzcd}$$

The following proposition is well-known result.

Proposition 4. [3, X.2.4] *Let Λ be an algebra with radical square zero and KQ^s the path algebra of the separated quiver of the quiver of Λ. Then two algebras Λ and KQ^s are stably equivalent, that is, there is an equivalent between the associated module categories modulo projectives.*

We have the following famous theorem characterizing representation-finiteness.

Theorem 5. [6] *Let Λ be an algebra with radical square zero and Q the quiver of Λ. The following are equivalent:*

(1) *Λ is representation-finite.*
(2) *The separated quiver Q^s is a disjoint union of Dynkin quivers.*

The following theorem is an analog of Theorem 5 for τ-rigid-finiteness. A full subquiver Q' of Q^s is called a *single subquiver* if, for any $i \in Q_0$, the vertex set Q'_0 contains at most one of i^+ or i^-.

Theorem 6. [1] *Let Λ be an algebra with radical square zero and Q the quiver of Λ. The following are equivalent:*

(1) *Λ is τ-rigid-finite.*
(2) *Each single subquiver of Q^s is a disjoint union of Dynkin quivers.*

We give some comment for loops of a quiver.

Remark 7. Let $Q = (Q_0, Q_1)$ be a quiver with a loop ℓ, and $Q' = (Q'_0, Q'_1)$ the quiver with $Q'_0 = Q_0$ and $Q'_1 = Q_1 \setminus \{\ell\}$. Then there is a natural bijection between the set of single subquiver of Q^s and those of Q'^s. Hence Λ_Q is τ-rigid-finite if and only if $\Lambda_{Q'}$ is τ-rigid-finite.

$$\begin{tikzcd} Q: 1 \arrow[loop right] \end{tikzcd} \qquad \begin{tikzcd} Q^s: 1^+ \arrow[r] & 1^- \end{tikzcd}$$

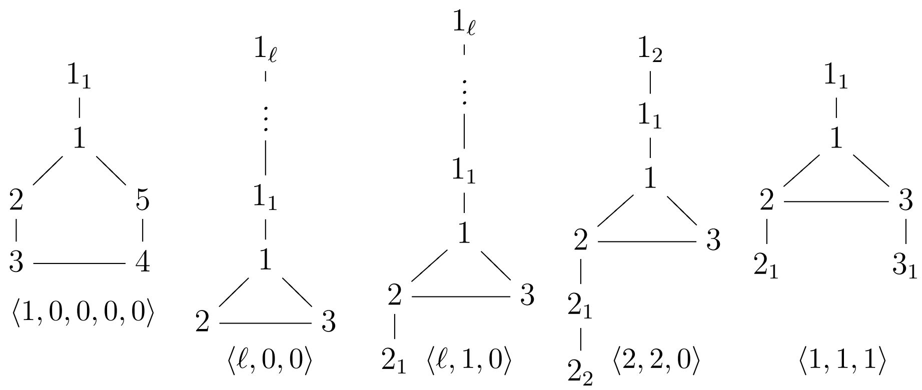
We give a main result of this note. Let $G = (V, E)$ be a connected graph, where V is the vertex set and E is the edge set. We define a quiver $Q_G = ((Q_G)_0, (Q_G)_1)$, called *the double quiver of* G, as follows:

$$(Q_G)_0 := V, \quad (Q_G)_1 := \{i \to j,\ i \leftarrow j \mid (i - j) \in E\}.$$

For non-negative integers $\ell_1, \ell_2, \dots, \ell_n$, we define a graph $G := \langle \ell_1, \dots, \ell_n \rangle$ as follows. G is an n-cycle such that each vertex v_i in the n-cycle is attached to a Dynkin graph A_{l_i} and the degree of v_i is at most three.

Theorem 8. *Let G be a connected graph with no loop. Then the following are equivalent:*

(1) Λ_{Q_G} *is* τ*-rigid-finite.*
(2) G *is one of the following graphs:*
  (a) *Dynkin graphs of type* A, D, *and* E,
  (b) *odd-cycles,*
  (c) $\langle 1,0,0,0,0\rangle$,
  (d) $\langle \ell,0,0\rangle$ $(1 \le \ell)$,
  (e) $\langle \ell,1,0\rangle$ $(1 \le \ell \le 4)$,
  (f) $\langle 2,2,0\rangle$,
  (g) $\langle 1,1,1\rangle$.



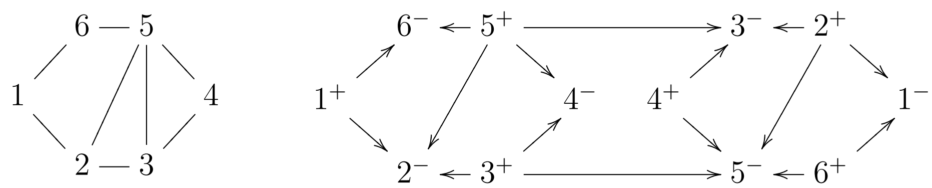
We can extend our theorem to the case of quivers/graphs with loops.

