ON THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE OF THE QUIVER ALGEBRA DEFINED BY c CYCLES AND A QUANTUM-LIKE RELATION

DAIKI OBARA

Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Osaka City University, Japan, 13–15 September 2014.

In this paper, we consider the quiver algebra A over a field K defined by c cycles and a quantum-like relation. We describe the minimal projective bimodule resolution of A, and determine the ring structure of the Hochschild cohomology ring of A modulo nilpotence. And we give some examples of the support variety of A-modules.

1. Introduction

Let K be a field and A an indecomposable finite dimensional algebra over K. We denote by A^e the enveloping algebra $A \otimes_K A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The n-th Hochschild cohomology group is given by $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$ and the Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \oplus_{n\geq 0}\mathrm{HH}^n(A, A)$ with Yoneda product. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. In this paper, we consider the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

The Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ was used in [5] to define a support variety for any finitely generated module over a finite dimensional algebra A. In [5], Snashall and Solberg defined the support variety $V(M)$ of an A-module M by

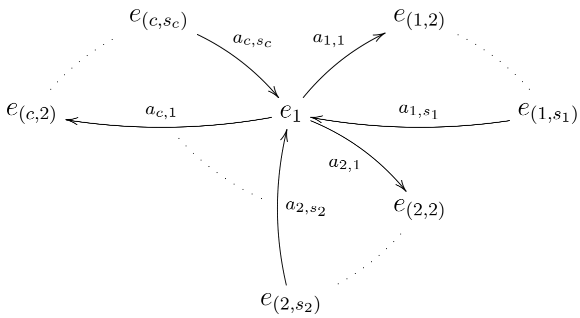
$$V(M) = \{m \in \mathrm{MaxSpec}\mathrm{HH}^*(A)/\mathcal{N} | \operatorname{Ann} \mathrm{Ext}^*_A(M, A/\operatorname{rad} A) \subseteq m'\}.$$

where m' is the inverse image of m in $\mathrm{HH}^*(A)$.

Let c be an integer with $c \geq 2$ and $q_{i,j} \in K$ nonzero elements for $1 \leq i < j \leq c$. We consider the quiver algebra KQ/I defined by c cycles and a quantum-like relation where

The detailed version of this paper will be submitted for publication elsewhere.

Q is the following quiver:



where $1 \leq j \leq c$ and $s_j \geq 2$, and where I is the ideal of KQ generated by

$$\begin{aligned} &X_i^{n_i} \text{ for } 1 \leq i \leq c, \\ &X_i X_j - q_{i,j} X_j X_i \text{ for } 1 \leq i < j \leq c. \end{aligned}$$

where $X_i := (\sum_{k_i=1}^{s_i} a_{i,k_i})^{s_i}$ and n_i are integers with $n_i \geq 2$ for $1 \leq i \leq c$.

In the case $c = 2$, we determined the Hochschild cohomology ring of A modulo nilpotence in [2] and [3]. In the case $s_i = 1$ for $1 \leq i \leq c$, the Hochschild cohomology ring of A modulo nilpotence was described by Oppermann in [4]. In this paper, we describe the minimal projective bimodule resolution of A, and determine explicitly the ring structure of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ by giving the K-basis and the multiplication.

2. Precedent results

In this section, we introduce the precedent results about the quiver algebra A. In the case of $s_i = 1$ for $1 \leq i \leq c$, A is called a quantum complete intersection. In this case, the projective bimodule resolution of A and the Hochschild cohomology ring modulo nilpotence of A was given by Oppermann in [4] as follows.

