GORENSTEINNESS ON THE PUNCTURED SPECTRUM

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Abstract. In this article, we shall characterize torsionfreeness of modules with respect to a semidualizing module in terms of the Serre's condition (S_n). As an application we give a characterization of Cohen-Macaulay rings R such that $R_\mathfrak{p}$ is Gorenstein for all prime ideals $\mathfrak{p}$ with height less than n.

1. Introduction

Auslander and Bridger introduce a notion of *n-torsionfree* as generalization of reflexive [1]. Evans and Griffith give a characterization of n-torsionfree modules [3].

The notion of *n-torsionfree* with respect to a semidualizing module has been introduced by Takahashi [6]. In this article, we study an n-torsionfreeness of modules with respect to a semidualizing module in terms of the Serre's condition (S_n). Recently, Dibaei and Sadeghi [2] give a similar property independently.

Proposition 1. *Let n be a non-negative integer. Assume that R satisfies the conditions (G_{n-1}^C) and (S_n). Then the following statements are equivalent for an R-module M:*

(1) *M is n-C-torsionfree,*
(2) *There exists a exact sequence $0 \to M \to P_C^1 \to \cdots \to P_C^n$ such that each P_C^i is a direct summand of direct sum of finite copies of C and that C-dual sequence ${P_C^n}^\dagger \to \cdots \to {P_C^1}^\dagger \to M^\dagger \to 0$ is exact. Here, $(-)^\dagger = \mathrm{Hom}(-, C)$.*
(3) *M is n-C-syzygy,*
(4) *M satisfies the condition (S_n).*

The following throrem is a main theorem of this article.

Theorem 2. *Let R be a Cohen-Macaulay local ring with a dualizing module ω. For non-negative integer n, the following conditions are equivalent:*

(1) *$C_\mathfrak{p}$ is dualizing $R_\mathfrak{p}$-module for all prime ideal $\mathfrak{p}$ of hight at most n,*
(2) *$(S_{n+1})(R) = \Omega_C^{n+1}(\mathrm{mod} R)$,*
(3) *$\omega \in \Omega_C^{n+1}(\mathrm{mod} R)$.*

This theorem recovers a result of Leuschke and Wiegand [5] which gives a characterization of Cohen-Macaulay rings R such that $R_\mathfrak{p}$ is Gorenstein for all prime ideals $\mathfrak{p}$ with height less than n.

The detailed version of this paper will be submitted for publication elsewhere.

The first author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 26400056.

2. Preliminaries

Throughout the rest of this article, let R be a commutative noetherian ring. All modules are assumed to be finitely generated. In this section, we give some notions and properties.

An R-module C is called *semidualizing* if the homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and if $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. A rank 1 free module R and a dualizing module ω over Cohen-Macaulay local rings are typical examples of semidualizing modules. From now on, we fix a semidualizing module C and put $(-)^\dagger = \mathrm{Hom}_R(-, C)$.

Let $\cdots \to P_1 \xrightarrow{\partial} P_0 \to M \to 0$ be a projective resolution of an R-module M. We define a *C-transpose* module $\mathrm{Tr}_C M$ of M the cokernel of $P_0^\dagger \xrightarrow{\partial^\dagger} P_1^\dagger$. We remark that $\mathrm{Tr}_C M$ is uniquely determined up to direct summands of finite direct sums of copy of C. Note that if C is isomorphic to R then C-transpose coincides with ordinary (Auslander) transpose. An R-module M is called *n-C-torsionfree* if $\mathrm{Ext}^i_R(\mathrm{Tr}_C M, C) = 0$ for all $1 \le i \le n$.

We denote by λ_M the natural map $M \to M^{\dagger\dagger}$. n-C-torsionfreeness has following properties similar to ordinary n-torsionfreeness [1]. One can show this by diagram chasing (c.f. [6]).

Proposition 3. *Let M be an R-module.*

1. *M is 1-C-torsionfree if and only if λ_M is a monomorphism,*
2. *M is 2-C-torsionfree if and only if λ_M is an isomorphism,*
3. *Let $n \ge 3$. M is n-C-torsionfree if and only if λ_M is an isomorphism and if $\mathrm{Ext}^i_R(M^\dagger, C) = 0$ for all $1 \le i \le n-2$.*

An R-module M is called *n-C-syzygy* if there exists an exact sequence $0 \to M \to P_C^1 \to P_C^2 \to \cdots \to P_C^n$ such that each P_C^i is a direct summand of finite direct sums of copy of C. We set $\Omega^n_C(\mathrm{mod}R)$ the class of n-C-syzygy modules.

