TILTED ALGEBRAS AND CONFIGURATIONS OF SELF-INJECTIVE ALGEBRAS OF DYNKIN TYPE

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Abstract. All algebras are assumed to be basic, connected finite-dimensional algebras over an algebraically closed field. We give an easier way to calculate a bijection from the set of isoclasses of tilted algebras of Dynkin type Δ to the set of configurations on the translation quiver $\mathbb{Z}\Delta$.

Introduction

This work is a generalization of Hironobu Suzuki's Master thesis [7] that dealt with representation-finite self-injective algebras of type A in a combinatorial way. Throughout this paper n is a positive integer and $\Bbbk$ is an algebraically closed field, and all algebras considered here are assumed to be basic, connected, finite-dimensional associative $\Bbbk$-algebras.

Let Δ be a Dynkin graph of type A, D, E with the set $\Delta_0 := \{1, \dots, n\}$ of vertices. We set $\mathbf{C}_n$ to be the set of configurations on the translation quiver $\mathbb{Z}\Delta$ (see Definition 1.6), and $\mathbf{T}_n$ to be the set of isoclasses of tilted algebras of type Δ. Then Bretscher, Läser and Riedtmann have given a bijection $c\colon \mathbf{T}_n \to \mathbf{C}_n$ in [1]. But the map c is not given in a direct way, it needs a long computation of a function on $\mathbb{Z}\Delta$. In this paper we will give an easier way to calculate the map c by giving a map sending each projective A-module over a tilted algebra A in $\mathbf{T}_n$ to an element of the configuration $c(A)$.

We fix an orientation of each Dynkin graph Δ to have a quiver $\vec{\Delta}$ as in the following table.

Δ	A_n $(n \geq 1)$	D_n $(n \geq 4)$	E_n $(n = 6, 7, 8)$
$\vec{\Delta}$	$1 \to 2 \to \cdots \to n$	$1 \to \cdots \to n-2 \to n-1$, $n-2 \to n$	$1 \to \cdots \to n-3 \to n-2 \to n-1$, $n-3 \to n$
m_Δ	n	$2n-3$	11, 17, 29, respectively

This orientation of Δ gives us a coordinate system on the set $(\mathbb{Z}\Delta)_0 := \mathbb{Z} \times \Delta_0$ of vertices of $\mathbb{Z}\Delta := \mathbb{Z}\vec{\Delta}$ as presented in [1, fig. 1] and in [3, Fig. 13], and by definition the full subquiver $\mathcal{S}$ of $\mathbb{Z}\Delta$ consisting of $\{(0, i) \mid i \in \Delta_0\}$ is isomorphic to $\vec{\Delta}$.

Let A be a tilted algebra of type Δ. Then by identify A with the $(0,0)$-entry of the repetitive category $\hat{A}$, the vertex set of AR-quiver Γ_A is embedded into the vertex set of the stable AR-quiver ${}_s\Gamma_{\hat{A}}$ $(\cong \mathbb{Z}\Delta)$ of $\hat{A}$. Further the configuration $\mathcal{C} := c(A)$ of $\mathbb{Z}\Delta$ computed in [1] is given by the vertices of $\mathbb{Z}\Delta$ corresponding to radicals of projective

The detailed version of this paper will be submitted for publication elsewhere.

indecomposable $\hat{A}$-modules. Note that the configuration $\mathcal{C}$ has a period m_Δ listed in the table, thus $\mathcal{C} = \tau^{m_\Delta \mathbb{Z}}\mathcal{F}$ for some subset $\mathcal{F}$ of $\mathcal{C}$. By $\mathcal{P} = \{(p(i), i) \mid i \in \Delta_0\}$ we denote the set of images of the projective vertices of Γ_A in $\mathbb{Z}\Delta$ and set

$$\mathbb{N}\mathcal{P} := \{(m, i) \in (\mathbb{Z}\Delta)_0 \mid p(i) \leq m, i \in \Delta_0\}.$$

