ON SILTING-DISCRETE TRIANGULATED CATEGORIES

TAKUMA AIHARA

Abstract. The aim of this paper is to study silting-discrete triangulated categories. We establish a simple criterion for silting-discreteness in terms of 2-term silting objects. This gives a powerful tool to prove silting-discreteness of finite dimensional algebras. Moreover, we will show Bongartz-type Lemma for silting-discrete triangulated categories.

1. Introduction

In the study of triangulated categories, the class of tilting objects is one of the most important classes of objects, and tilting mutation for tilting objects often plays a crucial role, e.g. categorification of cluster algebras [8, 10] and Broué's conjecture in modular representation theory of finite groups [11]. From viewpoint of mutation, it was pointed out in [5] that one should deal with a more general class of silting objects than tilting objects, and silting mutation for silting objects were introduced. Moreover, the set of silting objects naturally has the structure of a partially orderd set which is closely related with silting mutation [5]. When a silting object is fixed, the partial order yields the notion of lengths of objects [3].

A problem is to understand the whole context of silting objects; e.g. to give a combinatorial description of silting objects. A triangulated category is called *silting-connected* provided all silting objects are reachable each other by iterated silting mutation. In this case, we can describe the combinatorial structure of the triangulated category in terms of silting objects and the relationship given by silting mutation. The *silting-discrete* triangulated categories are in some sense the simplest kinds of silting-connected triangulated categories [3], that is, the triangulated category admits a silting object A such that for any positive integer $\ell > 0$, there exist only finitely many silting objects of the length ℓ with respect to A: a finite dimensional algebra is also said to be *silting-discrete* if the perfect derived category of the algebra is silting-discrete. For example, we know that local algebras, path algebras of Dynkin type and representation-finite symmetric algebras are silting-discrete [5, 3].

We investigate silting-discrete triangulated categories and study the following question:

Question 1. *When is a triangulated category silting-discrete?*

The first aim of this pape is to give an answer to this question. A triangulated category is said to be *2-silting-finite* if for every silting object T, there exist only finitely many silting objects of the length 2 with respect to T.

A main result of this paper is the following theorem.

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Theorem 2 (Theorem 16). *A triangulated category is silting-discrete if and only if it is 2-silting-finite.*

A great advantage of this theorem is that we can let Question 1 come down to the question of the finiteness of certain modules for algebras: For a silting object A, there is a one-to-one correspondence between silting objects of the length 2 with respect to A and support τ-tilting modules for the endomorphism algebra of A [2, 9].

Therefore, Theorem 2 gives a powerful tool to prove that a given finite dimensional algebra is silting-discrete. In fact, Theorem 2 will be applied in [1] and [4] to show that the following algebras are silting-discrete:

- Brauer graph algebras of type odd;
- Preprojective algebras of Dynkin type D_{2n}, E_7, E_8.

The second aim of this paper is to study a generalization of famous Bongartz's Lemma [6], which says that every (classical) pretilting module is partial tilting. On the other hand, a naive generalization of Bongartz's Lemma for tilting objects in a triangulated category fails: an easy example [12] shows that a pretilting object in a triangulated category is not necessarily partial tilting. In the previous paper [3], we observed that it is reasonable to consider Bongartz-type Lemma for silting objects in a triangulated category. Thus, we discuss the following question:

Question 3. *Is any presilting object partial silting?*

In this paper, we give a positive answer to Question 3 for silting-discrete triangulated categories.

Theorem 4 (Theorem 17). *Any presilting object of a silting-discrete triangulated category is partial silting.*

A point for the proofs of Theorem 2 and Theorem 17 is to use a kind of induction on the length ℓ of a (pre)silting object T. To do this, we introduce the notion of "minimal silting objects" for T, which is a minimal element in a poset consisting of certain silting objects (see Definition 10 for details). The key result for the proofs of Theorem 2 and Theorem 17 is the following theorem.

Theorem 5 (Theorem 11). *Let A be a silting object and T a presilting object of the length ℓ with respect to A. If there exists a minimal silting object P for T, then the length of T with respect to P is at most $\ell - 1$.*

This paper is organized as follows. In section 2, we introduce the notion of minimal silting objects and state a main theorem of this paper (Theorem 11). In section 3, we study silting-discrete triangulated categories and give the theorems on equivalent conditions of and Bongartz-type Lemma for silting-discrete triangulated categories (Theorem 16 and Theorem 17). In section 4, we give several examples of silting-discrete triangulated categories. Furthermore, we will know from the final example (Example 23) that the finiteness of silting objects of length 2 is not derived invariant.

