Abstract. In this note, we study $\tau$-rigid-finite algebras with radical square zero.

Throughout this note, by an algebra we mean a basic connected finite dimensional algebra over an algebraically closed field $K$. By a module we mean a finite dimensional right module. Let $\Lambda$ be an algebra. For a $\Lambda$-module $M$ with a minimal projective presentation $P^{-1} \xrightarrow{p} P^0 \rightarrow M \rightarrow 0$, we define a $\Lambda$-module $\tau M$ by an exact sequence

$$0 \rightarrow \tau M \rightarrow \nu P^{-1} \xrightarrow{\nu p} \nu P^0,$$

where $\nu := \text{Hom}_K(\text{Hom}_\Lambda(-, \Lambda), K)$ is the Nakayama functor.

The following module plays an important role in this note.

Definition 1. A $\Lambda$-module $M$ is $\tau$-rigid if $\text{Hom}_\Lambda(M, \tau M) = 0$. We denote by $\tau$-rigid the set of isomorphism classes of indecomposable $\tau$-rigid $\Lambda$-modules.

In 1980’s, Auslander-Smalø [4] have already studied $\tau$-rigid modules from the viewpoint of torsion theory. Recently, from the perspective of tilting mutation theory, the authors in [2] introduced the notion of (support) $\tau$-tilting modules as a special class of $\tau$-rigid modules. They correspond bijectively with many important objects in representation theory, i.e., functorially finite torsion classes, two-term silting complexes and cluster-tilting objects in a special cases. By the following proposition, finiteness of these objects is induced by that of $\tau$-rigid $\Lambda$.

Proposition 2. [5] Let $\Lambda$ be an algebra. The following are equivalent:

1. The set $\tau$-rigid $\Lambda$ is finite.
2. There are finitely many isomorphism classes of basic support $\tau$-tilting $\Lambda$-modules.

Definition 3. An algebra $\Lambda$ is called $\tau$-rigid-finite if it satisfies the equivalent conditions in Proposition 2.

Our aim of this note is to study $\tau$-rigid-finite algebras with radical square zero. In the rest of this note, let $\Lambda$ be an algebra with radical square zero and $Q = (Q_0, Q_1)$ the quiver of $\Lambda$, where $Q_0$ is the vertex set and $Q_1$ is the arrow set. Namely, $\Lambda = \Lambda_Q$ is the path algebra of a quiver $Q$ modulo the ideal generated by all paths of length 2. In representation theory of algebras with radical square zero, the notion of the separated quiver play a central role. For a quiver $Q = (Q_0, Q_1)$, we define a new quiver $Q^s = (Q_0^s, Q_1^s)$, called the

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The detailed version of this paper will be submitted for publication elsewhere.
separated quiver of $Q$, as follows:

\[ Q^*_0 := \{i^+, i^- \mid i \in Q_0\}, \quad Q^*_1 := \{i^+ \to j^- \mid (i \to j) \in Q_1\}. \]

Note that the separated quiver $Q^*$ is bipartite and not connected even if $Q$ is connected.

\[ Q: \begin{array}{c|c}
1 & 2 \\
\hline
3 & 1^+ \\
\end{array} \quad \begin{array}{c|c|c}
2^- & 3^+ & Q^*:
1^- & 4 \\
\hline
2 & 3 \\
\end{array} \quad \begin{array}{c|c|c|c|c}
Q^* : 1 & 4^- & 1^- & 4^+ \\
\hline
2^- & 3^+ & 2^+ & 3^- \\
\end{array} \]

The following proposition is well-known result.

**Proposition 4.** [3, X.2.4] Let $\Lambda$ be an algebra with radical square zero and $KQ^*$ the path algebra of the separated quiver of the quiver of $\Lambda$. Then two algebras $\Lambda$ and $KQ^*$ are stably equivalent, that is, there is an equivalent between the associated module categories modulo projectives.

We have the following famous theorem characterizing representation-finiteness.

**Theorem 5.** [6] Let $\Lambda$ be an algebra with radical square zero and $Q$ the quiver of $\Lambda$. The following are equivalent:

1. $\Lambda$ is representation-finite.
2. The separated quiver $Q^*$ is a disjoint union of Dynkin quivers.

The following theorem is an analog of Theorem 5 for $\tau$-rigid-finiteness. A full subquiver $Q'$ of $Q^*$ is called a single subquiver if, for any $i \in Q_0$, the vertex set $Q'_0$ contains at most one of $i^+$ or $i^-$. A single subquiver.

**Theorem 6.** [1] Let $\Lambda$ be an algebra with radical square zero and $Q$ the quiver of $\Lambda$. The following are equivalent:

1. $\Lambda$ is $\tau$-rigid-finite.
2. Each single subquiver of $Q^*$ is a disjoint union of Dynkin quivers.