Since the mesh category $\Bbbk(\mathbb{Z}\Delta)$ is a Frobenius category, it has the Nakayama permutation $\hat{\nu}$ on $(\mathbb{Z}\Delta)_0$ that is defined by the isomorphism

$$\Bbbk(\mathbb{Z}\Delta)(x, \text{-}) \cong \mathrm{Hom}_{\Bbbk}(\Bbbk(\mathbb{Z}\Delta)(\text{-}, \hat{\nu}x), \Bbbk)$$

for all $x \in (\mathbb{Z}\Delta)_0$. The explicit formula of $\hat{\nu}$ is given in [3, pp. 48–50]. (Note that it should be corrected as $\hat{\nu}(p, q) = (p + q + 2, 6 - q)$ if $q \leq 5$ when $\Delta = E_6$ as pointed out in [1, 1.1]). In this paper we will define a map $\nu' \colon \mathcal{P} \to \mathbb{N}\mathcal{P}$ using the supports of starting functions $\dim_{\Bbbk} \Bbbk(\mathbb{Z}\Delta)(x, \text{-}) \colon \mathbb{N}\mathcal{P} \to \mathbb{Z}$ for $x \in \mathbb{N}\mathcal{P}$ (cf. [3, Fig. 15]). Then ν' has the following property.

Lemma 0.1. *Let $x \in \mathcal{P}$ and P be the projective indecomposable A-module corresponding to x. Then $\nu' x$ corresponds to the simple module* $\mathrm{top}\, P$.

In this paper, we make use of modules over the algebra

$$B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$$

to compute an $\mathcal{F}$ above (the configuration (see Definition 3.9) of B gives $\mathcal{F}$.) We will define a map $\nu := \nu_B$ from the set of isoclasses of simple A-modules to $\mathcal{C}$, which coincides with the restriction of the Nakayama permutation $\hat{\nu}$ if A is hereditary.

Lemma 0.2. *Assume that a vertex $x \in \mathbb{Z}\Delta$ corresponds to a simple A-module S and let Q be the injective hull of S over $\hat{A}$. Then $\nu(x)$ corresponds to* $\mathrm{rad}\, Q$*, and hence $\nu(x) \in \mathcal{C}$.*

Combining the lemmas above we obtain the following.

Proposition 0.3. *If $x \in \mathcal{P}$, then $\nu(\nu' x) \in \mathcal{C}$.*

This leads us to the following definition.

Definition 0.4. We define a map $c_A \colon \mathcal{P} \to \mathcal{C}$ by $c_A(x) := \nu(\nu' x)$ for all $x \in \mathcal{P}$.

The image of the map c_A gives us an $\mathcal{F}$ above, namely we have the following.

Theorem 0.5. *The map c_A is an injection, and we have $c(A) = \tau^{m_\Delta \mathbb{Z}}\, \mathrm{Im}\, c_A$.*

Corollary 0.6. *If A is hereditary, then $c_A = \hat{\nu}\nu'$ and we have $c(A) = \tau^{m_\Delta \mathbb{Z}}\, \mathrm{Im}\, \hat{\nu}\nu'$.*

Section 1 is devoted to preparations. In Section 2 we will give the complete list of indecomposable projectives and indecomposable injectives over the triangular matrix algebra B. In Section 3 we state our main results.

1. Preliminaries

1.1. **Algebras and categories.** A category $\mathcal{C}$ is called a $\Bbbk$-*category* if the morphism sets $\mathcal{C}(x, y)$ are $\Bbbk$-vector spaces, and the compositions $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ are $\Bbbk$-bilinear for all $x, y, z \in \mathcal{C}_0$ ($\mathcal{C}_0$ is the class of objects of $\mathcal{C}$, we sometimes write $x \in \mathcal{C}$ for $x \in \mathcal{C}_0$). In the sequel *all categories are assumed to be* $\Bbbk$*-categories* unless otherwise stated.

To construct repetitive categories and to make use of a covering theory we need to extend the range of considerations from algebras to categories. First we regard an algebra as a special type of categories by constructing a category $\operatorname{cat} A$ from an algebra A as follows.

1. We fix a decomposition $1 = e_1 + \cdots + e_n$ of the identity element 1 of A as a sum of orthogonal primitive idempotents.
2. We set the object class of $\operatorname{cat} A$ to be the set $\{e_1, \dots, e_n\}$.
3. For each pair (e_i, e_j) of objects, we set $(\operatorname{cat} A)(e_i, e_j) := e_j A e_i$.
4. We define the composition of $\operatorname{cat} A$ by the multiplication of A.