Notation. Throughout this paper, let $\mathcal{T}$ be a Krull-Schmidt triangulated category and assume that it satisfies the following property:

(F) For any object X of $\mathcal{T}$, the additive closure $\mathsf{add}\, X$ is functorially finite in $\mathcal{T}$.

For example, let R be a complete local Noetherian ring and $\mathcal{T}$ an R-linear idempotent-complete triangulated category such that $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is a finitely generated R-module for any object X and Y of $\mathcal{T}$. Then $\mathcal{T}$ is a Krull-Schmidt triangulated category satisfying the property (F).

2. Minimal silting objects

In this section, we study silting mutation and a main theorem of this paper is stated.

Let us start with recalling the definition of silting objects.

Definition 6. (1) We say that an object T in $\mathcal{T}$ is *presilting* (*pretilting*) if it satisfies $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any $i > 0$ ($i \neq 0$).
(2) An object T is said to be *silting* (*tilting*) if it is presilting (pretilting) and generates $\mathcal{T}$ by taking direct summands, mapping cones and shifts.
(3) A presilting object T is called *partial silting* provided it is a direct summand of some silting object.

We denote by $\mathsf{silt}\,\mathcal{T}$ the set of non-isomorphic basic silting objects in $\mathcal{T}$.

In the rest of this paper, we assume that $\mathcal{T}$ has a silting object.

It is known that the number of non-isomorphic indecomposable summands of any silting object does not depend on the choice of silting objects.

Proposition 7. [5] *Let T and U be silting objects of $\mathcal{T}$. Then the number of non-isomorphic indecomposable summands of T coincides with that of U.*

For objects M and N of $\mathcal{T}$, we write $M \geq N$ if $\mathrm{Hom}_{\mathcal{T}}(M, N[n]) = 0$ for any $n > 0$. Note that $\geq$ is not a partial order on $\mathcal{T}$. According to [5], we have that $\geq$ gives a partial order on $\mathsf{silt}\,\mathcal{T}$.

We also recall silting mutation for silting objects.

Definition 8. Let T be a basic silting object of $\mathcal{T}$. For a decomposition $T := X \oplus M$, we take a triangle

$$X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left $\mathsf{add}\,M$-approximation f of X. Then $\mu_X^-(T) := Y \oplus M$ is again silting, and we call it the *left mutation* of T with respect to X. Dually, define the right mutation $\mu_X^+(T)$. *(Silting) mutation* will mean either left or right mutation. Mutation is said to be *irreducible* if X is indecomposable.

We get basic properties of silting mutation.

Proposition 9. [5, 3] *With the notations as in Definition 8, the following hold:*

(1) We have the inequality $T > \mu_X^-(T)$.
(2) The right mutation $\mu_Y^+(\mu_X^-(T))$ of $\mu_X^-(T)$ with respect to Y is isomorphic to T.
(3) If X is indecomposable, then there is no silting object U satisfying $T > U > \mu_X^-(T)$.

(4) Let U be a presilting object with $T \geq U$ which does not belong to $\mathsf{add}\, T$. For $U_0 := U$, take triangles

$$\begin{array}{l} U_1 \longrightarrow T_0 \xrightarrow{f_0} U_0 \longrightarrow U_1[1] \\ \cdots \\ U_\ell \longrightarrow T_{\ell-1} \xrightarrow{f_{\ell-1}} U_{\ell-1} \longrightarrow U_\ell[1] \\ 0 \longrightarrow T_\ell \xrightarrow{f_\ell} U_\ell \longrightarrow 0 \end{array}$$

where f_i is a minimal right $\mathsf{add}\, T$-approximation of U_i for $0 \leq i \leq \ell$. Let X be an indecomposable summand of T. If X belongs to $\mathsf{add}\, T_\ell$, then we have $\mu_X^-(T) \geq U$.

We always use the following terminology.

Definition 10. We define a subset of $\mathsf{silt}\, \mathcal{T}$ as follows:

$$\nabla(A; T) := \{U \in \mathsf{silt}\, \mathcal{T} \mid A \geq U \geq A[1] \text{ and } U \geq T\},$$

where A is a silting object and T is a presilting object with $A \geq T$. We can take a non-negative interger ℓ such that $T \geq A[\ell]$. Thus, one visualize such a U as follows:

$$\begin{tikzcd} & & A[1] \arrow[rr, dash] & & A[\ell] \\ A \arrow[r, dash] & U \arrow[ur, dash] \arrow[rr, dash] & & T \arrow[ur, dash] & \end{tikzcd}$$

Now we state the main theorem of this paper.

