TILTED ALGEBRAS AND CONFIGURATIONS OF SELF-INJECTIVE ALGEBRAS OF DYNKIN TYPE

HIDETO ASASHIBA AND KEN NAKASHIMA

Abstract. All algebras are assumed to be basic, connected finite-dimensional algebras over an algebraically closed field. We give an easier way to calculate a bijection from the set of isoclasses of tilted algebras of Dynkin type $\Delta$ to the set of configurations on the translation quiver $Z\Delta$.

Introduction

This work is a generalization of Hironobu Suzuki’s Master thesis [7] that dealt with representation-finite self-injective algebras of type $A$ in a combinatorial way. Throughout this paper $n$ is a positive integer and $k$ is an algebraically closed field, and all algebras considered here are assumed to be basic, connected, finite-dimensional associative $k$-algebras.

Let $\Delta$ be a Dynkin graph of type $A$, $D$, $E$ with the set $\Delta_0 := \{1, \ldots, n\}$ of vertices. We set $C_n$ to be the set of configurations on the translation quiver $Z\Delta$ (see Definition 1.6), and $T_n$ to be the set of isoclasses of tilted algebras of type $\Delta$. Then Bretscher, Läser and Riedtmann have given a bijection $c: T_n \rightarrow C_n$ in [1]. But the map $c$ is not given in a direct way, it needs a long computation of a function on $Z\Delta$. In this paper we will give an easier way to calculate the map $c$ by giving a map sending each projective $A$-module over a tilted algebra $A$ in $T_n$ to an element of the configuration $c(A)$.

We fix an orientation of each Dynkin graph $\Delta$ to have a quiver $\vec{\Delta}$ as in the following table.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$A_n \ (n \geq 1)$</th>
<th>$D_n \ (n \geq 4)$</th>
<th>$E_n \ (n = 6, 7, 8)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{\Delta}$</td>
<td>$\circ \circ \cdots \circ$</td>
<td>$\circ \circ \cdots \circ$</td>
<td>$\circ \circ \cdots \circ$</td>
</tr>
<tr>
<td>$m_{\Delta}$</td>
<td>$n$</td>
<td>$2n - 3$</td>
<td>$11, 17, 29$, respectively</td>
</tr>
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</table>

This orientation of $\Delta$ gives us a coordinate system on the set $(Z\Delta)_0 := Z \times \Delta_0$ of vertices of $Z\Delta := Z\vec{\Delta}$ as presented in [1, fig. 1] and in [3, Fig. 13], and by definition the full subquiver $S$ of $Z\Delta$ consisting of $\{(0, i) \mid i \in \Delta_0\}$ is isomorphic to $\vec{\Delta}$.

Let $A$ be a tilted algebra of type $\Delta$. Then by identify $A$ with the $(0,0)$-entry of the repetitive category $\hat{A}$, the vertex set of AR-quiver $\Gamma_A$ is embedded into the vertex set of the stable AR-quiver $s\Gamma_A$ ($\cong Z\Delta$) of $\hat{A}$. Further the configuration $C := c(A)$ of $Z\Delta$ computed in [1] is given by the vertices of $Z\Delta$ corresponding to radicals of projective $A$-modules.

The detailed version of this paper will be submitted for publication elsewhere.
indecomposable \(\hat{A}\)-modules. Note that the configuration \(C\) has a period \(m_\Delta\) listed in the table, thus \(\mathcal{C} = \tau^{m_\Delta Z}\mathcal{F}\) for some subset \(\mathcal{F}\) of \(\mathcal{C}\). By \(\mathcal{P} = \{(p(i), i) \mid i \in \Delta_0\}\) we denote the set of images of the projective vertices of \(\Gamma_A\) in \(Z\Delta\) and set
\[
\mathbb{N}\mathcal{P} := \{(m, i) \in (Z\Delta)_0 \mid p(i) \leq m, i \in \Delta_0\}.
\]
Since the mesh category \(k(Z\Delta)\) is a Frobenius category, it has the Nakayama permutation \(\hat{\nu}\) on \((Z\Delta)_0\) that is defined by the isomorphism
\[
k(Z\Delta)(x, -) \cong \text{Hom}_k(k(Z\Delta)(-, \hat{\nu}x), k)
\]
for all \(x \in (Z\Delta)_0\). The explicit formula of \(\hat{\nu}\) is given in [3, pp. 48–50]. (Note that it should be corrected as \(\hat{\nu}(p, q) = (p + q + 2, 6 - q)\) if \(q \leq 5\) when \(\Delta = E_6\) as pointed out in [1, 1.1]). In this paper we will define a map \(\nu' : \mathcal{P} \to \mathbb{N}\mathcal{P}\) using the supports of starting functions \(\text{dim}_k k(Z\Delta)(x, -) : \mathbb{N}\mathcal{P} \to \mathbb{Z}\) for \(x \in \mathbb{N}\mathcal{P}\) (cf. [3, Fig. 15]). Then \(\nu'\) has the following property.

**Lemma 0.1.** Let \(x \in \mathcal{P}\) and \(P\) be the projective indecomposable \(A\)-module corresponding to \(x\). Then \(\nu'x\) corresponds to the simple module \(\text{top} P\).

In this paper, we make use of modules over the algebra
\[
B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}
\]
to compute an \(\mathcal{F}\) above (the configuration (see Definition 3.9) of \(B\) gives \(\mathcal{F}\).) We will define a map \(\nu := \nu_B\) from the set of isoclasses of simple \(A\)-modules to \(\mathcal{C}\), which coincides with the restriction of the Nakayama permutation \(\hat{\nu}\) if \(A\) is hereditary.