We give some comment for loops of a quiver.

**Remark 7.** Let $Q = (Q_0, Q_1)$ be a quiver with a loop $\ell$, and $Q' = (Q'_0, Q'_1)$ the quiver with $Q'_0 = Q_0$ and $Q'_1 = Q_1 \setminus \{\ell\}$. Then there is a natural bijection between the set of single subquiver of $Q^*$ and those of $Q'^*$. Hence $\Lambda_Q$ is $\tau$-rigid-finite if and only if $\Lambda_{Q'}$ is $\tau$-rigid-finite.

\[ Q: \begin{array}{c}
1 \\
\hline
1^- \\
\end{array} \quad Q^*: \begin{array}{c|c|c}
1^+ & 1^- \\
\hline
\end{array} \]

We give a main result of this note. Let $G = (V, E)$ be a connected graph, where $V$ is the vertex set and $E$ is the edge set. We define a quiver $Q_G = ((Q_G)_0, (Q_G)_1)$, called the double quiver of $G$, as follows:

\[(Q_G)_0 := V, \quad (Q_G)_1 := \{i \to j, \; i \leftarrow j \mid (i, j) \in E\}.\]

For non-negative integers $\ell_1, \ell_2, \ldots, \ell_n$, we define a graph $G := \langle \ell_1, \ldots, \ell_n \rangle$ as follows. $G$ is an $n$-cycle such that each vertex $v_i$ in the $n$-cycle is attached to a Dynkin graph $A_{\ell_i}$ and the degree of $v_i$ is at most three.

**Theorem 8.** Let $G$ be a connected graph with no loop. Then the following are equivalent:
(1) $\Lambda_{Q_G}$ is $\tau$-rigid-finite.
(2) $G$ is one of the following graphs:
   (a) Dynkin graphs of type $A$, $D$, and $E$,
   (b) odd-cycles,
   (c) $\langle 1, 0, 0, 0, 0 \rangle$,
   (d) $\langle \ell, 0, 0 \rangle$ $(1 \leq \ell)$,
   (e) $\langle \ell, 1, 0 \rangle$ $(1 \leq \ell \leq 4)$,
   (f) $\langle 2, 2, 0 \rangle$,
   (g) $\langle 1, 1, 1 \rangle$.

We can extend our theorem to the case of quivers/graphs with loops.

Remark 9. Assume that the quiver $Q$ of $\Lambda$ has a loop. By Remark 7, if there exists a graph $G$ in Theorem 8 (2) such that $Q_G$ is isomorphic to $Q$ up to all loops, then $\Lambda_Q$ is also $\tau$-rigid-finite.

In the rest of this section, we give a proof of Theorem 8 by removing extended Dynkin graphs from connected single subquivers of the separated quiver. First we remove extended Dynkin graphs of type $A$ from the separated quiver. A graph is called an $n$-cycle if it is a cycle with exactly $n$ vertices. In particular, it is called an odd-cycle if $n$ is odd, and an even-cycle if $n$ even. We write by $Q_s$ the underlying graph of a quiver $Q$.

Lemma 10. A graph $G$ contains an even-cycle as a subgraph if and only if there exists a single subquiver $Q'$ of $Q^+_G$ such that $Q^+_G$ is an extended Dynkin graph of type $A$.

Proof. Since $Q^+_G$ is bipartite, all cycles as a subgraph in $Q^+_G$ are even-cycles. Hence $G$ contains an even-cycle as a subgraph. Conversely, assume that $G$ contains an even-cycle as a subgraph. By taking a minimal even-cycle $G'$ in $G$ as a subgraph, $Q^+_G$ includes $G'$ as a full subgraph. Hence the assertion follows. □

By Lemma 10, we may assume that $G$ contains no even-cycle as a subgraph. Since $G$ is also bipartite, we have the following connection between $G$ and $Q^+_G$. A spanning tree of...
Proposition 11. Let $G$ be a graph with no even-cycle as a subgraph. Let $G'$ be a graph. Then $G'$ is a subtree of $G$ if and only if there exists a connected single subquiver $Q'$ of $Q^*_G$ such that $Q' = G'$. In particular, there is a naturally one-to-two correspondence between the set of subtrees of $G$ and the set of connected single subquivers of $Q^*_G$.

Proof. If $G'$ is a subtree of $G$, then there exists a connected subquiver $Q'$ of $Q^*_G$ with $Q' = G'$. By Lemma 10, $Q'$ is clearly a full subquiver, and hence it is a single subquiver. Conversely, assume that $Q'$ is a single subquiver $Q'$ of $Q^*_G$ with $Q' = G'$. By Lemma 10, $Q'$ is a tree. Since $Q'$ is a full subquiver, $Q'$ is a subtree of $G$ by the definition of separated quivers. □

By Proposition 11, to remove non-Dynkin quivers from single subquivers of the separated quiver, we have only to concentrate on observing subtrees of graphs. For a tree, we have the following result.

