JACOBIAN ALGEBRAS AND DEFORMATION QUANTIZATIONS

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Abstract. Let V be a 3-dimensional vector space over an algebraically closed field k of characteristic 0. In this paper, we study the following two classes of algebras: (1) the Jacobian algebra $J(\omega)$ of a potential $0 \neq \omega \in V^{\otimes 3}$, and (2) the algebra S_f^λ induced by the deformation quantization of the polynomial algebra $S := S(V) = k[x, y, z]$ in three variables whose semi-classical limit has a quadratic unimodular Poisson bracket on S determined by $f \in S_3$. It is known that every noetherian quadratic Calabi-Yau algebra of dimension 3 is of the form $J(\omega)$, however, it is not easy to see for which potential $0 \neq \omega \in V^{\otimes 3}$, $J(\omega)$ is a Calabi-Yau algebra of dimension 3. In this paper, we try to answer this question by relating $J(\omega)$ to S_f^λ.

1. Jacobian Algebras

This is a report on a joint work in progress with S. Paul Smith. Throughout this paper, let k be an algebraically closed field of characteristic 0, and V a finite dimensional vector space over k. We denote by $T(V)$ the tensor algebra and $S(V)$ the symmetric algebra.

We define the action of $\theta \in \mathfrak{S}_m$ on $V^{\otimes m}$ by

$$\theta(v_1 \otimes \cdots \otimes v_m) := v_{\theta(1)} \otimes \cdots \otimes v_{\theta(m)}.$$

Specializing to the m-cycle $\phi \in \mathfrak{S}_m$, we define

$$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{m-1} \otimes v_m) := v_m \otimes v_1 \otimes \cdots \otimes v_{m-2} \otimes v_{m-1}.$$

We define linear maps $c, s, a : V^{\otimes m} \to V^{\otimes m}$ by

$$c(\omega) := \frac{1}{m} \sum_{i=0}^{m-1} \phi^i(\omega)$$

$$s(\omega) := \frac{1}{m!} \sum_{\theta \in \mathfrak{S}_m} \theta(\omega)$$

$$a(\omega) := \frac{1}{m!} \sum_{\theta \in \mathfrak{S}_m} (\operatorname{sgn} \theta)\theta(\omega).$$

We define the following subspaces of $V^{\otimes m}$:

$$\operatorname{Sym}^m V := \{\omega \in V^{\otimes m} \mid \theta(\omega) = \omega \text{ for all } \theta \in \mathfrak{S}_m\}$$

$$\operatorname{Alt}^m V := \{\omega \in V^{\otimes m} \mid \theta(\omega) = (\operatorname{sgn} \theta)\omega \text{ for all } \theta \in \mathfrak{S}_m\}.$$

It is easy to see that $\operatorname{Sym}^m V = \operatorname{Im} s$ and $\operatorname{Alt}^m V = \operatorname{Im} a$.

The following is a key lemma in this paper.

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Lemma 1. *Suppose that* $\dim V = 3$. *For every choice of a basis* x, y, z *for* V, $\mathrm{Alt}^3 V = k\omega_0$ *where*

$$\omega_0 = 2a(xyz) = c(xyz - zyx) = \frac{1}{3}(xyz + zxy + yzx - zyx - xzy - yxz).$$

By Lemma 1, we can define a linear map $\mu : V^{\otimes 3} \to k$ by the formula $a(\omega) = \mu(\omega)\omega_0$ when $\dim V = 3$.

We define three kinds of derivatives: Choose a basis $x_1, \dots, x_n$ for V so that $S(V) = k[x_1, \dots, x_n]$ and $T(V) = k\langle x_1, \dots, x_n\rangle$. For $f \in k[x_1, \dots, x_n]$, the usual partial derivative of f with respect to x_i is denoted by f_{x_i}. For a monomial $\omega = x_{i_1}x_{i_2}\cdots x_{i_{m-1}}x_{i_m} \in k\langle x_1, \dots, x_n\rangle_m$ of degree m, we define

$$x_i^{-1}\omega := \begin{cases} x_{i_2}\cdots x_{i_{m-1}}x_{i_m} & \text{if } i_1 = i, \\ 0 & \text{if } i_1 \neq i, \end{cases} \textit{ and}$$

$$\partial_{x_i}(\omega) := m x_i^{-1}c(\omega).$$

We extend the map $\partial_{x_i} : k\langle x_1, \dots, x_n\rangle \to k\langle x_1, \dots, x_n\rangle$ by linearity. We call ∂_{x_i} the cyclic derivative with respect to x_i.