Remark 9. Assume that the quiver Q of Λ has a loop. By Remark 7, if there exists a graph G in Theorem 8 (2) such that Q_G is isomorphic to Q up to all loops, then Λ_Q is also τ-rigid-finite.

In the rest of this section, we give a proof of Theorem 8 by removing extended Dynkin graphs from connected single subquivers of the separated quiver. First we remove extended Dynkin graphs of type $\tilde{A}$ from the separated quiver. A graph is called an *n-cycle* if it is a cycle with exactly n vertices. In particular, it is called an *odd-cycle* if n is odd, and an *even-cycle* if n even. We write by $\overline{Q}$ the underlying graph of a quiver Q.

Lemma 10. *A graph* G *contains an even-cycle as a subgraph if and only if there exists a single subquiver* Q' *of* Q_G^s *such that* $\overline{Q'}$ *is an extended Dynkin graph of type* $\tilde{A}$.

Proof. Since Q_G^s is bipartite, all cycles as a subgraph in Q_G^s are even-cycles. Hence G contains an even-cycle as a subgraph. Conversely, assume that G contains an even-cycle as a subgraph. By taking a minimal even-cycle G' in G as a subgraph, $\overline{Q_G^s}$ includes G' as a full subgraph. Hence the assertion follows. $\square$



By Lemma 10, we may assume that G contains no even-cycle as a subgraph. Since G is also bipartite, we have the following connection between G and Q_G^s. A *spanning tree of*

G is a subgraph of G that includes all of the vertices of G and is a tree. A *subtree of* G is a connected full subgraph of a spanning tree of G.

Proposition 11. *Let G be a graph with no even-cycle as a subgraph. Let G' be a graph. Then G' is a subtree of G if and only if there exists a connected single subquiver Q' of Q_G^s such that $\overline{Q'} = G'$. In particular, there is a naturally one-to-two correspondence between the set of subtrees of G and the set of connected single subquivers of Q_G^s.*

Proof. If G' is a subtree of G, then there exists a connected subquiver Q' of Q_G^s with $\overline{Q'} = G'$. By Lemma 10, Q' is clearly a full subquiver, and hence it is a single subquiver. Conversely, assume that Q' is a single subquiver Q' of Q_G^s with $\overline{Q'} = G'$. By Lemma 10, $\overline{Q'}$ is a tree. Since Q' is a full subquiver, $\overline{Q'}$ is a subtree of G by the definition of separated quivers. □

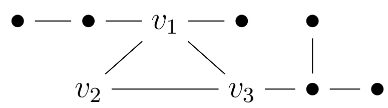
By Proposition 11, to remove non-Dynkin quivers from single subquivers of the separated quiver, we have only to concentrate on observing subtrees of graphs. For a tree, we have the following result.

Corollary 12. *Let G be a tree. Then the following are equivalent:*

(1) *Λ_{Q_G} is τ-rigid-finite.*
(2) *G is a Dynkin graph.*

Proof. Assume that G is a tree. G is Dynkin if and only if all subtrees of G are Dynkin. Thus the assertion follows from Theorem 6 and Proposition 11. □

By Corollary 12, we may assume that G contains exactly one odd-cycle and no even-cycles. Namely, G is an odd-cycle such that each vertex v in the odd-cycle is attached to a tree T_v.



We remove extended Dynkin graphs of type $\tilde{D}$ from the separated quiver Q_G^s.

Lemma 13. *Fix a positive integer k and $n := 2k + 1$. Let G be an n-cycle such that each vertex v in the n-cycle is attached to a tree T_v. Then G contains an extended Dynkin graph of type $\tilde{D}$ as a subgraph if and only if it satisfies one of the following conditions:*

(a) *There is a vertex v in the n-cycle such that the degree is at least four.*
(b) *There is a vertex v in the n-cycle such that the degree is exactly three and T_v is not Dynkin graph of type A.*
(c) *$k > 1$ and there are at least two vertices in the n-cycle such that the degrees are at least three.*

Proof. Clearly, if G satisfies one of the conditions (a), (b), and (c), then it contains an extended Dynkin graph of type $\tilde{D}$. Conversely, assume that G contains an extended Dynkin graph of type $\tilde{D}$. Then $\tilde{D}_4$ has exactly one vertex whose degree is exactly four and $\tilde{D}_l$ has exactly two vertices whose degree is exactly three for any integer $\ell > 4$. We can check that G satisfies one of (a), (b), and (c). □