Theorem 11. *If there exists a minimal element P in the poset $\nabla(A; T)$, then we have $T \geq P[\ell - 1]$.*

We can inductively get silting objects.

Corollary 12. *With the notation as in Definition 10, assume that for any silting object B with $A \geq B \geq T$, the poset $\nabla(B; T)$ admits a minimal element. Then there exists a silting object P in $\mathcal{T}$ satisfying $P \geq T \geq P[1]$.*

Proof. We may assume $\ell \geq 2$. Since we have a minimal element A_1 in $\nabla(A; T)$, by Theorem 11 it is obtained that $A_1 \geq T \geq A_1[\ell - 1]$. As our assumption, we can repeat this argument and have a sequence

$$A \geq A_1 \geq \cdots \geq A_{\ell-1} \geq T \geq A_{\ell-1}[1] \geq \cdots \geq A_1[\ell - 1] \geq A[\ell]$$

of silting objects with A_{i+1} minimal in $\nabla(A_i; T)$ for $0 \leq i \leq \ell - 2$. Thus, we get the desired silting object $P := A_{\ell-1}$. $\square$

From Corollary 12 and [3, Proposition 2.16], we immediately obtain the following corollary.

Corollary 13. *Under the assumption as in Corollary 12, T is a partial silting object.*

3. Silting-discrete triangulated categories

In this section, we discuss silting-discrete triangulated categories.

We begin with recalling the definition of silting-discrete triangulated categories.

Definition 14. A triangulated category $\mathcal{T}$ is said to be *silting-discrete* if there exists a silting object A such that for any $\ell > 0$, the subset $\{T \in \mathsf{silt}\,\mathcal{T} \mid A \geq T \geq A[\ell]\}$ of $\mathsf{silt}\,\mathcal{T}$ is a finite set.

For a silting object A of $\mathcal{T}$, we denote by $\mathsf{2silt}_A\,\mathcal{T}$ the subset of $\mathsf{silt}\,\mathcal{T}$ consisting of all basic silting objects T with $A \geq T \geq A[1]$.

We can easily check the following lemma.

Lemma 15. *Let A be a silting object of $\mathcal{T}$. If $\mathsf{2silt}_A\,\mathcal{T}$ is a finite set, then for every presilting object T of $\mathcal{T}$ with $A \geq T$, the poset $\nabla(A;T)$ has a minimal element.*

We say that $\mathcal{T}$ is *2-silting-finite* if $\mathsf{2silt}_T\,\mathcal{T}$ is a finite set for any silting object T of $\mathcal{T}$.

Now the first main theorem of this section is stated.

Theorem 16. *The following are equivalent:*

(1) *$\mathcal{T}$ is silting-discrete.*
(2) *It is 2-silting-finite.*
(3) *It admits a silting object A such that $\mathsf{2silt}_P\,\mathcal{T}$ is a finite set for any iterated irreducible left mutation P of A.*

Proof. It is obvious that the implications (1)⇒(2)⇒(3) hold.

We show that the implication (3)⇒(1) holds. Let T be a silting object with $A \geq T \geq A[\ell]$ for some $\ell > 0$. Since $\mathsf{2silt}_A\,\mathcal{T}$ is a finite set, we observe that the poset $\nabla(A;T)$ has a minimal element P by Lemma 15. It follows from Theorem 11 that the inequalities $P \geq T \geq P[\ell-1]$ hold, whence one has

$$\{T \in \mathsf{silt}\,\mathcal{T} \mid A \geq T \geq A[\ell]\} \subseteq \bigcup_{P \in \mathsf{2silt}_A\,\mathcal{T}} \{U \in \mathsf{silt}\,\mathcal{T} \mid P \geq U \geq P[\ell-1]\}.$$

By [3, Theorem 3.5], the finiteness of $\mathsf{2silt}_A\,\mathcal{T}$ leads to the conclusion that P can be obtained from A by iterated irreducible left mutation. Therefore, our assumption yields that $\mathsf{2silt}_P\,\mathcal{T}$ is also a finite set. Repeating this argument leads to the assertion. $\square$

We remark that the finiteness of $\mathsf{2silt}_P\,\mathcal{T}$ depends on the choice of silting objects P: For a left mutation P of a silting object A, the set $\mathsf{2silt}_P\,\mathcal{T}$ is not necessarily a finite set even if $\mathsf{2silt}_A\,\mathcal{T}$ is finite (see Example 23).