**Lemma 0.2.** Assume that a vertex \(x \in Z\Delta\) corresponds to a simple \(A\)-module \(S\) and let \(Q\) be the injective hull of \(S\) over \(\hat{A}\). Then \(\nu(x)\) corresponds to \(\text{rad} Q\), and hence \(\nu(x) \in \mathcal{C}\).

Combining the lemmas above we obtain the following.

**Proposition 0.3.** If \(x \in \mathcal{P}\), then \(\nu(\nu'x) \in \mathcal{C}\).

This leads us to the following definition.

**Definition 0.4.** We define a map \(c_A : \mathcal{P} \to \mathcal{C}\) by \(c_A(x) := \nu(\nu'x)\) for all \(x \in \mathcal{P}\).

The image of the map \(c_A\) gives us an \(\mathcal{F}\) above, namely we have the following.

**Theorem 0.5.** The map \(c_A\) is an injection, and we have \(c(A) = \tau^{m_\Delta Z}\text{Im} c_A\).

**Corollary 0.6.** If \(A\) is hereditary, then \(c_A = \hat{\nu}\nu'\) and we have \(c(A) = \tau^{m_\Delta Z}\text{Im} \hat{\nu}\nu'\).

Section 1 is devoted to preparations. In Section 2 we will give the complete list of indecomposable projectives and indecomposable injectives over the triangular matrix algebra \(B\). In Section 3 we state our main results.
1. Preliminaries

1.1. Algebras and categories. A category $\mathcal{C}$ is called a $k$-category if the morphism sets $\mathcal{C}(x, y)$ are $k$-vector spaces, and the compositions $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ are $k$-bilinear for all $x, y, z \in \mathcal{C}_0$ ($\mathcal{C}_0$ is the class of objects of $\mathcal{C}$, we sometimes write $x \in \mathcal{C}$ for $x \in \mathcal{C}_0$). In the sequel all categories are assumed to be $k$-categories unless otherwise stated.

To construct repetitive categories and to make use of a covering theory we need to extend the range of considerations from algebras to categories. First we regard an algebra as a special type of categories by constructing a category $\text{cat} \ A$ from an algebra $A$ as follows.

1. We fix a decomposition $1 = e_1 + \cdots + e_n$ of the identity element 1 of $A$ as a sum of orthogonal primitive idempotents.
2. We set the object class of $\text{cat} \ A$ to be the set $\{e_1, \ldots, e_n\}$.
3. For each pair $(e_i, e_j)$ of objects, we set $(\text{cat} \ A)(e_i, e_j) := e_j A e_i$.
4. We define the composition of $\text{cat} \ A$ by the multiplication of $A$.

The obtained category $\text{cat} \ A$ is uniquely determined up to isomorphisms not depending on the decomposition of 1. The category $C = \text{cat} \ A$ is a small category having the following three properties.

1. Distinct objects are not isomorphic.
2. For each object $x$ of $\mathcal{C}$ the algebra $\mathcal{C}(x, x)$ is local.
3. For each pair $(x, y)$ of objects of $\mathcal{C}$ the morphism space $\mathcal{C}(x, y)$ is finite-dimensional.

A small category with these three properties is called a spectroid and its objects are sometimes called points. A spectroid with only a finite number of points is called finite. Conversely we can construct a matrix algebra from a finite spectroid $C$ as follows.

\[ \text{alg} C := \{(m_{yx})_{x, y \in C} \mid m_{yx} \in \mathcal{C}(x, y), \forall x, y \in C\}. \]

Here we have $\text{alg} \ \text{cat} \ A \cong A$, $\text{cat} \ \text{alg} \ C \cong C$. Therefore we can identify the class of algebras and the class of finite spectroids by using $\text{cat}$ and $\text{alg}$.

A spectroid $C$ is called locally bounded if for each point $x$ the set $\{y \in C \mid C(x, y) \neq 0 \text{ or } C(y, x) \neq 0\}$ is a finite set. Of course algebras (= finite spectroids) are locally bounded. In the range of locally bounded spectroids we can freely construct repetitive categories or consider coverings.

Remark 1.1. We can construct the “path-category” $kQ$ from a locally finite quiver $Q$ by the same way as in the definition of the path-algebra. The only different part is in the following definition of compositions: For paths $\mu, \nu$ with $s(\mu) \neq t(\nu)$, it was defined as $\mu \nu = 0$ in the path-algebra, but in contrast the composition $\mu \nu$ is not defined in the path-category.

A locally bounded spectroid $C$ is also presented as the form $kQ/I$ for some locally finite quiver $Q$ and for some ideal $I$ of the path-category $kQ$ such that $I$ is included in the ideal

\[ I = \{m_{yx} \mid m_{yx} \neq 0, x, y \in C\}. \]

Here $s(\mu)$ and $t(\nu)$ stand for the source of $\mu$ and the target of $\nu$ and compositions are written from the right to the left.

\[ ^1 \text{a terminology used in [4]} \]

\[ ^2 \text{Here } s(\mu) \text{ and } t(\nu) \text{ stand for the source of } \mu \text{ and the target of } \nu \text{ and compositions are written from the right to the left.} \]
of \( kQ \) generated by the set of paths of length 2. Here the quiver \( Q \) is uniquely determined by \( C \) up to isomorphisms. This \( Q \) is called the quiver of \( C \).