Definition 2. The Jacobian algebra of $\omega \in k\langle x_1, \dots, x_n\rangle$ is the algebra of the form

$$J(\omega) := k\langle x_1, \dots, x_n\rangle/(\partial_{x_1}\omega, \dots, \partial_{x_n}\omega).$$

We call ω the potential of $J(\omega)$.

It is easy to see that the Jacobian algebra is independent of the choice of a basis $x_1, \dots, x_n$ for V. Note that if ω is homogeneous, then $J(\omega)$ is a graded algebra. In this paper, we focus on the case that $\dim V = 3$ and $0 \neq \omega \in V^{\otimes 3}$. In this case, $J(\omega) = T(V)/(R)$ is a quadratic algebra where $R \subset V \otimes V$.

A Calabi-Yau algebra defined below plays an important role in many branches of mathematics. For an algebra A, we denote by $A^e := A \otimes A^{op}$ the enveloping algebra of A.

Definition 3. An algebra A is called Calabi-Yau of dimension d (d-CY for short) if

(1) A has a resolution of finite length consisting of finitely generated projective A^e-modules, and

(2) $\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$ as A^e-modules.

Bocklandt [3] showed that every graded Calabi-Yau algebra is a Jacobian algebra. Specializing to the noetherian quadratic case, we have the following result, which is the main motivation of this paper.

Theorem 4. [3] *Every noetherian quadratic Calabi-Yau algebra of dimension 3 is of the form* $J(\omega)$ *where* $\dim V = 3$ *and* $0 \neq \omega \in V^{\otimes 3}$.

By the above theorem, it is interesting to know for which potential $0 \neq \omega \in V^{\otimes 3}$, $J(\omega)$ is a Calabi-Yau algebra of dimension 3. Some criteria were given by [4], [2], however, these criteria are difficult to check in practice. The purpose of this paper is to give a more effective criterion by using geometry.

2. Deformation Quantizations

Let A be a commutative algebra.

Definition 5. A Poisson algebra is an algebra A together with a bilinear map $\{-,-\}: A\times A\to A$, called the Poisson bracket, satisfying the following axioms:

(1) $\{a,b\}=-\{b,a\}$ for all $a,b\in A$.
(2) $\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0$ for all $a,b,c\in A$.
(3) $\{a,bc\}=\{a,b\}c+b\{a,c\}$ for all $a,b,c\in A$.

Definition 6. A formal deformation of A is a $k[[t]]$-algebra $A[[t]]$ with the multiplication $\varphi : A[[t]]\times A[[t]]\to A[[t]]$ of the form $\varphi=\sum_{i\in\mathbb{N}}\varphi_i t^i$ where $\varphi_0: A\times A\to A$ is the original multiplication of A and each $\varphi_i: A\times A\to A$ is a k-bilinear map extended to be $k[[t]]$-bilinear.

Since A is commutative, for all $a,b\in A$, $\varphi_0(a,b)=\varphi_0(b,a)$, so

$$\begin{aligned}\varphi(a,b)-\varphi(b,a)&=\sum_{i\in\mathbb{N}}\varphi_i(a,b)t^i-\sum_{i\in\mathbb{N}}\varphi_i(b,a)t^i\\&=\sum_{i\in\mathbb{N}}(\varphi_i(a,b)-\varphi_i(b,a))t^i\\&=(\varphi_1(a,b)-\varphi_1(b,a))t+O(t^2).\end{aligned}$$

It is easy to see that $(A,\{-,-\}_\varphi)$ where $\{a,b\}_\varphi:=\varphi_1(a,b)-\varphi_1(b,a)$ for $a,b\in A$ is a Poisson algebra. We call $(A,\{-,-\}_\varphi)$ the semi-classical limit of $(A[[t]],\varphi)$. It is not easy to see which Poisson algebra can be realized as a semi-classical limit of a formal deformation. If this is the case, we call it a deformation quantization.

Definition 7. Let $(A,\{-,-\})$ be a Poisson algebra. A formal deformation $(A[[t]],\varphi)$ of A is called a deformation quantization of $(A,\{-,-\})$ if $\{-,-\}=\{-,-\}_\varphi$.

We now focus on the case $A=S(V)$. For $m\geq 2$, $S(V)_m=V^{\otimes m}/\sum_{i+j=m-2}V^i\otimes R\otimes V^j$ is the quotient space where $R=\{u\otimes v-v\otimes u\in V\otimes V\mid u,v\in V\}$. We denote the quotient map by $\overline{(-)}: V^{\otimes m}\to S(V)_m$. Since $s(\omega)=0$ for every $\omega\in V^i\otimes R\otimes V^j$, the linear map $s: V^{\otimes m}\to V^{\otimes m}$ induces a linear map $\widetilde{(-)}: S(V)_m\to V^{\otimes m}$, called the symmetrization map.