Theorem 1. [4] *In the case of $s_i = 1$ for $1 \leq i \leq c$, the projective bimodule resolution of A is total complex* $\mathrm{Tot}(\mathbb{P}_1 \otimes \mathbb{P}_2 \otimes \cdots \otimes \mathbb{P}_c)$ *where $\mathbb{P}_i$ is the projective bimodule resolution of $A_i = K[\alpha_i]/(\alpha_i^{n_i})$:*

$$\mathbb{P}_i : A_i^e \xleftarrow{1\otimes x - x_i \otimes 1} A_i^e \xleftarrow{\sum_{k=0}^{n_i-1} x_i^k \otimes x_i^{n_i-1-k}} A_i^e \xleftarrow{1\otimes x - x_i \otimes 1} A_i^e \longleftarrow \cdots .$$

Theorem 2. [4] $\mathrm{HH}^*(A)/\mathcal{N}$ *is isomorphic to the following finitely generated K-algebra.*

$$\begin{aligned} {}_K\langle y_1^{p_1 n_1/2} \cdots y_c^{p_c n_c/2} \in K[y_1, \ldots, y_c] \,|\, &\textstyle\prod_{j=1}^{c} q_{i,j}^{p_j n_j/2} = 1 \text{ for all } i \text{ with } p_i \text{ even}, \\ &\textstyle\prod_{j=1}^{c} q_{i,j}^{(p_j-1)n_j/2+1} = -1 \text{ and } n_i = 2 \text{ for all } i \text{ with } p_i \text{ odd}\rangle. \end{aligned}$$

where $q_{i,i} = 1$ and $q_{i,j} = q_{j,i}^{-1}$ for $1 \leq j < i \leq c$.

In the case of $c = 2$, we determined the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ in [2] and [3] as follows.

Theorem 3. *Let r be an integer with $r > 0$. In the case of $c = 2$, if $q_{1,2}$ is a primitive r-th root of unity, then $\mathrm{HH}^*(A)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables:*

$$\mathrm{HH}^*(A)/\mathcal{N} \cong \begin{cases} K[x^{2r}, y^{2r}] & \text{if } n_1, n_2 \not\equiv 0 \bmod r, \\ K[x^2, y^{2r}] & \text{if } n_1 \equiv 0 \bmod r, n_i \not\equiv 0 \bmod r, \\ K[x^{2r}, y^2] & \text{if } n_1 \not\equiv 0 \bmod r, n_2 \equiv 0 \bmod r, \\ K[x^2, y^2] & \text{if } n_1, n_2 \equiv 0 \bmod r, \end{cases}$$

where $x^n = \sum_{k_1=1}^{s_1} e_{(1,k_1)}$, $y^n = \sum_{k_2=1}^{s_2} e_{(2,k_2)}$ in $\mathrm{HH}^n(A)$.

Theorem 4. *In the case of $c = 2$, if $q_{1,2}$ is not a root of unity, then $\mathrm{HH}^*(A)/\mathcal{N} \cong K$.*

3. Projective bimodule resolution of A

In this section, we describe the minimal projective bimodule resolution of the quiver algebra $A = KQ/I$ defined by c cycles and a quantum-like relation.

Let c and n be integers with $c \geq 2$ and $n \geq 1$. We set

$$L_n = \{(l_1, l_2, \dots, l_c) \in (\mathbb{N} \cup \{0\})^c \mid \sum_{k=1}^{c} l_k = n\} \text{ for any integer } n \geq 1.$$

We define projective left A^e-modules, equivalently A-bimodules:

$$P_0 = A\varepsilon_0^0 A \oplus \coprod_{i=1}^{c} \coprod_{k_i=2}^{s_i} A\varepsilon_{(i,k_i)}^0 A \text{ and,}$$

$$Q_{(l_1,\dots,l_c)}^n = \begin{cases} \coprod_{k_i=1}^{s_i} A\varepsilon_{(i,k_i)}^n A & \text{if } l_i = n \text{ for some } 1 \leq i \leq c, \\ A\varepsilon_{(l_1,\dots,l_c)}^n A & \text{if } l_i < n \text{ for all } 1 \leq i \leq c, \end{cases}$$

for $(l_1, \dots, l_c) \in L_n$, where $\varepsilon_{(l_1,\dots,l_c)}^n = e_1 \otimes e_1$ and

$$\varepsilon_{(i,k_i)}^n = \begin{cases} e_{(i,k_i)} \otimes e_{(i,k_i)} & \text{if } n \text{ is even,} \\ e_{(i,k_i+1)} \otimes e_{(i,k_i)} & \text{if } n \text{ is odd.} \end{cases}$$

Then, we have the minimal projective A-bimodule resolution of A as the total complex of the following complexes.