A (right) module over a spectroid \( C \) is a contravariant functor \( C \to \text{Mod} k \). From a usual (right) module over an algebra \( A \) we can construct a contravariant functor \( \text{cat} A \to \text{Mod} k \) by the correspondence \( e_i \to Me_i \) for each point \( e_i \) in cat \( A \), and \( f \to (f : Me_j \to Me_i) \) for each \( f \in e_j A e_i = (\text{cat} A)(e_i, e_j) \). Conversely, from a contravariant functor \( F : \text{cat} A \to \text{Mod} k \) we can construct an \( A \)-module \( \bigoplus_{i=1}^n F(e_i) \); and these constructions are inverse to each other. In this way we can identify \( A \)-modules and modules over cat \( A \).

The set of projective indecomposable modules over a spectroid \( C \) is given by \( \{C(\nu, x)\}_{x \in C} \) up to isomorphism, and finitely generated projective \( C \)-modules are nothing but finite direct sums of these. Using this we can define finitely generated modules or finitely presented modules over \( C \) by the same way as those over algebras.

The dimension of a \( C \)-module \( M \) is defined to be the dimension of \( \bigoplus_{x \in C} M(x) \). When \( C \) is locally bounded, a \( C \)-module is finitely presented if and only if it is finitely generated if and only if it is finite-dimensional.

### 1.2. Repetitive category.

**Definition 1.2.** Let \( A \) be an algebra with a basic set of local idempotents \( \{e_1, \ldots, e_n\} \).

1. The repetitive category \( \hat{A} \) of \( A \) is a spectroid defined as follows.
   - **Objects:** \( \hat{A}_0 := \{x[i] := (x, i) \mid x \in \{e_1, \ldots, e_n\}, i \in \mathbb{Z}\} \).
   - **Morphisms:** Let \( x[i], y[j] \in \hat{A}_0 \). Then we set
     \[
     \hat{A}(x[i], y[j]) := \begin{cases} 
     \{f[i] := (f, i) \mid f \in A(x, y)\} & (j = i) \\
     \{\varphi[i] := (\varphi, i) \mid \varphi \in \text{DA}(y, x)\} & (j = i + 1) \\
     0 & \text{otherwise},
     \end{cases}
     \]
   - **Compositions:** The composition \( \hat{A}(y[i], z[k]) \times \hat{A}(x[i], y[j]) \to \hat{A}(x[i], z[k]) \) is defined as follows.
     - (i) If \( j = i, k = j \), then we use the composition of \( A \):
       \[ A(y, z) \times A(x, y) \to A(x, z). \]
     - (ii) If \( j = i, k = j + 1 \), then we use the right \( A \)-module structure of \( \text{DA}(-, ?) \):
       \[ DA(z, y) \times A(x, y) \to DA(z, x). \]
     - (iii) If \( j = i + 1, k = j \), then we use the left \( A \)-module structure of \( \text{DA}(-, ?) \):
       \[ A(y, z) \times DA(y, x) \to DA(z, x). \]
     - (iv) Otherwise the composition is zero.

2. For each \( i \in \mathbb{Z} \), we denote by \( A[i] \) the full subcategory of \( \hat{A} \) whose object class is \( \{x[i] \mid x \in \{e_1, \ldots, e_n\}\} \).

3. We define the Nakayama automorphism \( \nu_A \) of \( \hat{A} \) as follows: for each \( i \in \mathbb{Z}, x, y \in A, f \in A(x, y) \) and \( \phi \in \text{DA}(y, x) \),
   \[
   \nu_A(x[i]) := x[i+1], \nu_A(f[i]) := f[i+1], \nu_A(\varphi[i]) := \varphi[i+1].
   \]
Remark 1.3. (1) If a spectroid $A$ is locally bounded, then so is $\hat{A}$.

(2) When $A$ is an algebra, the set of all $\mathbb{Z} \times \mathbb{Z}$-matrices with only a finite number of nonzero entries whose diagonal entries belong to $A$, $(i + 1, i)$ entries belong to $DA$ for all $i \in \mathbb{Z}$, and other entries are zero forms an infinite-dimensional algebra without identity element, which is called the repetitive algebra of $A$. The repetitive category $\hat{A}$ is nothing but this repetitive algebra regarded as a spectroid in a similar way. This is not an algebra (= a finite spectroid) any more, but a locally bounded spectroid.

Definition 1.4 (Gabriel [2]). Let $C$ be a locally bounded spectroid with a free$^3$ action of a group $G$. Then we define the orbit category $C/G$ of $C$ by $G$ as follows.

(1) The objects of $C/G$ are the $G$-orbits $Gx$ of objects $x$ of $C$.
(2) For each pair $Gx, Gy$ of objects of $C/G$ we set

$$(C/G)(Gx, Gy) := \left\{ (bfa)_{a,b} \in \prod_{(a,b) \in Gx \times Gy} C(a,b) \mid gbfa = g(bfa), \text{ for all } g \in G \right\}.$$

(3) The composition is defined by

$$(dhc)_{c,d} \cdot (bfa)_{a,b} := \left( \sum_{b \in Gy} dhb \cdot bfa \right)_{a,d}$$

for all $(bfa)_{a,b} \in (C/G)(Gx, Gy), (dhc)_{c,d} \in (C/G)(Gy, Gz)$. Note that each entry of the right hand side is a finite sum because $C$ is locally bounded.

A functor $F: C \to C'$ is called a Galois covering with group $G$ if it is isomorphic to the canonical functor $\pi: C \to C/G$, namely if there exists an isomorphism $H: C/G \to C'$ such that $F = H\pi$.