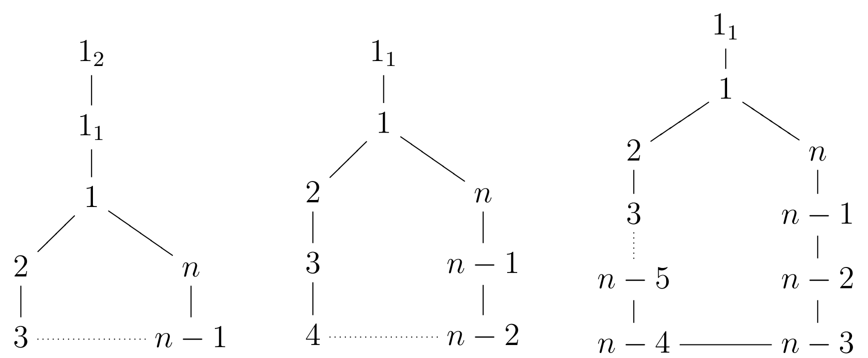
Fix a positive integer k and $n := 2k+1$. By Lemma 13, we may assume that G is one of the following graphs:

(a) $\langle \ell_1, 0 \ldots, 0 \rangle$ if $k \geq 2$.
(b) $\langle \ell_1, \ell_2, \ell_3 \rangle$ with $\ell_1 \geq \ell_2 \geq \ell_3$ if $k = 1$.

Finally, we remove extended Dynkin graphs of type $\tilde{E}$ from the separated quiver Q_G^s.

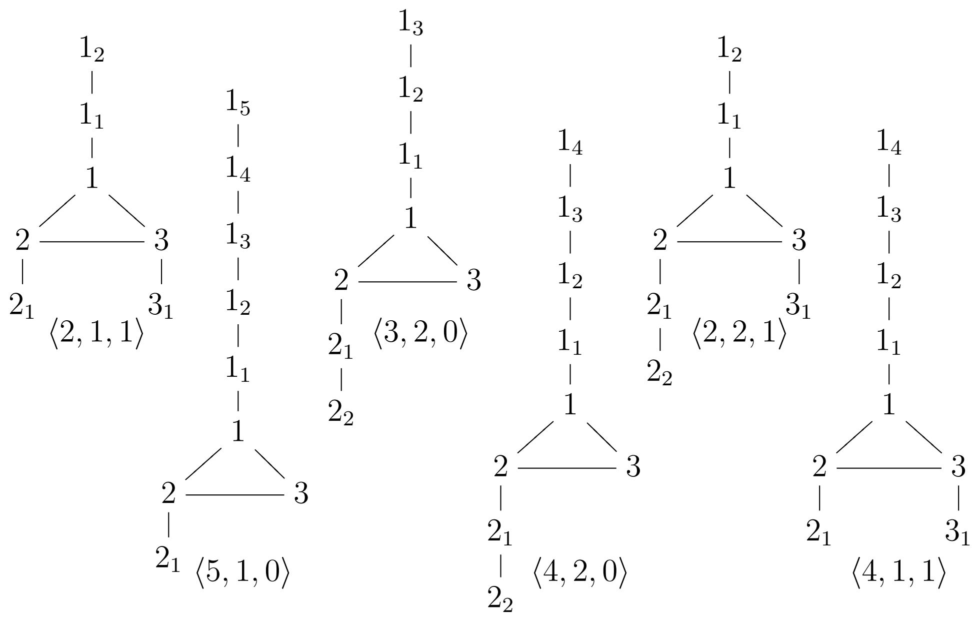
Lemma 14. *Fix a positive integer k and $n := 2k+1$. Assume that $G = \langle \ell_1, \ell_2, \cdots, \ell_n \rangle$.*

(1) *Assume that $k \geq 2$. The following graphs* (a), (b) *and* (c) *are the minimal graphs containing extended Dynkin graphs $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ respectively in the forms $\langle \ell_1, 0, \ldots, 0 \rangle$.*
 (a) $\langle 2, 0, \ldots, 0 \rangle$ $(k \geq 2)$
 (b) $\langle 1, 0, \ldots, 0 \rangle$ $(k \geq 3)$
 (c) $\langle 1, 0, \ldots, 0 \rangle$ $(k \geq 4)$



(a) $\ell_1 = 2$ *(b)* $\ell_1 = 1$ *if* $k \geq 3$ *(c)* $\ell_1 = 1$ *if* $k \geq 4$

(2) *Assume that $k = 1$. The following graphs* (d), (e) *and* (f) *are the minimal graphs containing extended Dynkin graphs $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ respectively in the forms $\langle \ell_1, \ell_2, \ell_3 \rangle$.*
 (d) $\langle 2, 1, 1 \rangle$.
 (e) $\langle 3, 2, 0 \rangle$, $\langle 2, 2, 1 \rangle$.
 (f) $\langle 5, 1, 0 \rangle$, $\langle 4, 2, 0 \rangle$, $\langle 4, 1, 1 \rangle$.



Proof. We can check from the pictures above. □

Now we are ready to prove Theorem 8.

Proof of Theorem 8. If G is a tree, then the assertion follows from Corollary 12. We assume that G is not a tree. By the argument above, we have the minimal set of graphs including extended Dynkin graphs of type $\tilde{A}$, $\tilde{D}$, or $\tilde{E}$. Thus Λ_{Q_G} is τ-rigid-finite if and only if G is one of nontrivial full subgraphs with the n-cycle of graphs in Lemma 14. The assertion follows from that G is the desired graph. □

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