We say that an R-module M satisfies the Serre's condition (S_n) if $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \ge \min\{n, \dim R_\mathfrak{p}\}$ for each prime ideal $\mathfrak{p}$ of R. We denote by $(S_n)(R)$ the class of modules which satisfies (S_n)-condition.

We say that R satisfies the condition (G_n^C) if injective dimension of $C_\mathfrak{p}$ (as an $R_\mathfrak{p}$-module) is finite for all prime ideal $\mathfrak{p}$ of height at most n. In this case, $R_\mathfrak{p}$ is Cohen-Macaulay local ring with canonical module $C_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$ of height at most n. Note that R satisfies (G_n^R) if and only if $R_\mathfrak{p}$ is Gorenstein local ring for all prime ideal $\mathfrak{p}$ of height at most n.

3. Proofs

In this section, we give a proof of the Proposition 1 and the Theorem 2.

Proof of Proposition 1.

(1) $\Rightarrow$ (2). We prove by induction on n. We assume $n = 1$. Let $f : R^r \to M^\dagger$ be a left add$R$-approximation of M. Then f is epimorphism. Since M is 1-C-torsionfree, λ_M is monomorphism and so is $f^\dagger \lambda_M : M \to M^{\dagger\dagger} \to (R^r)^\dagger = C^r$. One can check $(f^\dagger \lambda_M)^\dagger = f$.

Assume $n \ge 2$. Since M is 1-C-torsionfree, there exists a short exact sequence $0 \to M \to P_C^1 \to N \to 0$ such that the daggar dual sequence $0 \to N^\dagger \to (P_C^1)^\dagger \to M^\dagger \to 0$ is exact. Then we have a following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\lambda_M"'] & P_C^1 \arrow[r] \arrow[d, "\lambda_{P_C^1}"'] & N \arrow[r] \arrow[d, "\lambda_N"'] & 0 \\
0 \arrow[r] & M^{\dagger\dagger} \arrow[r] & {P_C^1}^{\dagger\dagger} \arrow[r] & N^{\dagger\dagger} \arrow[r] & \operatorname{Ext}_R^1(M^\dagger, C) \arrow[r] & 0.
\end{tikzcd}$$

Since $\operatorname{Ext}_R^i(N^\dagger, C) \cong \operatorname{Ext}_R^{i+1}(M^\dagger, C)$ for each $i > 0$, N is $(n-1)$-C-torsionfree. By induction assumption, there exists a exact sequence $0 \to N \to P_C^2 \to \cdots \to P_C^n$ such that the daggar dual sequence $(P_C^n)^\dagger \to \cdots \to (P_C^2)^\dagger \to N^\dagger \to 0$ is exact. Conbining exact sequences, we get an exact sequence $0 \to M \to P_C^1 \to P_C^2 \to \cdots \to P_C^n$ such that the daggar dual sequence $(P_C^n)^\dagger \to \cdots \to (P_C^1)^\dagger \to M^\dagger \to 0$ is exact.

The implication (2) $\Rightarrow$ (3) is obvious by the definition.

Since $\operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} R_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$, C satisfies (S_n). Thus one can check the implication (3) $\Rightarrow$ (4) by using depth lemma.

We prove the implication (4) $\Rightarrow$ (1) by induction n. Assume $n = 1$. Let $\mathfrak{p}$ be an associated prime ideal of M. Since M satisfies the condition (S_1), we have $\dim R_\mathfrak{p} = 0$. Furthermore, the assumption that R satisfies (G_0^C) implies that $C_\mathfrak{p}$ is a dualizing module and that $\operatorname{Hom}_R(M, C)_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) \neq 0$. In particular, $\operatorname{Hom}_R(M, C) \neq 0$.

Let $f_1, f_2, \ldots, f_m$ be a generating system of $\operatorname{Hom}(M, C)$ and put $f = {}^t(f_1, f_2, \ldots, f_m) : M \to C^{\oplus m}$. Suppose that $N = \ker f$ is not zero. Let $\mathfrak{q}$ be an associated prime ideal of N. Since $\mathfrak{q}$ is also an associated prime ideal of M, we have $\dim R_\mathfrak{q} = 0$. Noting that $C_\mathfrak{q}$ is dualizing module over $R_\mathfrak{q}$, we see that $f_\mathfrak{q}$ is a monomorphism. This yields that $N_\mathfrak{q} = 0$. This contradicts that $\mathfrak{q}$ is an associated prime ideal of N. Hence f is a monomorphism.