Lemma 8. *The linear maps $\overline{(-)}: V^{\otimes m}\to S(V)_m$ and $\widetilde{(-)}: S(V)_m\to V^{\otimes m}$ induce isomorphisms $\overline{(-)}: \operatorname{Sym}^m V\to S(V)_m$ and $\widetilde{(-)}: S(V)_m\to \operatorname{Sym}^m V$ inverses to each other.*

For the rest of the paper, we assume that $\dim V=3$ and we write $S=S(V)=k[x,y,z]$. In this case, every Poisson bracket on S is uniquely determined by $\{y,z\},\{z,x\},\{x,y\}\in S$. A Poisson algebra $(S,\{-,-\})$ is called quadratic if $\{y,z\},\{z,x\},\{x,y\}\in S_2$.

Theorem 9. [5] *If $(S,\{-,-\})$ is a quadratic Poisson algebra, then*

$$k[[t]]\langle x,y,z\rangle/([y,z]-t\widetilde{\{y,z\}},[z,x]-t\widetilde{\{z,x\}},[x,y]-t\widetilde{\{x,y\}})$$

is a deformation quantization of $(S,\{-,-\})$.

For every $f \in S$,

$$\{y,z\}_f := f_x,\ \{z,x\}_f := f_y,\ \{x,y\}_f := f_z$$

defines a Poisson bracket on S. In fact, it is known that $\{-,-\}$ is a unimodular Poisson bracket on S if and only if $\{-,-\} = \{-,-\}_f$ for some $f \in S$. If $f \in S_3$, then $(S, \{-,-\}_f)$ is a quadratic Poisson algebra, so

$$k[[t]]\langle x,y,z\rangle/([y,z] - t\widetilde{f_x}, [z,x] - t\widetilde{f_y}, [x,y] - t\widetilde{f_z})$$

is a deformation quantization of $(S, \{-,-\}_f)$ by Theorem 9. For $f \in S_3$ and $\lambda \in k$, we define the algebra induced by the above deformation quantization as

$$S_f^\lambda := k\langle x,y,z\rangle/([y,z] - \lambda\widetilde{f_x}, [z,x] - \lambda\widetilde{f_y}, [x,y] - \lambda\widetilde{f_z}).$$

The next two results show that Jacobian algebras and deformation quantizations are strongly ralated.

Theorem 10. *For every $f \in S_3$ and every $\lambda \in k$, $S_f^\lambda = J\left(\omega_0 - \lambda\widetilde{f}\right)$.*

Theorem 11. *For $J(\omega) = T(V)/(R)$ where $0 \neq \omega \in V^{\otimes 3}$ and $R \subset V \otimes V$, the following are equivalent:*

(1) *$J(\omega) = S_f^\lambda$ for some $f \in S_3, \lambda \in k$.*
(2) *$R \cap \operatorname{Sym}^2 V = \{0\}$.*
(3) *$R \not\subset \operatorname{Sym}^2 V$.*
(4) *$c(\omega) \notin \operatorname{Sym}^3 V$.*
(5) *$a(\omega) \neq 0$.*
(6) *$\mu(\omega) \neq 0$.*

If any of the above equivalent condition holds, then $J(\omega) = S_{\overline{\omega}}^{-1/\mu(\omega)}$.

The above theorem shows that majority of Jacobian algebras are induced by deformation quantizations.

3. A Criterion for the Calabi-Yau Property

In this section, we will give a criterion for which potential $0 \neq \omega \in V^{\otimes 3}$, $J(\omega)$ is 3-CY. By the previous section, we divide into two cases (1) $a(\omega) \neq 0$ (majority), and (2) $a(\omega) = 0$ (minority).

Let $H(f) := \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$ be the Hessian of $f \in S$. Since $H(f) \in S$, we can define $H^{i+1}(f) := H(H^i(f))$ for every $i \in \mathbb{N}$. The classification of cubic divisors in $\mathbb{P}^2$ is well-known. There are eight singular ones and one family of smooth ones (elliptic curves) up to isomorphisms. The Hessian gives a rough classification of cubic divisors in $\mathbb{P}^2$.