Remark 1.5. If $A$ is an algebra and a group $G$ acts freely on the category $\hat{A}$, then $\hat{A}/G$ turns out to be a self-injective spectroid. In particular, when $\hat{A}/G$ is a finite spectroid, it becomes a self-injective algebra. In this way we can construct a great number of self-injective algebras.

Definition 1.6. From a quiver $Q$ we can construct a translation quiver $ZQ$ as follows.

• $(ZQ)_0 := \mathbb{Z} \times Q_0$,
• $(ZQ)_1 := \mathbb{Z} \times Q_1 \cup \{(i, \alpha') \mid i \in \mathbb{Z}, \alpha \in Q_1\}$,
• We define the sources and the targets of arrows by

$$(i, \alpha): (i, s(\alpha)) \to (i, t(\alpha)), (i, \alpha'): (i, t(\alpha)) \to (i+1, s(\alpha))$$

for all $(i, \alpha) \in \mathbb{Z} \times Q_1$.
• We take the bijection $\tau: (ZQ)_0 \to (ZQ)_0, (i, x) \mapsto (i - 1, x)$ as the translation.

In addition, we can define a polarization by $(i + 1, \alpha) \mapsto (i, \alpha'), (i, \alpha') \mapsto (i, \alpha)$. Note that by construction the translation quiver $ZQ$ does not have any projective or injective vertices.

$^3g \neq x \in Q_0$ implies $gx \neq x$
Proposition 2.2. \( Z \) does not depend on orientations of \( \Delta \), therefore we set \( Z \).

Remark 1.7. When \( Q \) is a Dynkin quiver with the underlying graph \( \Delta \), the isoclass of \( ZQ \) does not depend on orientations of \( \Delta \), therefore we set \( Z\Delta := ZQ \).

2. TRIANGULAR MATRIX ALGEBRAS

Definition 2.1. Let \( R \) and \( S \) be algebras, \( M \) be an \( S-R \)-bimodule. We define a category \( \mathcal{C} = \mathcal{C}(R, S, M) \) as follows.

- **Objects:** \( \mathcal{C}_0 := \{ (X, Y, f) \mid X_R \in \text{mod} R, Y_S \in \text{mod} S, f \in \text{Hom}_{A}(Y \otimes_{S} M, X) \} \).

- **Morphisms:** Let \((X, Y, f), (X', Y', f') \in \mathcal{C}_0\). Then we set
  \[
  \mathcal{C}((X, Y, f), (X', Y', f')) := \left\{ (\phi_0, \phi_1) \in \text{Hom}_{R}(X, X') \times \text{Hom}_{S}(Y, Y') \mid \begin{array}{c}
Y \otimes_{S} M \xrightarrow{f} X \\
\phi_1 \otimes 1_M \rightarrow \phi_0
\end{array} \right\}.
  \]

- **Compositions:** Let \((X, Y, f), (X', Y', f'), (X'', Y'', f'') \in \mathcal{C}_0\) and let
  \[(\phi_0, \phi_1) \in \mathcal{C}((X, Y, f), (X', Y', f')), (\phi'_0, \phi'_1) \in \mathcal{C}((X', Y', f'), (X'', Y'', f'')).\]
  Then we set
  \[(\phi'_0, \phi'_1)(\phi_0, \phi_1) := (\phi'_0 \phi_0, \phi'_1 \phi_1) \in \mathcal{C}((X, Y, f), (X'', Y'', f'')).\]

Then the following is well known.

Proposition 2.2. Let \( R \) and \( S \) be algebras, \( M \) be an \( S-R \)-bimodule. Then
  \[
  \text{mod} \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \simeq \mathcal{C}(R, S, M).
  \]

Recall that an equivalence \( F : \text{mod} \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \rightarrow \mathcal{C}(R, S, M) \) is given as follows.

- **Objects:** For each \( L \in (\text{mod} T)_0 \),
  \[
  F(L) := (L \varepsilon_1, L \varepsilon_2, f_L),
  \]
  where \( \varepsilon_1 := \begin{bmatrix} 1_R & 0 \\ 0 & 0 \end{bmatrix}, \varepsilon_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1_S \end{bmatrix} \) and \( f_L : L \varepsilon_2 \otimes_S M \rightarrow L \varepsilon_1 \) is defined by
  \[
  f_L(l \varepsilon_2 \otimes m) := \begin{bmatrix} 0 \\ m \end{bmatrix} \]
  for all \( l \in L \) and \( m \in M \).

- **Morphisms:** For each \( \alpha \in \text{Hom}_T(L, L') \),
  \[
  F(\alpha) := (\alpha \big|_{L \varepsilon_1}, \alpha \big|_{L \varepsilon_2}).
  \]
Let $A$ be a tilted algebra of type $\Delta$, and set $B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$, $C := C(A, A, DA)$.

Then we have $\text{mod } B \simeq C$ by Proposition 2.2. By this equivalence, we identify $\text{mod } B$ with $C$.

Let $\{e_1, \ldots, e_n\}$ be a complete set of orthogonal local idempotents of $A$. Then as is easily seen $\{e_1, \ldots, e_n, e_1^{[0]}, \ldots, e_1^{[1]}, \ldots, e_n^{[0]}, e_n^{[1]}, \ldots, e_n^{[1]}\}$ is a complete set of orthogonal local idempotents of $B$, and $\{e_1^{[0]}B, \ldots, e_n^{[0]}B, e_1^{[1]}B, \ldots, e_n^{[1]}B\}$ is a complete set of isoclasses of projective indecomposable $B$-modules. The following is immediate.