The obtained category $\operatorname{cat} A$ is uniquely determined up to isomorphisms not depending on the decomposition of 1. The category $C = \operatorname{cat} A$ is a small category having the following three properties.

1. Distinct objects are not isomorphic.
2. For each object x of C the algebra $C(x, x)$ is local.
3. For each pair (x, y) of objects of C the morphism space $C(x, y)$ is finite-dimensional.

A small category with these three properties is called a *spectroid*[1] and its objects are sometimes called *points*. A spectroid with only a finite number of points is called *finite*. The category $\operatorname{cat} A$ is a finite spectroid. Conversely we can construct a matrix algebra from a finite spectroid C as follows.

$$\operatorname{alg} C := \{(m_{yx})_{x,y \in C} \mid m_{yx} \in C(x, y), \forall x, y \in C\}.$$

Here we have $\operatorname{alg} \operatorname{cat} A \cong A, \operatorname{cat} \operatorname{alg} C \cong C$. Therefore we can identify the class of algebras and the class of finite spectroids by using cat and alg.

A spectroid C is called *locally bounded* if for each point x the set $\{y \in C \mid C(x, y) \neq 0 \text{ or } C(y, x) \neq 0\}$ is a finite set. Of course algebras (= finite spectroids) are locally bounded. In the range of locally bounded spectroids we can freely construct repetitive categories or consider coverings.

Remark 1.1. We can construct the "path-category" $\Bbbk Q$ from a locally finite quiver Q by the same way as in the definition of the path-algebra. The only different part is in the following definition of compositions: For paths μ, ν with[2] $s(\mu) \neq t(\nu)$, it was defined as $\mu\nu = 0$ in the path-algebra, but in contrast the composition $\mu\nu$ is not defined in the path-category.

A locally bounded spectroid C is also presented as the form $\Bbbk Q/I$ for some locally finite quiver Q and for some ideal I of the path-category $\Bbbk Q$ such that I is included in the ideal

[1] a terminology used in [4]

[2] Here $s(\mu)$ and $t(\nu)$ stand for the source of μ and the target of ν and compositions are written from the right to the left.

of $\Bbbk Q$ generated by the set of paths of length 2. Here the quiver Q is uniquely determined by C up to isomorphisms. This Q is called *the quiver* of C.

A (right) *module* over a spectroid C is a contravariant functor $C \to \operatorname{Mod} \Bbbk$. From a usual (right) module over an algebra A we can construct a contravariant functor $\operatorname{cat} A \to \operatorname{Mod} \Bbbk$ by the correspondence $e_i \mapsto Me_i$ for each point e_i in $\operatorname{cat} A$, and $f \mapsto (\cdot f\colon Me_j \to Me_i)$ for each $f \in e_j A e_i = (\operatorname{cat} A)(e_i, e_j)$. Conversely, from a contravariant functor $F\colon \operatorname{cat} A \to \operatorname{Mod} \Bbbk$ we can construct an A-module $\bigoplus_{i=1}^n F(e_i)$; and these constructions are inverse to each other. In this way we can identify A-modules and modules over $\operatorname{cat} A$.

The set of projective indecomposable modules over a spectroid C is given by $\{C(\text{-}, x)\}_{x \in C}$ up to isomorphism, and finitely generated projective C-modules are nothing but finite direct sums of these. Using this we can define finitely generated modules or finitely presented modules over C by the same way as those over algebras.

The dimension of a C-module M is defined to be the dimension of $\bigoplus_{x \in C} M(x)$. When C is locally bounded, a C-module is finitely presented if and only if it is finitely generated if and only if it is finite-dimensional.

1.2. **Repetitive category.**

Definition 1.2. Let A be an algebra with a basic set of local idempotents $\{e_1, \dots, e_n\}$.

(1) The *repetitive category* $\hat{A}$ of A is a spectroid defined as follows.

Objects: $\hat{A}_0 := \{x^{[i]} := (x, i) \mid x \in \{e_1, \dots, e_n\}, i \in \mathbb{Z}\}$.

Morphisms: Let $x^{[i]}, y^{[j]} \in \hat{A}_0$. Then we set

$$\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} := (f, i) \mid f \in A(x, y)\} & (j = i) \\ \{\varphi^{[i]} := (\varphi, i) \mid \varphi \in DA(y, x)\} & (j = i + 1) \\ 0 & \text{otherwise.} \end{cases}$$

Compositions: The composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is defined as follows.