Lemma 5. *Let n be an integer with $n \geq 1$ and $E_{i,k_i}^n = \sum_{l=0}^{s_i-1} x_i^l \varepsilon_{(i,k_i-l)}^n x_i^{s_i-1-l}$ for $1 \leq i \leq c$ and $0 \leq k_i \leq s_i - 1$. For $(l_1, \dots, l_c) \in L_n$, we set the integers μ_i by*

$$\mu_i = \begin{cases} n_i(l_i - 1)/2 + 1 & \text{if } l_i \text{ is odd,} \\ n_i l_i/2 & \text{if } l_i \text{ is even,} \end{cases} \quad \text{for } 1 \leq i \leq c.$$

Then, we have the following complexes.

(1) *For* $(l_1,\dots,l_c)\in L_n$ *such that* $l_i=n$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),i}: Q^n_{(l_1,\dots,l_c)}\to Q^{n-1}_{(l_1,\dots,l_i-1,\dots,l_c)}$ *by*

$$\partial^n_{(l_1,\dots,l_c),i}:\varepsilon^n_{(i,k_i)}\mapsto\begin{cases}\varepsilon^{n-1}_{(i,k_i+1)}x_i-x_i\varepsilon^{n-1}_{(i,k_i)} & \text{if } n \text{ is odd,}\\ \sum_{l=0}^{n_i-1}X_i^lE^{n-1}_{i,k_i-1}X_i^{n_i-1-l} & \text{if } n \text{ is even,}\end{cases}\quad \text{for } 1\le k_i\le s_i.$$

Then, since $\partial^n_{(l_1,\dots,l_c),i}\circ\partial^{n+1}_{(l_1,\dots,l_i+1,\dots,l_c),i}=0$, *we have the complex* $\mathbb{P}_i$:

$$P_0\xleftarrow{\partial^1_{(0,\dots,1,\dots,0),i}}Q^1_{(0,\dots,1,\dots,0)}\xleftarrow{\partial^2_{(0,\dots,2,\dots,0),i}}\cdots\xleftarrow{\partial^n_{(0,\dots,n,\dots,0),i}}Q^n_{(0,\dots,n,\dots,0)}\leftarrow\cdots.$$

(2) *Let* $m=\min\{i\,|\,l_i>0\}$ *for* $(l_1,\dots,l_c)\in L_n$. *For* $m\le j\le c$ *and* $(l_1,\dots,l_c)\in L_n$ *such that* $l_i<n-1$ *for* $1\le i\le c$ *and* $l_j\neq 0$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),j}: Q^n_{(l_1,\dots,l_c)}\to Q^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}$ *as follows*:

$$\partial^n_{(l_1,l_2,\dots,l_c),j}:\varepsilon^n_{(l_1,\dots,l_c)}\mapsto$$
$$\begin{cases}(-1)^{\sum_{k=j+1}^c l_k}\Big(\prod_{h_1=1}^{c-j}q^{\mu_{j+h_1}}_{j,j+h_1}\varepsilon^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}X_j-\prod_{h_2=1}^{j-1}q^{\mu_{h_2}}_{h_2,j}X_j\varepsilon^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}\Big) & \text{if } l_j \text{ is odd,}\\ (-1)^{\sum_{k=j+1}^c l_k}\sum_{k_j=0}^{n_j-1}\prod_{h_1=1}^{c-j}q^{\mu_{j+h_1}(n_j-1-k_j)}_{j,j+h_1}\prod_{h_2=1}^{j-1}q^{\mu_{h_2}k_j}_{h_2,j}X_j^{k_j}\varepsilon^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}X_j^{n_j-1-k_j} & \text{if } l_j \text{ is even}(\neq 0).\end{cases}$$