Lemma 12. *For $0 \neq f \in S_3$, the exactly one of the following occurs:*

(1) *$H(f) = 0$. In this case,* $\operatorname{Proj} S/(f)$ *is either triple lines, the union of double line and a line, or the union of three lines meeting at one point.*
(2) *$H(f) \neq 0$, but $H^2(f) = 0$. In this case,* $\operatorname{Proj} S/(f)$ *is either the union of a conic and a line meeting at one point, or a cuspidal curve.*

(3) *$H^i(f) \neq 0$ for every $i \in \mathbb{N}$, In this case, $\operatorname{Proj} S/(f)$ is either a triangle, the union of a conic and a line meeting at two points, a nodal curve or an elliptic curve.*

Recall that $a(\omega) \neq 0$ if and only if $J(\omega) = S_f^\lambda$ for some $f \in S_3$ and $\lambda \in k$ by Theorem 11, so it is essential to ask which S_f^λ is 3-CY.

Theorem 13. *Let $f \in S_3$.*

(1) *If $H^2(f) = 0$, then S_f^λ is 3-CY for every $\lambda \in k$.*
(2) *If $H^2(f) \neq 0$ and $\operatorname{Proj} S/(f)$ is singular, then S_f^λ is 3-CY except for exactly two values of $\lambda \in k$ for each $f \in S_3$.*
(3) *If $H^2(f) \neq 0$ and $\operatorname{Proj} S/(f)$ is smooth, then S_f^λ is 3-CY for every $\lambda \in k$.*

The above theorem shows that majority of S_f^λ is 3-CY. In fact, there are only three exceptions up to isomorphisms.

Theorem 14. *Let $f \in S_3$ and $\lambda \in k$. If S_f^λ is not 3-CY, then it is isomorphic to one of the following algebras:*

- $k\langle x, y, z\rangle/(yz, zx, xy)$.
- $k\langle x, y, z\rangle/(yz + x^2, zx, xy)$.
- $k\langle x, y, z\rangle/(yz + x^2, zx + y^2, xy)$.

On the other hand, if $a(\omega) = 0$, then there are not much choice for ω (minority), so we can show the following theorem by case-by-case analysis.

Theorem 15. *Let $0 \neq \omega \in V^{\otimes 3}$ such that $a(\omega) = 0$.*

(1) *If $H^2(\overline{\omega}) = 0$, then $J(\omega)$ is not 3-CY.*
(2) *If $H^2(\overline{\omega}) \neq 0$ and $\operatorname{Proj} S/(\overline{\omega})$ is singular, then $J(\omega)$ is 3-CY.*
(3) *If $H^2(\overline{\omega}) \neq 0$ and $\operatorname{Proj} S/(\overline{\omega})$ is smooth, then $J(\omega)$ is 3-CY if and only if the j-invariant of $\operatorname{Proj} S/(\overline{\omega})$ is not 0.*

There are six exceptions up to isomorphisms.

Theorem 16. *Let $0 \neq \omega \in V^{\otimes 3}$ such that $a(\omega) = 0$. If $J(\omega)$ is not 3-CY, then it is isomorphic to one of the following algebras:*

- $k\langle x, y, z\rangle/(x^2)$.
- $k\langle x, y, z\rangle/(xy + yx, x^2)$.
- $k\langle x, y, z\rangle/(y^2, x^2)$.
- $k\langle x, y, z\rangle/(xz + zx + y^2, xy + yx, x^2)$.
- $k\langle x, y, z\rangle/(xz + zx, y^2, x^2)$.
- $k\langle x, y, z\rangle/(z^2, y^2, x^2)$.

These nine exceptional algebras in Theorem 14 and Theorem 16 are in one-to-one correspondence with eight singular cubics together with the elliptic curve of j-invariant 0. By [1], every noetherian quadratic Calabi-Yau algebra of dimension 3 is a domain. On the other hand, none of the nine exceptional algebras above is a domain, so we have a rather surprising result:

Theorem 17. *Let $0 \neq \omega \in V^{\otimes 3}$. Then $J(\omega)$ is 3-CY if and only if it is a domain.*

The point scheme is an essential ingredient to study noetherian quadratic Calabi-Yau algebras of dimension 3 in noncommutative algebraic geometry.

Theorem 18. *Let $f \in S_3$ and $\lambda \in k$. If S_f^λ is 3-CY, then the point scheme of S_f^λ is given by* $\operatorname{Proj} S/(24\lambda f + \lambda^3 H(f))$.

It follows that, for a generic choice of $f \in S_3$ and $\lambda \in k$, the point scheme of S_f^λ parameterizes 0-dimensional symplectic leaves for the unimodular Poisson structure on $\mathbb{P}^2 = \operatorname{Proj} S$ induced by f.

A few more calculations for minority show the following theorem:

Theorem 19. *Let $0 \neq \omega \in V^{\otimes 3}$. If $J(\omega)$ is 3-CY, then the point scheme of $J(\omega)$ is given by* $\operatorname{Proj} A/(24\mu(\omega)^2\overline{\omega} + H(\overline{\omega}))$.