Finally, we have the second main theorem of this section, which is a direct consequence of Corollary 13.

Theorem 17. *If $\mathcal{T}$ is silting-discrete, then every presilting object is partial silting.*

4. Examples

This section is devoted to giving several examples of silting-discrete triangulated categories.

The first example is an observation from the viewpoint of triangle dimensions in the sense of Rouquier [13]: a triangulated category $\mathcal{T}$ has *triangle dimension 0* ($\dim \mathcal{T} = 0$) if $\mathcal{T} = \mathsf{add}\{M[i] \mid i \in \mathbb{Z}\}$ for some object M of $\mathcal{T}$.

Example 18. If $\dim \mathcal{T} = 0$, then $\mathcal{T}$ is silting-discrete.

In the rest of this paper, let Λ be a finite dimensional algebra over an algebraically closed field k which is indecomposable and basic. We denote by $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} \Lambda)$ the bounded homotopy category of finitely generated projective Λ-modules. Then it is a Krull-Schmidt triangulated category satisfying the property (F).

An algebra Λ is said to be *silting-discrete* if $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} \Lambda)$ is silting-discrete.

We give several examples of silting-discrete algebras. The most easiest example of silting-discrete algebras is the class of local algebras [5].

We characterize silting-discrete hereditary algebras.

Example 19. Assume that Λ is hereditary. Then the following are equivalent:

(1) Λ is silting-discrete;
(2) It is of Dynkin type A, D, E;
(3) $\mathsf{2silt}_\Lambda(\mathsf{K}^{\mathrm{b}}(\operatorname{proj} \Lambda))$ is a finite set.

Proof. We can easily show the implications $(2) \overset{\text{Ex.18}}{\Longrightarrow} (1) \overset{\text{Def.}}{\Longrightarrow} (3) \overset{\text{Easy}}{\Longrightarrow} (2)$. $\square$

A concept of derived-discrete algebras was introduced in [14]: an algebra Λ is said to be *derived-discrete* if for every positive element x of $K_0(A)^{(\mathbb{Z})}$, there exist only finitely many isomorphism classes of indecomposable objects X of the bounded derived category $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ such that $(\underline{\dim} H^i(X))_{i \in \mathbb{Z}} = x$ where $K_0(A), \underline{\dim} M$ and H^i stand for the Grothendieck group of $\operatorname{mod} \Lambda$, the dimension vecter of a module M and the i-th cohomological functor.

Recently, the following result was proved by Broomhead-Pauksztello-Ploog.

Example 20. [7]Any derived-discrete algebra with finite global dimension is silting-discrete.

We know two classes of silting-discrete symmetric algebras.

Example 21. [3, 1] An algebra Λ is silting-discrete if it is either

(1) a representation-finite symmetric algebra or
(2) a Brauer graph algebra of type odd.

The following example was shown by a joint work with Y. Mizuno.

Example 22. [4] The preprojective algebra of Dynkin type $D_{2n} (n \geq 2), E_7, E_8$ is silting-discrete.

We close this paper by giving an example which says that the finiteness of $\mathsf{2silt}_P \mathcal{T}$ depends on the choice of silting objects P.

Example 23. Let Λ be the algebra presented by the quiver

$$\begin{tikzcd}[row sep=small] & 2 \arrow[dr, "x_2"] & \\ 1 \arrow[ur, "x_1"] \arrow[dr, "y_1"'] & & 4 \\ & 3 \arrow[ur, "y_2"'] & \end{tikzcd}$$

with relations $x_1x_2 = 0 = y_1y_2$. Then $\mathsf{2silt}_\Lambda(\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda))$ is a finite set. Now, let $T := \mu^-_{P_2}\mu^-_{P_3}\mu^-_{P_4}(\Lambda)$, which is isomorphic to a tilting module whose endomorphism algebra Γ is the path algebra obtained by the quiver

$$\begin{tikzcd}[row sep=small] & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 4 \\ & 3 \arrow[ur] & \end{tikzcd} .$$

We conclude from Example 19 that $\mathsf{2silt}_\Gamma(\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Gamma))$, hence $\mathsf{2silt}_T(\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda))$, is not a finite set.

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Faculty of Liberal Arts and Sciences
Osaka Prefecture University
1-1 Gakuen-cho, Nakaku, Sakai,
Osaka 599-8531, Japan

E-mail address: `aihara@las.osakafu-u.ac.jp`