For $(l_1,\dots,l_c)\in L_n$ *such that* $l_m=n-1$ *and* $l_j=1$ *for* $m\le j\le c$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),j}$ *by*

$$\partial^n_{(l_1,\dots,l_c),j}:\varepsilon^n_{(l_1,\dots,l_c)}\mapsto\begin{cases}E^{n-1}_{m,0}X_j-q^{\mu_m}_{m,j}X_jE^{n-1}_{m,0} & \text{if } n \text{ is even,}\\ \varepsilon^{n-1}_{(m,1)}X_j-q^{\mu_m}_{m,j}X_j\varepsilon^{n-1}_{(m,1)} & \text{if } n \text{ is odd,}\end{cases}$$

For $(l_1,\dots,l_c)\in L_n$ *such that* $l_m=1$ *and* $l_j=n-1$ *for* $m\le j\le c$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),j}$ *by*

$$\partial^n_{(l_1,\dots,l_c),j}:\varepsilon^n_{(l_1,\dots,l_c)}\mapsto\begin{cases}E^{n-1}_{j,0}X_m-q^{\mu_j}_{m,j}X_mE^{n-1}_{j,0} & \text{if } n \text{ is even,}\\ \varepsilon^{n-1}_{(j,1)}X_m-q^{\mu_j}_{m,j}X_m\varepsilon^{n-1}_{(j,1)} & \text{if } n \text{ is odd,}\end{cases}$$

Then, since $\partial^n_{(l_1,\dots,l_c),j}\circ\partial^{n+1}_{(l_1,\dots,l_j+1,\dots,l_c),j}=0$, *for* $(l_1,\dots,l_c)\in L_n$ *such that* $l_j=0$, *we have the complex* $\mathbb{Q}_{(l_1,\dots,l_c),j}$:

$$Q^n_{(l_1,\dots,0,\dots,l_c)}\xleftarrow{\partial^{n+1}_{(l_1,\dots,1,\dots,l_c),j}}Q^{n+1}_{(l_1,\dots,1,\dots,l_c)}\leftarrow\cdots\xleftarrow{\partial^{n+n'}_{(l_1,\dots,n',\dots,l_c),j}}Q^{n+n'}_{(l_1,\dots,n',\dots,l_c)}\leftarrow\cdots.$$

Theorem 6. *The following total complex* $\mathbb{P}$ *is the minimal projective resolution of the left* A^e*-module* A.

$$\mathbb{P}:0\leftarrow A\xleftarrow{\pi}P_0\xleftarrow{d_1}P_1\leftarrow\cdots\xleftarrow{d_n}P_n\leftarrow\cdots$$

where π is the multiplication map and

$$P_n = \coprod_{(l_1,\dots,l_c)\in L_n} Q^n_{(l_1,\dots,l_c)} \text{ and } d_n = \sum_{j=1}^{c} \sum_{(l_1,\dots,l_c)\in L_n} \partial^n_{(l_1,\dots,l_c),j},$$

for $n \geq 1$, where $\partial^n_{(l_1,\dots,l_c),j}$ are the A^e-homomorphisms given in Lemma 5.

Now we consider the complex $\mathbb{P} \otimes_A A/\operatorname{rad} A$. We can prove that $\mathbb{P}$ is exact, by the following Lemma.

Lemma 7. [1] *If $\mathbb{P} \otimes_A A/\operatorname{rad} A$ is exact sequence then $\mathbb{P}$ is also exact sequence.*

We can prove that $\mathbb{P}\otimes_A A/\operatorname{rad} A$ is exact, that is $\dim_k \operatorname{Im} d_n \otimes_A \mathrm{id}_{A/\operatorname{rad} A} + \dim_k \operatorname{Im} d_{n+1} \otimes_A \mathrm{id}_{A/\operatorname{rad} A} = \dim_k P_n \otimes_A A/\operatorname{rad} A$ by the following Lemma.