**Proposition 2.3.** For each $i = 1, \ldots, n$, we have
\[
F(e_i^{[0]}B) \cong (e_iA, 0, 0),
\]
\[
F(e_i^{[1]}B) \cong (e_i(DA), e_iA, \text{can}).
\]

In addition $\{D(Be_1^{[0]}), \ldots, D(Be_n^{[0]}), D(Be_1^{[1]}), \ldots, D(Be_n^{[1]})\}$ is a complete set of isoclasses of injective indecomposable $B$-modules. The following two statements are obvious.

**Lemma 2.4.** For each $i = 1, \ldots, n$, we have

(1) $D \begin{bmatrix} Ae_i \\ (DA)e_i \end{bmatrix} \cong \begin{bmatrix} 0 \\ D(Ae_i) \end{bmatrix}$, and

(2) $D \begin{bmatrix} 0 & 0 \\ 0 & Ae_i \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ 0 & D(Ae_i) \end{bmatrix}$.

**Proposition 2.5.** For each $i = 1, \ldots, n$, we have
\[
F(D(Be_i^{[0]})) \cong (e_i(DA), e_iA, \text{can}) \cong e_i^{[1]}B,
\]
\[
F(D(Be_i^{[1]})) \cong (0, e_i(DA), 0).
\]

3. Configurations

**Definition 3.1.** Let $\Lambda$ be a standard representation-finite self-injective algebra. Then we set
\[
C_{\Lambda} := \{ [\text{rad } P] \in \Gamma_{\Lambda} \mid P : \text{projective(-injective) } \Lambda\text{-module} \},
\]
which is called a configuration of $\Lambda$.

**Definition 3.2.** Let $\Gamma$ be a stable translation quiver, and $C$ be a subset of $\Gamma_0$. Then we define a translation quiver $\Gamma_C$ by
\[
(\Gamma_C)_0 := \Gamma_0 \cup \{ p_x \mid x \in C \},
\]
\[
(\Gamma_C)_1 := \Gamma_1 \cup \{ x \rightarrow p_x, \ p_x \rightarrow \tau^{-1}x \},
\]
where the translation of $\Gamma_C$ is the same as that of $\Gamma$. In particular, $p_x$ are projective-injective\(^4\) vertices for all $x \in C$.

\(^4\)The word “projective-injective” stands for projective and injective.
Remark 3.3. The quiver of \( \text{mod} \Lambda \) is the full subquiver \( s\Gamma_\Lambda \) of \( \Gamma_\Lambda \) with
\[
(s\Gamma_\Lambda)_0 := \{ x \mid x \text{ is a stable vertex of } \Gamma_\Lambda \}
\]
(namely \( s\Gamma_\Lambda \) is obtained from \( \Gamma_\Lambda \) by removing all projective vertices), which is a stable translation quiver. Then it holds that \( \mathcal{C}_\Lambda \subseteq (s\Gamma_\Lambda)_0 \), and we have
\[
(s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong \Gamma_\Lambda. \tag{3.1}
\]

Theorem 3.4. Let \( \Lambda \) be a standard representation-finite self-injective algebra and \( \Delta \) the Dynkin type of \( \Lambda \). Then the following hold.

1. (Waschb"{u}sch [5, 8]) There exist a tilted algebra \( A \) of type \( \Delta \) and an automorphism \( \phi \) of \( \hat{A} \) without fixed vertices such that \( \Lambda \cong \hat{A}/\langle \phi \rangle \).
2. (Riedtmann [6]) There is an isomorphism \( f : s\Gamma_{\hat{A}} \rightarrow \mathbb{Z}\Delta \). Denote also by \( \phi \) the automorphism of \( s\Gamma_{\hat{A}} \) induced from \( \phi \) canonically, and define an automorphism \( \phi' \) of \( \mathbb{Z}\Delta \) by the following commutative diagram:

\[
\begin{array}{ccc}
s\Gamma_{\hat{A}} & \xrightarrow{f} & \mathbb{Z}\Delta \\
\phi \downarrow & & \phi' \downarrow \\
s\Gamma_{\hat{A}} & \xrightarrow{f} & \mathbb{Z}\Delta.
\end{array}
\]

Then we have \( s\Gamma_\Lambda \cong s\Gamma_{\hat{A}}/\langle \phi \rangle \cong \mathbb{Z}\Delta/\langle \phi' \rangle \).

By the formula (3.1) to compute \( \Gamma_\Lambda \), it is enough to solve the following problem.

Problem 1. Let \( \Lambda \) be a standard representation-finite self-injective algebra, which has the form \( \hat{A}/\langle \phi \rangle \) for some tilted algebra \( A \) of Dynkin type and an automorphism \( \phi \) of \( \hat{A} \) by Theorem 3.4. Then compute \( \mathcal{C}_\Lambda \) from \( A \).

Remark 3.5. Let \( f' : s\Gamma_\Lambda \rightarrow \mathbb{Z}\Delta/\langle \phi' \rangle \) be an isomorphism, and set \( C := f'(\mathcal{C}_\Lambda) \). Then we have
\[
\Gamma_\Lambda \cong (s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong (\mathbb{Z}\Delta/\langle \phi' \rangle)_C.
\]
Thus we can compute \( \Gamma_\Lambda \) by Theorem 3.4(2) if we can obtain the set \( C \).