(i) If $j = i, k = j$, then we use the composition of A:

$$A(y, z) \times A(x, y) \to A(x, z).$$

(ii) If $j = i, k = j+1$, then we use the right A-module structure of $DA(\text{-}, ?)$:

$$DA(z, y) \times A(x, y) \to DA(z, x).$$

(iii) If $j = i+1, k = j$, then we use the left A-module structure of $DA(\text{-}, ?)$:

$$A(y, z) \times DA(y, x) \to DA(z, x).$$

(iv) Otherwise the composition is zero.

(2) For each $i \in \mathbb{Z}$, we denote by $A^{[i]}$ the full subcategory of $\hat{A}$ whose object class is $\{x^{[i]} \mid x \in \{e_1, \dots, e_n\}\}$.

(3) We define the *Nakayama automorphism* ν_A of $\hat{A}$ as follows: for each $i \in \mathbb{Z}, x, y \in A, f \in A(x, y)$ and $\phi \in DA(y, x)$,

$$\nu_A(x^{[i]}) := x^{[i+1]}, \nu_A(f^{[i]}) := f^{[i+1]}, \nu_A(\varphi^{[i]}) := \varphi^{[i+1]}.$$

Remark 1.3. (1) If a spectroid A is locally bounded, then so is $\hat{A}$.

(2) When A is an algebra, the set of all $\mathbb{Z} \times \mathbb{Z}$-matrices with only a finite number of nonzero entries whose diagonal entries belong to A, $(i+1, i)$ entries belong to DA for all $i \in \mathbb{Z}$, and other entries are zero forms an infinite-dimensional algebra without identity element, which is called the *repetitive algebra* of A. The repetitive category $\hat{A}$ is nothing but this repetitive algebra regarded as a spectroid in a similar way. This is not an algebra (= a finite spectroid) any more, but a locally bounded spectroid.

Definition 1.4 (Gabriel [2])**.** Let C be a locally bounded spectroid with a free[3] action of a group G. Then we define the *orbit category* C/G of C by G as follows.

(1) The objects of C/G are the G-orbits Gx of objects x of C.
(2) For each pair Gx, Gy of objects of C/G we set

$$(C/G)(Gx, Gy) := \left\{ ({}_bf_a)_{a,b} \in \prod_{(a,b)\in Gx\times Gy} C(a,b) \,\middle|\, {}_{gb}f_{ga} = g({}_bf_a), \text{ for all } g \in G \right\}.$$

(3) The composition is defined by

$$({}_dh_c)_{c,d} \cdot ({}_bf_a)_{a,b} := \left(\sum_{b\in Gy} {}_dh_b \cdot {}_bf_a \right)_{a,d}.$$

for all $({}_bf_a)_{a,b} \in (C/G)(Gx, Gy), ({}_dh_c)_{c,d} \in (C/G)(Gy, Gz)$. Note that each entry of the right hand side is a finite sum because C is locally bounded.

A functor $F\colon C \to C'$ is called a *Galois covering* with group G if it is isomorphic to the canonical functor $\pi\colon C \to C/G$, namely if there exists an isomorphism $H\colon C/G \to C'$ such that $F = H\pi$.

Remark 1.5. If A is an algebra and a group G acts freely on the category $\hat{A}$, then $\hat{A}/G$ turns out to be a self-injective spectroid. In particular, when $\hat{A}/G$ is a finite spectroid, it becomes a self-injective algebra. In this way we can construct a great number of self-injective algebras.

Definition 1.6. From a quiver Q we can construct a translation quiver $\mathbb{Z}Q$ as follows.

- $(\mathbb{Z}Q)_0 := \mathbb{Z} \times Q_0$,
- $(\mathbb{Z}Q)_1 := \mathbb{Z} \times Q_1 \cup \{(i, \alpha') \mid i \in \mathbb{Z}, \alpha \in Q_1\}$,
- We define the sources and the targets of arrows by

$$(i,\alpha)\colon (i, s(\alpha)) \to (i, t(\alpha)),\ (i,\alpha')\colon (i, t(\alpha)) \to (i+1, s(\alpha))$$

  for all $(i, \alpha) \in \mathbb{Z} \times Q_1$.
- We take the bijection $\tau\colon (\mathbb{Z}Q)_0 \to (\mathbb{Z}Q)_0, (i, x) \mapsto (i-1, x)$ as the translation.