Corollary 12. Let $G$ be a tree. Then the following are equivalent:

1. $\Lambda_{Q_G}$ is $\tau$-rigid-finite.
2. $G$ is a Dynkin graph.

Proof. Assume that $G$ is a tree. $G$ is Dynkin if and only if all subtrees of $G$ are Dynkin. Thus the assertion follows from Theorem 6 and Proposition 11. □

By Corollary 12, we may assume that $G$ contains exactly one odd-cycle and no even-cycles. Namely, $G$ is an odd-cycle such that each vertex $v$ in the odd-cycle is attached to a tree $T_v$.

We remove extended Dynkin graphs of type $\tilde{D}$ from the separated quiver $Q^*_G$.

Lemma 13. Fix a positive integer $k$ and $n := 2k + 1$. Let $G$ be an $n$-cycle such that each vertex $v$ in the $n$-cycle is attached to a tree $T_v$. Then $G$ contains an extended Dynkin graph of type $\tilde{D}$ as a subgraph if and only if it satisfies one of the following conditions:

1. There is a vertex $v$ in the $n$-cycle such that the degree is at least four.
2. There is a vertex $v$ in the $n$-cycle such that the degree is exactly three and $T_v$ is not Dynkin graph of type $A$.
3. $k > 1$ and there are at least two vertices in the $n$-cycle such that the degrees are at least three.

Proof. Clearly, if $G$ satisfies one of the conditions (a), (b), and (c), then it contains an extended Dynkin graph of type $\tilde{D}$. Conversely, assume that $G$ contains an extended Dynkin graph of type $\tilde{D}$. Then $\tilde{D}_4$ has exactly one vertex whose degree is exactly four and $\tilde{D}_l$ has exactly two vertices whose degree is exactly three for any integer $\ell > 4$. We can check that $G$ satisfies one of (a), (b), and (c). □
Fix a positive integer $k$ and $n := 2k + 1$. By Lemma 13, we may assume that $G$ is one of the following graphs:

(a) $\langle \ell_1, 0, \ldots, 0 \rangle$ if $k \geq 2$.
(b) $\langle \ell_1, \ell_2, \ell_3 \rangle$ with $\ell_1 \geq \ell_2 \geq \ell_3$ if $k = 1$.

Finally, we remove extended Dynkin graphs of type $\tilde{E}$ from the separated quiver $Q_{\ell_2}$.

Lemma 14. Fix a positive integer $k$ and $n := 2k + 1$. Assume that $G = \langle \ell_1, \ell_2, \cdots, \ell_n \rangle$.

(1) Assume that $k \geq 2$. The following graphs (a), (b) and (c) are the minimal graphs containing extended Dynkin graphs $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ respectively in the forms $\langle \ell_1, 0, \ldots, 0 \rangle$.

(a) $\langle 2, 0, \ldots, 0 \rangle$ ($k \geq 2$)
(b) $\langle 1, 0, \ldots, 0 \rangle$ ($k \geq 3$)
(c) $\langle 1, 0, \ldots, 0 \rangle$ ($k \geq 4$)

(2) Assume that $k = 1$. The following graphs (d), (e) and (f) are the minimal graphs containing extended Dynkin graphs $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ respectively in the forms $\langle \ell_1, \ell_2, \ell_3 \rangle$.

(d) $\langle 2, 1, 1 \rangle$.
(e) $\langle 3, 2, 0 \rangle$, $\langle 2, 2, 1 \rangle$.
(f) $\langle 5, 1, 0 \rangle$, $\langle 4, 2, 0 \rangle$, $\langle 4, 1, 1 \rangle$. 

\[ \text{Diagram for Lemma 14} \]
Proof. We can check from the pictures above.

Now we are ready to prove Theorem 8.

Proof of Theorem 8. If \( G \) is a tree, then the assertion follows from Corollary 12. We assume that \( G \) is not a tree. By the argument above, we have the minimal set of graphs including extended Dynkin graphs of type \( \tilde{A}, \tilde{D}, \) or \( \tilde{E}. \) Thus \( \Lambda_{Q_G} \) is \( \tau \)-rigid-finite if and only if \( G \) is one of nontrivial full subgraphs with the \( n \)-cycle of graphs in Lemma 14. The assertion follows from that \( G \) is the desired graph.

References


Graduate School of Mathematics
Nagoya University
Frocho, Chikusaku, Nagoya 464-8602 Japan

E-mail address: m09002b@math.nagoya-u.ac.jp