On the other hand, the following holds by [2, Theorem 3.6].

Theorem 3.6 (Gabriel). Let \( R \) be a locally representation-finite and locally bounded \( k \)-category, and \( G \) be a group consisting of automorphisms of \( R \) that acts freely on \( R \). Then the AR-quiver \( \Gamma_R \) of \( R \) has an induced \( G \)-action, and we have \( \Gamma_R/G \cong \Gamma_{R/G} \).

Definition 3.7. Let \( A \) be a tilted algebra of Dynkin type. Then we set
\[
\mathcal{C}_A := \{ \text{rad } P \in \Gamma_{\hat{A}} \mid P : \text{projective(-injective) } \hat{A}\text{-module} \},
\]
which is called the configuration of \( \hat{A} \).

Corollary 3.8. Let \( A \) be a tilted algebra of Dynkin type, and \( \phi \) be an automorphism of \( \hat{A} \) without fixed vertices. Then we have
\[
\mathcal{C}_A/\langle \phi \rangle \cong \mathcal{C}_\Lambda.
\]

Therefore to solve Problem 1, it is enough to consider the following.
**Problem 2.** In the same setting as in Problem 1, compute $C\hat{A}$ from $A$.

Throughout the rest of their section

(1) let $A$ be a tilted algebra of Dynkin type $\Delta$, and set
(2) $B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$.

By (1), $\Gamma_A$ has a section $S$ whose underlying graph is isomorphic to $\Delta$.

**Definition 3.9.** We call the following set the *configuration* of $B$:
$$C_B := \{ \text{rad } P \in \Gamma_B \mid P : \text{projective-injective } B\text{-module} \}.$$  

3.1. **Relationship among $\hat{A}$, $B$ and $A$.** We set as follows:

$$I_{0,1} = \langle e_i^j \mid i \in \mathbb{Z} \setminus \{0, 1\}, j \in \{1, \ldots, n\} \rangle,$$

$$I_0 = \langle e_i^j \mid i \in \mathbb{Z} \setminus \{0\}, j \in \{1, \ldots, n\} \rangle,$$

$$I_1 = \langle e_i^j \mid i \in \mathbb{Z} \setminus \{1\}, j \in \{1, \ldots, n\} \rangle.$$  

Then $\hat{A}/I_{0,1} \cong B$, $\hat{A}/I_0 \cong A^{[0]}(\cong A)$ and $\hat{A}/I_1 \cong A^{[1]}(\cong A)$. We also have

$$B/ \begin{bmatrix} 0 & 0 \\ DA & 0 \end{bmatrix} \cong A^{[0]} \times A^{[1]}.$$  

We have the following surjective algebra homomorphisms

$$\hat{A} \xymatrix{ \ar[r] & B \ar[r] & A^{[0]} \times A^{[1]} \ar[r] & A^{[1]}},$$

which induce the following embeddings of categories

$$\text{mod } A^{[0]} \xymatrix{ \ar[r]^-{\sigma} & \text{mod } B \ar[r] & \text{mod } A^{[1]}.}$$
We regard $\text{mod } A \subseteq \text{mod } B$ by the embedding $\text{mod } A = \text{mod } A^{[0]} \hookrightarrow \text{mod } B$. The embeddings above give us the following embeddings of vertex sets of AR-quivers:

$$\begin{array}{c}
(\Gamma^{[0]}_A)_0 = (\Gamma_A)_0 \\
(\Gamma^{[0]}_B)_0
\end{array}$$

We define an ideal $k(\mathbb{Z}\Delta)^+$ of the mesh category $k(\mathbb{Z}\Delta)$ as follows:

$$k(\mathbb{Z}\Delta)^+ := \bigoplus (\mathbb{Z}\Delta)_1 + I_{\mathbb{Z}\Delta}.$$  

Then the values of $m_\Delta := \min \{m \in \mathbb{N} \mid (k(\mathbb{Z}\Delta)^+)_i = 0, \forall i \geq m\}$ are known as follows:

$$m_\Delta = \begin{cases} 
  n & (\Delta = A_n) \\
  2n - 3 & (\Delta = D_n) \\
  11 & (\Delta = E_6) \\
  17 & (\Delta = E_7) \\
  29 & (\Delta = E_8)
\end{cases}$$

We see the following by [1].

**Proposition 3.10.** Let $i = 0, 1$.

1. The full subquiver $S^{[i]}_B$ of $\Gamma_B$ with the vertex set $\sigma_i(S_0)$ forms a section of $\sigma \Gamma_B$.

2. The full subquiver $S^{[i]}_A$ of $\Gamma_A$ with the vertex set $\sigma_i(S_0)$ forms a section of $\sigma \Gamma_A$.

**Remark 3.11.** A quiver $Q$ without oriented cycles will be regarded as a poset by the order defined as follows:

For each $x, y \in Q_0$, $x \preceq y$ if and only if there is a path in $Q$ from $x$ to $y$.

**Definition 3.12.**

1. We set $\mathcal{H}_B$ to be the full subquiver of $\Gamma_B$ defined by the set

$$(\mathcal{H}_B)_0 := \{ x \in (\Gamma_B)_0 \mid a \preceq x \preceq b \text{ for some } a \in (S^{[0]}_B)_0, b \in (S^{[1]}_B)_0 \}$$

of vertices.