Lemma 8. *Let $(l_1, \dots, l_c) \in L_n$ such that $l_i < n-1$ for $1 \leq i \leq c$, and $m = min\{i \,|l_i > 0\}$ for $(l_1, \dots, l_c) \in L_n$.*

(1) *If l_m is even, then the left A-module $AX_m d_n \otimes_A \mathrm{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_c)})$ is generated by*
$$d_n \otimes_A \mathrm{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_m+1,\dots,l_j-1,\dots,l_c)}) \quad \text{for } m+1 \leq j \leq c \text{ such that } l_j \neq 0.$$

(2) *If l_m is odd, then the left A-module $AX_m^{n_m-1}\widehat{d}_n(e_{(l_1,\dots,l_c)})$ is generated by*
$$d_n \otimes_A \mathrm{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_m+1,\dots,l_j-1,\dots,l_c)}) \quad \text{for } m+1 \leq j \leq c \text{ such that } l_j \neq 0.$$

(3) *For $1 \leq i \leq m-1$, the left A-module $AX_i\widehat{d}_n(e^n_{(l_1,\dots,l_c)})$ is generated by*
$$d_n \otimes_A \mathrm{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_i+1,\dots,l_j-1,\dots,l_c)}) \quad \text{for } m+1 \leq j \leq c \text{ such that } l_j \neq 0.$$

4. The Hochschild cohomology ring modulo nilpotence

In this section, we give a K-basis of the Hochschild cohomology ring modulo nilpotence. Applying the functor $\operatorname{Hom}_{A^e}(-, A)$ to the A^e-projective resolution $\mathbb{P}$ given in Theorem 6, we have the following complex:

$$\mathbb{P}^* : 0 \to P_0^* \xrightarrow{d_1^*} P_1^* \to \cdots \to P_{n-1}^* \xrightarrow{d_n^*} P_n^* \to \cdots,$$

where

$$P_n^* = \coprod_{(l_1,\dots,l_c)\in L_n} \operatorname{Hom}_{A^e}(Q^n_{(l_1,\dots,l_c)}, A) \text{ and } d_n^* = \sum_{i=1}^{c} \sum_{(l_1,\dots,l_c)\in L_n} \operatorname{Hom}_{A^e}(\partial^n_{(l_1,\dots,l_c),i}, A),$$

for $n \geq 1$. Then we have the following isomorphisms:

$$P_0^* = \operatorname{Hom}_{A^e}(P_0, A) \simeq e_1Ae_0^0 \oplus \coprod_{i=1}^{c}\coprod_{k_i=2}^{s_i} e_{i,k_i}Ae^0_{(i,k_i)},$$

$$\operatorname{Hom}_{A^e}(Q^n_{(l_1,\dots,l_c)}, A) \simeq \begin{cases} \displaystyle\coprod_{k_i=1}^{s_i} e_{(i,k_i)}Ae^n_{(i,k_i)} & \text{if } n \text{ is even and } l_i = n, \\ \displaystyle\coprod_{k_i=1}^{s_i} e_{(i,k_i+1)}Ae^n_{(i,k_i)} & \text{if } n \text{ is odd and } l_i = n, \\ e_1Ae^n_{(l_1,\dots,l_c)} & \text{if } l_i < n \text{ for } 1 \leq i \leq c, \end{cases}$$

for $(l_1, \ldots, l_c) \in L_n$. Since we give the Hochschild cohomology ring modulo nilpotence, we only consider the elements, which are trivial passes in A, in $\mathrm{HH}^n(A) = \mathrm{Ker}\, d^*_{n+1}/\mathrm{Im}\, d^*_n$. Now, we give the image of $e^n_{(l_1,\ldots,l_c)}$ in P^*_n by $\partial^{n+1*}_{(l_1,\ldots,l_j+1,\ldots,l_c),j}$ for $(l_1, \ldots, l_j+1, \ldots, l_c) \in L_{n+1}$ and $1 \leq j \leq c$.