4. Examples

We claim that the criterion given in this paper is effective. In fact, given $\omega \in V^{\otimes 3}$, it is routine to calculate $a(\omega)$. Moreover, given $f \in S_3$, it is routine to calculate $H^2(f)$, and it is easy to check if $\operatorname{Proj} S/(f)$ is singular or smooth because $\operatorname{Proj} S/(f)$ is singular if and only if the system of polynomial equations $f_x = f_y = f_z = 0$ has a non-trivial solution. Alternately, by sketching the curve, we can fit $\operatorname{Proj} S/(f)$ into one of the cubic divisors in the classification. Then we can see if it is singular or smooth and we can determine if $H^2(f) = 0$ or not by Lemma 12.

Example 20. If $f = x^2z + xy^2$, then it is easy to see that $\operatorname{Proj} S/(f)$ is the union of a conic and a line meeting at one point, so $H^2(f) = 0$ by Lemma 12, hence S_f^λ is 3-CY for every $\lambda \in k$ by Theorem 13.

Example 21. If $f = xyz + (1/3)x^3 \in S_3$, then it is easy to see that $\operatorname{Proj} S/(f)$ is the union of a conic and a line meeting at two points, so $H^2(f) \neq 0$ by Lemma 12. Since $\operatorname{Proj} S/(f)$ is singular, S_f^λ is 3-CY except for exactly two values of $\lambda \in k$ by Theorem 13. These exceptional values can also be determined by a geometric condition as follows. Since

$$H(f) = \begin{vmatrix} 2x & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix} = 2(xyz - x^3),$$

if S_f^λ is 3-CY, then the point scheme of S_f^λ is $\operatorname{Proj} S/(g)$ where

$$g = 24\lambda f + \lambda^3 H(f) = 2\lambda\{(12 + \lambda^2)xyz + (4 - \lambda^2)x^3\}$$

by Theorem 18. It is easy to see that

$$\operatorname{Proj} S/(g) = \begin{cases} \text{the union of a conic and a line meeting at two points} & \text{if } \lambda^2 \neq 0, -12, 4, \\ \mathbb{P}^2 & \text{if } \lambda = 0, \\ \text{a triple line} & \text{if } \lambda^2 = -12, \\ \text{a triangle} & \text{if } \lambda^2 = 4. \end{cases}$$

We can show that S_f^λ is 3-CY if and only if $\operatorname{Proj} S/(g)$ is not a triangle. In fact, the defining relations of S_f^λ are

$$\begin{aligned}
[y,z]-\lambda\widetilde{f_x} &= yz-zy-\lambda\left(\frac{yz+zy}{2}+x^2\right)=\frac{2-\lambda}{2}yz-\frac{2+\lambda}{2}zy-\lambda x^2\\
[z,x]-\lambda\widetilde{f_y} &= zx-xz-\lambda\left(\frac{zx+xz}{2}\right)=\frac{2-\lambda}{2}zx-\frac{2+\lambda}{2}xz\\
[x,y]-\lambda\widetilde{f_z} &= xy-yx-\lambda\left(\frac{xy+yx}{2}\right)=\frac{2-\lambda}{2}yx-\frac{2+\lambda}{2}yx,
\end{aligned}$$

so if $\lambda=\pm 2$, then S_f^λ is not a domain, hence it is not 3-CY.

Example 22. If $\omega = x^3+y^3+z^3+(3\alpha/2)(xyz+zyx)\in V^{\otimes 3}$ where $\alpha\in k$, then it is easy to see that $a(\omega)=0$, so we apply Theorem 15 to this example. Since $f:=\overline{\omega}=x^3+y^3+z^3+3\alpha xyz\in S_3$, it is well-known that

$$\operatorname{Proj} S/(f)=\begin{cases}\text{a triangle} & \text{if } \alpha^3=-1,\\ \text{an elliptic curve} & \text{if } \alpha^3\neq -1,\end{cases}$$

so $H^2(f)\neq 0$ in either case by Lemma 12. If $\alpha^3=-1$, then $\operatorname{Proj} S/(f)$ is singular, so $J(\omega)$ is 3-CY by Theorem 15. On the other hand, if $\alpha^3\neq -1$, then $\operatorname{Proj} S/(f)$ is smooth (an elliptic curve) and the j-invariant of $\operatorname{Proj} S/(f)$ is given by the formula

$$\frac{\alpha^3(8-\alpha^3)}{(1+\alpha^3)^3},$$

so $J(\omega)$ is 3-CY if and only if $\alpha^3\neq 0,8$ by Theorem 15.

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