2. We set $\mathcal{H}^{[0,1]}_A$ to be the full subquiver of $\Gamma_A$ defined by the set

$$(\mathcal{H}^{[0,1]}_A)_0 := \{ x \in (\Gamma_A)_0 \mid a \preceq x \preceq b \text{ for some } a \in (S^{[0]}_A)_0, b \in (S^{[1]}_A)_0 \}$$

of vertices.

**Proposition 3.13.**

1. The map $\sigma : (\Gamma_B)_0 \rightarrow (\Gamma_A)_0$ is uniquely extended to a quiver isomorphism $\mathcal{H}_B \rightarrow \mathcal{H}^{[0,1]}_A$.

2. We have $S^{[1]}_A = \tau^{-m_\Delta} S^{[0]}_A$. We set $S^{[n]}_A := \tau^{-nm_\Delta} S^{[0]}_A$ for all $n \in \mathbb{Z}$.
(3) Set $\mathcal{H}_A^{[n,n+1]} := \tau^{-nm}(\mathcal{H}_A^{[0,1]})$ for all $n \in \mathbb{Z}$. Then for each $i = 0, 1$

$$(\Gamma_A)_i = \bigcup_{n \in \mathbb{Z}} (\mathcal{H}_A^{[n,n+1]})_i$$

$$(S_A^{[n+1]})_i = (\mathcal{H}_A^{[n,n+1]})_i \cap (\mathcal{H}_A^{[n+1,n+2]})_i$$

Roughly speaking, $\Gamma_A$ is obtained by connecting infinite copies of $\mathcal{H}_B$ on both sides.

**Example 3.14.** Let $A$ be the path algebra of the following quiver.

$$
\begin{array}{c}
1^{[0]} \longrightarrow 2^{[0]} \longrightarrow 3^{[0]} \\
\end{array}
$$

Then $\Gamma_A$ is given as follows (double arrows present a section).

Therefore $A$ is a tilted algebra of type $A_3$. Moreover $B = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} = \begin{bmatrix} A^{[0]} & 0 \\ (DA)^{[0]} & A^{[1]} \end{bmatrix}$ is an algebra given by following quiver with relations.

$$
\begin{array}{c}
1^{[0]} \longrightarrow 2^{[0]} \longrightarrow 3^{[0]} \\
\end{array}
$$
Then $\Gamma_B$ is given as follows (elements of $C_B$ are encircled).

In the above, $H_B$ is given by the full subquiver consisting of vertices between the left section and the right section. $A$ is given by the following quiver with relations.
Then $\Gamma_A$ is follows (each element of $C_A$ is encircled by a broken or solid line, in particular solid circles present elements of $C_B$). In this case we have $m_\Delta = 3$.

The following is immediate from Proposition 3.13.

**Corollary 3.15.** We have $C_A = \tau^{m_\Delta \sigma}(C_B)$.

By this corollary, Problem 2 is reduced to the following.

**Problem 3.** Let $A$ be a tilted algebra of Dynkin type $\Delta$, and $B$ as above. Then give the configuration $C_B$ from $A$.

The purpose of this section is to solve Problem 3.

**Definition 3.16.** (1) We define an ideal $\mathcal{P}I$ of mod $B$ as follows and set $\tilde{\text{mod}} B := (\text{mod} B)/\mathcal{P}I$. For each $X, Y \in \text{mod} B_0$

$$\mathcal{P}I(X, Y) := \{ f \in \text{Hom}_B(X, Y) \mid f \text{ factors through a projective-injective } B\text{-module} \}$$

Let $(\tilde{\cdot}) : \text{mod} B \to \tilde{\text{mod}} B$ be the canonical functor and set

$$\tilde{\text{Hom}}_B(\tilde{X}, \tilde{Y}) := (\tilde{\text{mod}} B)(\tilde{X}, \tilde{Y})$$

for all $X, Y \in \text{mod} B$. Thus $\tilde{X} = X$ for all $X \in (\text{mod} B)_0$ and $\tilde{f} = f + \mathcal{P}I(X, Y)$ for all $f \in \text{Hom}_B(X, Y)$.

(2) We denote by $\text{mod}_{\mathcal{P}I} B$ the full subcategory of mod $B$ consisting of $B$-modules without projective-injective direct summands.

(3) Let $X$ and $Y \in \text{mod}_{\mathcal{P}I} B$. Then it is well known that $\mathcal{P}I(X, Y) \subseteq \text{rad}_B(X, Y)$. We set $\text{rad}_B(X, Y) := \text{rad}_B(X, Y)/\mathcal{P}I(X, Y)$.

**Definition 3.17.** For AR-quiver $\Gamma_B$ of $B$, we define the full translation subquiver $\tilde{\Gamma}_B$ as follows.

$$(\tilde{\Gamma}_B)_0 := \{ X \in (\Gamma_B)_0 \mid X \text{ is not projective-injective} \}$$

Moreover we set

$$\text{supp}(s_X) := \{ Y \in (\tilde{\Gamma}_B)_0 \mid s_X(Y) \neq 0 \},$$
where the map \( s_X : (\tilde{\Gamma}_B)_0 \to \mathbb{Z}_{\geq 0} \) is defined by \( s_X(Y) := \dim \text{Hom}_B(\tilde{X}, \tilde{Y}) \) (\( Y \in (\tilde{\Gamma}_B)_0 \)) for all \( X \in (\tilde{\Gamma}_B)_0 \).