Since $f^{\dagger\dagger}\lambda_M = \lambda_{C^{\oplus m}} f$ is a monomorphism, we obtain that λ_M is a monomorphism. This means that M is 1-C-torsionfree by Proposition 3.

Assume $n \geq 2$. Since M satisfies the condition (S_1), M is 1-C-torsionfree. In particular, there exists a short exact sequence $0 \to M \to P_C \to N \to 0$ such that the daggar dual sequence $0 \to N^\dagger \to (P_C)^\dagger \to M^\dagger \to 0$ is exact. Then we get a following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\lambda_M"'] & P_C \arrow[r] \arrow[d, "\lambda_{P_C}"'] & N \arrow[r] \arrow[d, "\lambda_N"'] & 0 \\
0 \arrow[r] & M^{\dagger\dagger} \arrow[r] & P_C^{\dagger\dagger} \arrow[r] & N^{\dagger\dagger} \arrow[r] & \operatorname{Ext}_R^1(M^\dagger, C) \arrow[r] & 0.
\end{tikzcd}$$

Note that $\operatorname{Ext}_R^i(N^\dagger, C) \cong \operatorname{Ext}_R^{i+1}(M^\dagger, C)$ for each $i > 0$. It is enough to prove that N satisfies the condition (S_{n-1}). Indeed, if N satisfies the condition (S_{n-1}), N is $(n-1)$-C-torsionfree by induction assumption. Then we can show that M is n-C-torsionfree by the above commutative diagram.

From now on, we shall show that N satisfies the condition (S_{n-1}). Let $\mathfrak{p}$ be a prime ideal. If $\dim R_\mathfrak{p} \geq n$, we have $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min\{n, \dim R_\mathfrak{p}\} = n$. Therefore we obtain $\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} \geq n - 1$ by depth lemma.

Assume $\dim R_\mathfrak{p} \leq n - 1$. Since R satisfies the condition (G_{n-1}^C), $R_\mathfrak{p}$ is Cohen-Macaulay with canonical module $C_\mathfrak{p}$. Inequalities $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min\{n, \dim R_\mathfrak{p}\} = \dim R_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} R_\mathfrak{p}$ gives that $M_\mathfrak{p}$ is a maximal Cohen-Macaulay $R_\mathfrak{p}$-module. Thus so are $(M_\mathfrak{p})^{\dagger_\mathfrak{p}}$, $R_\mathfrak{p}$ and $(N_\mathfrak{p})^{\dagger_\mathfrak{p}}$.

It comes from a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M_{\mathfrak p} \arrow[r] \arrow[d, "\lambda_{M_{\mathfrak p}}"', "\cong"] & (P_C)_{\mathfrak p} \arrow[r] \arrow[d, "\lambda_{(P_C)_{\mathfrak p}}"', "\cong"] & N_{\mathfrak p} \arrow[r] \arrow[d, "\lambda_{N_{\mathfrak p}}"'] & 0 \\ 0 \arrow[r] & (M_{\mathfrak p})^{\dagger_{\mathfrak p}\dagger_{\mathfrak p}} \arrow[r] & (P_C)_{\mathfrak p}^{\dagger_{\mathfrak p}\dagger_{\mathfrak p}} \arrow[r] & (N_{\mathfrak p})^{\dagger_{\mathfrak p}\dagger_{\mathfrak p}} \arrow[r] & 0, \end{tikzcd}$$

we can see that $\lambda_{N_{\mathfrak p}}$ is an isomorphism and that $N_{\mathfrak p} \cong (N_{\mathfrak p})^{\dagger_{\mathfrak p}\dagger_{\mathfrak p}}$ is a maximal Cohen-Macaulay $R_{\mathfrak p}$-module. Therefore we have $\operatorname{depth}_{R_{\mathfrak p}} N_{\mathfrak p} = \dim R_{\mathfrak p} \geq \min\{n-1, \dim R_{\mathfrak p}\}$. Thus N satisfies the condition (S_{n-1}). $\square$

Now, we can prove the Main theorem.