$$\mathrm{Hom}_{A^e}(\partial^{n+1}_{(l_1,\ldots,l_j+1,\ldots,l_c),j}, A) :$$
$$\begin{cases} e^n_{(i,k_i)} \mapsto x_i e^{n+1}_{(i,k_i-1)} - x_i e^{n+1}_{(i,k_i)} \quad \text{for } 1 \leq k_i \leq s_i & \text{if } n \text{ is even}, l_i = n \text{ and } i = j, \\ e^n_{(i,1)} \mapsto (1 - q^{\mu_i}_{i,j}) X_j e^{n+1}_{(l_1,\ldots,l_j+1,\ldots,l_c)} & \text{if } l_i = n \text{ and } i < j, \\ e^n_{(i,1)} \mapsto (-1)^n (q^{\mu_i}_{j,i} - 1) X_j e^{n+1}_{(l_1,\ldots,l_j+1,\ldots,l_c)} & \text{if } l_i = n \text{ and } i > j, \\ e^n_{(l_1,\ldots,l_c)} \mapsto & \\ \begin{cases} (-1)^{\sum_{k=j+1}^c l_k} \Big(\prod\limits_{h_1=1}^{c-j} q^{\mu_{j+h_1}}_{j,j+h_1} - \prod\limits_{h_2=1}^{j-1} q^{\mu_{h_2}}_{h_2,j} \Big) X_j e^{n+1}_{(l_1,\ldots,l_j+1,\ldots,l_c)} & \text{if } l_j \text{ is even}, \\ (-1)^{\sum_{k=j+1}^c l_k} \sum\limits_{k_j=0}^{n_j-1} \prod\limits_{h_1=1}^{c-j} q^{\mu_{j+h_1}(n_j-1-k_j)}_{j,j+h_1} \prod\limits_{h_2=1}^{j-1} q^{\mu_{h_2} k_j}_{h_2,j} X_j^{n_j-1} e^{n+1}_{(l_1,\ldots,l_j+1,\ldots,l_c)} & \text{if } l_j \text{ is odd}, \end{cases} & \\ & \text{if } l_i < n \text{ for } 1 \leq i \leq c, \end{cases}$$

For homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have the Yoneda product $\eta\theta = \eta\sigma_m \in \mathrm{HH}^{m+n}(A)$ where σ_m is a lifting of θ in the following commutative diagram of A-bimodules.

$$\begin{tikzcd} \cdots \arrow[r] & P_{m+n} \arrow[r, "d_{m+n}"] \arrow[d, "\sigma_m"] & \cdots \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\sigma_1"] & P_n \arrow[d, "\sigma_0"] \arrow[dr, "\theta"] & & \\ \cdots \arrow[r] & P_m \arrow[r, "d_m"] & \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\pi"] & A \arrow[r] & 0. \end{tikzcd}$$

Proposition 9. *Let $(l_1, \ldots, l_c) \in L_n$, $(l'_1, \ldots, l'_c) \in L_{n'}$. Then we have the lifting of $e^n_{(l_1,\ldots,l_c)}$ as follows.*

$$\sigma_{n'} : \varepsilon^{n+n'}_{(l_1+l'_1,\ldots,l_c+l'_c)} \mapsto \sum_{\substack{0 \leq k_j \leq n_j - 2 \\ 1 \leq j \leq c \\ \text{such that} \\ l_j, l'_j \text{ are odd}}} Q \prod_{\substack{1 \leq j \leq c \\ \text{such that} \\ l_j, l'_j \text{ are odd}}} X_j^{k_j} \varepsilon^{n'}_{(l'_1,\ldots,l'_c)} \prod_{\substack{1 \leq j \leq c \\ \text{such that} \\ l_j, l'_j \text{ are odd}}} X_j^{n_j-2-k_j},$$

for $n' \geq 0$ where $Q \in K$ depending on $(l_1 + l'_1, \ldots, l_c + l'_c) \in L_{n+n'}$ and integers k_j.

By Proposition 9, if n is odd or l_j is odd for some $1 \leq j \leq c$, $e^n_{(l_1,\ldots,l_c)}$ is nilpotence.

By the complex $\mathbb{P}^*$ and Yoneda product given by Proposition 9, we have the K-basis of the Hochschild cohomology ring of A modulo nilpotence as follows.