**Definition 3.18.** Let \( P \) be a projective indecomposable \( A \)-module, and \( \text{rad} P = \bigoplus_{i=1}^r R_i \) with \( R_i \) indecomposable for all \( i \). Then we define a full subquiver \( \mathcal{R}_P \) of \( \tilde{\Gamma}_B \) by

\[
(\mathcal{R}_P)_0 := \text{supp}(s_P) \setminus \left( \bigcup_{i=1}^r \text{supp}(s_{R_i}) \right).
\]

**Definition 3.19.** We regard the subquiver \( \mathcal{R}_P \) as a poset by Remark 3.11. For a projective indecomposable \( A \)-module \( P \), we set

\[
\nu'(P) := \min \mathcal{R}_P.
\]

**Example 3.20.** In the following figure, the vertices inside broken lines form \( \text{supp}(s_P) \) and those inside dotted lines form \( \left( \bigcup_{i=1}^r \text{supp}(s_{R_i}) \right) \). Therefore the subquiver \( \mathcal{R}_P \) consists of the vertices inside solid lines, and \( \nu'(P) \) is the minimum element of \( \mathcal{R}_P \). Projective vertices are presented by white circles \( \circ \).

We have the following the proof of which is omitted.

**Proposition 3.21.** Let \( P \) be a projective indecomposable \( A \)-module. then \( \nu'(P) \cong \text{top } P \).

We will give an alternative definition of the map \( \nu' \) below, which is easier to compute than the first one.

**Definition 3.22.** Let \( P \in \text{mod } B \) be projective.

(1) Let \( \mathcal{P}_P \) be the full subcategory of \( \text{mod } B \) consisting of projective modules \( Q \) such that \( P \) is not a direct summand of \( Q \).
(2) We define an ideal $I_P$ of mod $B$ and the factor category $\text{mod}^P B := \text{mod} B / I_P$ of mod $B$ by setting

$$I_P(X,Y) := \{ f \in \text{Hom}_B(X,Y) \mid f \text{ factors through an object in } \mathcal{P}_P \},$$

and set

$$\text{Hom}^P_B(X,Y) := \text{Hom}_B(X,Y) / I_P(X,Y)$$

for all $X, Y \in \text{mod} B$. Let $\Phi : \text{mod} B \to \text{mod}^P B$ be the canonical functor. Thus $X = \hat{X}$ for all $X \in (\text{mod} B)_0$ and $\Phi f = f + I_P(X,Y)$ for all $f \in \text{Hom}_B(X,Y)$.

$$\text{supp}(s'_P) := \{ X \in (\hat{\Gamma} B)_0 \mid s'_P(X) \neq 0 \} \subseteq (\hat{\Gamma} B)_0$$

where the map $s'_P : (\hat{\Gamma} B)_0 \to \mathbb{Z}_{\geq 0}$ is defined by $s_P(X) := \dim \text{Hom}^P_B(P,X)$ ($X \in (\hat{\Gamma} B)_0$) for all $P \in (\hat{\Gamma} B)_0$.

The easier way to compute $\nu'$ is given by the following three statements, which we state without proofs.

**Lemma 3.23.** Let $Q$ and $X$ be in mod $B$. If $Q$ is projective and there is an epimorphism $Q \to X$, then the projective cover of $X$ is a direct summand of $Q$.

**Lemma 3.24.** If $f : X \to \text{top} P$ is nonzero in mod $B$, then $f \neq 0$.

**Proposition 3.25.** Let $P$ be a projective indecomposable $A$-module. Then we have

$$\max \text{supp}(s'_P) \cong \text{top} P.$$ 

Thus $\nu'(P) = \max \text{supp}(s'_P)$.

Next we define a map sending a simple $A$-module to an element of the configurations.

**Lemma 3.26.** Let $S$ be a simple $A$-module, and $Q$ the injective hull of $S$ in mod $B$. Then the left $(\text{mod} B)$-module $\tilde{\text{Hom}}_B(S,-)$ has a simple socle, and

$$\text{soc} \tilde{\text{Hom}}_B(S,-) \cong \tilde{\text{Hom}}_B(\text{rad} Q,-) / \tilde{\text{rad}}(\text{rad} Q,-).$$

It follows by the lemma above that the poset $\text{supp}(s_S)$ has the maximum element for each simple $A$-module $S$. We then set $\nu_B(S)$ to be the maximum element. The following is immediate.

**Proposition 3.27.** Let $S$ be a simple $A$-module, and $Q$ the injective hull of $S$ in mod $B$. Then we have $\nu_B(S) \cong \text{rad} Q$.

We finally obtain the following by Propositions 3.25 and 3.27.

**Theorem 3.28.** Let $\mathcal{P}$ be a complete set of representatives of isoclass of indecomposable projective $A$-modules. Then we have

$$\mathcal{C}_B = \nu_B(\nu'(\mathcal{P})).$$

Hence as is stated before, $\mathcal{C}_A$ is obtained as follows.

**Theorem 3.29.**

$$\mathcal{C}_A = \mathcal{C}_A / \langle \phi \rangle = (\tau^{Zm} \sigma(C_B)) / \langle \phi \rangle = (\tau^{Zm} \sigma \nu_B(\nu'(\mathcal{P}))) / \langle \phi \rangle.$$

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References


Department of Mathematics,
Graduate School of Science,
Shizuoka University, 836 Ohya, Suruga-ku,
Shizuoka, 422-8529, Japan
E-mail address: asashiba.hideto@shizuoka.ac.jp

Department of Mathematics,
Graduate School of Science and Technology,
Shizuoka University, 836 Ohya, Suruga-ku,
Shizuoka, 422-8529, Japan
E-mail address: gehotan@gmail.com