Proof of Theorem 2.

(1) $\Rightarrow$ (2) It is obvious by Proposition 1.

(2) $\Rightarrow$ (3) A dualizing module ω satisfies the Serre's condition (S_n), so we have $\omega \in \Omega_C^n($ mod $R)$.

(3) $\Rightarrow$ (1) There is an exact sequence

$$0 \to \omega \to P_C^1 \to P_C^2 \to \cdots \to P_C^n \to M \to 0$$

such that each P_C^i is a direct summand of direct sum of finite copy of C. For any prime ideal $\mathfrak p$ of height less than n, $(\Omega_C^{n-1} M)_{\mathfrak p}$ is a maximal Cohen-Macaulay $R_{\mathfrak p}$-module. Then the exact sequence $0 \to \omega_{\mathfrak p} \to (P_C^1)_{\mathfrak p} \to (\Omega_C^{n-1} M)_{\mathfrak p} \to 0$ splits. This indicates $\omega_{\mathfrak p} \cong C_{\mathfrak p}$. Thus we have $\operatorname{id}_{R_{\mathfrak p}} C_{\mathfrak p} = \operatorname{id}_{R_{\mathfrak p}} \omega_{\mathfrak p} < \infty$. $\square$

4. EXAMPLE

Jorgensen, Leuschke and Sather-Wagstaff [4] have been determined the structure of rings which admits non-trivial semidualizing modules.

We give a class of Cohen-Macaulay local rings R which have a non-trivial semidualizing module C by using their result. Moreover, $C_{\mathfrak p}$ is a dualizing $R_{\mathfrak p}$-module for all non-maximal prime ideal $\mathfrak p$ of R.

Proposition 4. *Let k be a field and $S = k[[x_1, x_2, \ldots, x_m, y_1, y_2]]$ be a formal power series ring. For $f_1, f_2, \ldots, f_r \in k[[x_1, x_2, \ldots, x_m]]$ and $\ell \geq 2$, we set ideals $I_1 = (f_1, f_2, \ldots, f_r)S$ and $I_2 = (y_1, y_2)^{\ell} S$. Assume that $T = S/I_1$ is a $(d+2)$-dimensional Cohen-Macaulay ring which is not Gorenstein and that T satisfies the condition (G_{n+2}^T). Putting $R = T/I_2$ and $C = \operatorname{Ext}_T^2(R, T)$, then the followings hold:*

1. *R is d-dimensional Cohen-Macaulay ring,*
2. *C is neither R nor dualizing R-module,*
3. *R satisfies the condition (G_n^C).*

Proof. (1) is clear. (2) is comes from [4]. We show (3). Let $\mathfrak p$ be a prime ideal of R with height at most n. Since $P = \mathfrak p S$ is a prime ideal of S with height at most $n+2$, we have that $S_{\mathfrak p} = S_P$ is Gorenstein. Therefore $C_{\mathfrak p} = \operatorname{Ext}_{S_P}^2(R_{\mathfrak p}, S_P)$ is a canonical $R_{\mathfrak p}$-module. $\square$

In the end of this article, we give examples of 1-dimensional Cohen-Macaulay rings R and semidualizing module C such that R satisfies the condition (G_0^C) but not the condition (G_n^R) for all n.

Example 5. Let k be a field and let $S = k[[x_1, x_2, x_3, y_1, y_2]]/(x_2^2 - x_1x_3, x_2x_3, x_3^2)$ be a 3-dimensional Cohen-Macaulay local ring which is not Gorenstein. We set $R = S/(y_1^2, y_1y_2, y_2^2)$ which is a 1-dimensional Cohen-Macaulay local ring. Note that all the prime ideals of R are $\mathfrak{p} = (x_2, x_3, y_1, y_2)$ and $\mathfrak{m} = (x_1, x_2, x_3, y_1, y_2)$. It is easy to see that $S_\mathfrak{p}$ is Gorenstein but $R_\mathfrak{p}$ is not Gorenstein. In particular, R does not satisfy the condition (G_0^R). Putting $C = \mathrm{Ext}_S^2(R, S)$, one can check that C is a semidualizing R-module which is neither R nor canonical module. Since $S_\mathfrak{p}$ is Gorenstein, we can see that $C_\mathfrak{p}$ is a canonical module over $R_\mathfrak{p}$. This yield that R satisfy the condition (G_0^C).

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