Theorem 10. *Let $q_{i,j} = q^{-1}_{j,i}$ for $1 \leq j < i \leq c$. The following elements form a K-basis of $\mathrm{HH}^*(A)/\mathbb{N}$.*

(1) $\sum_{k_i=1}^{s_i} e^n_{(i,k_i)} \in \mathrm{HH}^n(A)/\mathcal{N}$ *for the even integer n and the integer i with $1 \leq i \leq c$ which satisfy the following conditions*:

$$q^{n_i n/2}_{i,j} = 1 \quad \text{for } 1 \leq j \leq c.$$

(2) $e^n_{(l_1,\dots,l_c)} \in \mathrm{HH}^n(A)/\mathcal{N}$ *for the even integer* n *and* $(l_1,\dots,l_c) \in L_n$ *which satisfy the following conditions*:

$$l_i \text{ is even for } 1 \le i \le c,$$
$$\prod_{h=1}^{c} q_{j,h}^{n_h l_h/2} = 1 \quad \text{for } 1 \le j \le c \text{ such that } l_j \ne 0,$$

Remark 11. In the case of $n_i > 2$ for $1 \le i \le c$, the K-basis elements of $\mathrm{HH}^*(A)/\mathcal{N}$ given in Theorem 10 coincide with those of given in Theorem 2.

5. Examples of the support variety

In this section, we give the examples of the support variety of an A-module. In [5], Snashall and Solberg defined the support variety $V(M)$ of a A-module M by

$$V(M) = \{m \in \mathrm{MaxSpecHH}^*(A)/\mathcal{N} \,|\, \mathrm{Ann}\,\mathrm{Ext}^*_A(M, A/\mathrm{rad}\,A) \subseteq m'\}.$$

where m' is the inverse image of m in $\mathrm{HH}^*(A)$ and $\mathrm{Ann}\,\mathrm{Ext}^*_A(M, A/\mathrm{rad}\,A)$ is annihilator of $\mathrm{Ext}^*_A(M, A/\mathrm{rad}\,A)$.

Let K be an algebraically closed filed and $r \in \mathbb{N}$. We consider the case $c = 2$, $s_1 = s_2 = 1$,$q_{1,2}$ is a primitive r-th root of unity and $n_1, n_2 \not\equiv 0 \bmod r$ ([2]). Then we have

$$\mathrm{HH}^*(A)/\mathcal{N} = K[X, Y].$$

where $X = \sum_{k_1=1}^{s_1} e_{(1,k_1)}$, $Y = \sum_{k_2=1}^{s_2} e_{(2,k_2)}$ in $\mathrm{HH}^{2r}(A)$.

Example 12. Let $M_1 = Ax_1^{s_1 t} e_1$. We have $\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A)$ and the annihilator of $\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A)$ as follows:

$$\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A) = \textstyle\coprod_{n\ge 0} K e^n_{(1,1)},$$
$$\mathrm{Ann}\,\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A) = (Y).$$

And we have the support variety of M_1 as follows:

$$V(M_1) = \{(a_1, a_2) \in K^2 \,|\, a_2 = 0\} \text{ as an affine algebraic set.}$$

Example 13. Let $M_2 = AX_1^{s_1 t_1} X_2^{s_2 t_2}$ and $M_3 = AX_1^{s_1 t_1} e_1 + AX_2^{s_2 t_2} e_1$. We have the annihilator of $\mathrm{Ext}^*_A(M_i, A/\mathrm{rad}\,A)$ for $i = 2, 3$ as follows:

$$\mathrm{Ann}\,\mathrm{Ext}^*_A(M_i, A/\mathrm{rad}\,A) = 0.$$

And we have the support variety of M_i for $2 \le i \le 3$ as follows:

$$V(M_i) = K^2 \text{ as an affine plane.}$$

References

[1] E.L. Green, N. Snashall, *Projective bimodule resolutions of an algebra and vanishing of the second Hochschild cohomology group*, Forum. Math. **16** (2004), 17–36.

[2] D. Obara, *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation*, Comm. in Algebra **40** (2012), 1724–1761.

[3] D. Obara, *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation* II, to appear Comm. in Algebra.

[4] S. Oppermann, *Hochschild cohomology and homology of quantum complete intersections*, Algebra Number Theory **4** (2010), 821–838.
[5] N. Snashall, Ø. Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. **88** (2004), 705–732.

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Sinjuku-ku, Tokyo 162-8601 JAPAN
E-mail address: d_obara@rs.tus.ac.jp