In addition, we can define a polarization by $(i+1, \alpha) \mapsto (i, \alpha'), (i, \alpha') \mapsto (i, \alpha)$. Note that by construction the translation quiver $\mathbb{Z}Q$ does not have any projective or injective vertices.

[3] $1 \neq g \in G$, $x \in C_0$ implies $gx \neq x$

For example,

$$Q = \begin{tikzcd}[row sep=small] & 1 \\ 2 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & 3 \end{tikzcd} \quad \text{gives} \quad \mathbb{Z}Q = \begin{tikzcd}[column sep=small] \cdots \arrow[rr, dash, dashed] & & (-1,1) \arrow[dr, "{(-1,\alpha')}"] \arrow[rr, dash, dashed] & & (0,1) \arrow[dr, "{(0,\alpha')}"] \arrow[rr, dash, dashed] & & (1,1) \arrow[r, dash] & \cdots \\ \cdots \arrow[r, dash] & (-1,2) \arrow[ur, "{(-1,\alpha)}"] \arrow[dr, "{(-1,\beta)}"'] \arrow[rr, dash, dashed] & & (0,2) \arrow[ur, "{(0,\alpha)}"] \arrow[dr, "{(0,\beta)}"'] \arrow[rr, dash, dashed] & & (1,2) \arrow[ur, "{(1,\alpha)}"] \arrow[dr, "{(1,\beta)}"'] \arrow[rr, dash, dashed] & & \cdots \\ \cdots \arrow[rr, dash, dashed] & & (-1,3) \arrow[ur, "{(-1,\beta')}"'] \arrow[rr, dash, dashed] & & (0,3) \arrow[ur, "{(0,\beta')}"'] \arrow[rr, dash, dashed] & & (1,3) \arrow[r, dash] & \cdots \end{tikzcd}.$$

Remark 1.7. When Q is a Dynkin quiver with the underlying graph Δ, the isoclass of $\mathbb{Z}Q$ does not depend on orientations of Δ, therefore we set $\mathbb{Z}\Delta := \mathbb{Z}Q$.

2. Triangular Matrix Algebras

Definition 2.1. Let R and S be algebras, M be an S-R-bimodule. We define a category $\mathcal{C} = \mathcal{C}(R, S, M)$ as follows.

Objects: $\mathcal{C}_0 := \{(X, Y, f) \mid X_R \in \operatorname{mod} R, Y_S \in \operatorname{mod} S, f \in \operatorname{Hom}_A(Y \otimes_S M, X)\}$.

Morphisms: Let $(X, Y, f), (X', Y', f') \in \mathcal{C}_0$. Then we set

$$\mathcal{C}((X, Y, f), (X', Y', f')) := \left\{ (\phi_0, \phi_1) \in \operatorname{Hom}_R(X, X') \times \operatorname{Hom}_S(Y, Y') \;\middle|\; \begin{tikzcd} Y \otimes_S M \arrow[r, "f"] \arrow[d, "\phi_1 \otimes 1_M"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "\phi_0"] \\ Y' \otimes_S M \arrow[r, "f'"'] & X' \end{tikzcd} \right\}.$$

Compositions: Let $(X, Y, f), (X', Y', f'), (X'', Y'', f'') \in \mathcal{C}_0$ and let

$$(\phi_0, \phi_1) \in \mathcal{C}((X, Y, f), (X', Y', f')), (\phi'_0, \phi'_1) \in \mathcal{C}((X', Y', f'), (X'', Y'', f'')).$$

Then we set

$$(\phi'_0, \phi'_1)(\phi_0, \phi_1) := (\phi'_0\phi_0, \phi'_1\phi_1) \in \mathcal{C}((X, Y, f), (X'', Y'', f'')).$$

Then the following is well known.

Proposition 2.2. *Let R and S be algebras, M be an S-R-bimodule. Then*

$$\operatorname{mod} \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \simeq \mathcal{C}(R, S, M).$$

Recall that an equivalence $F : \operatorname{mod} \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \to \mathcal{C}(R, S, M)$ is given as follows.

Objects: For each $L \in (\operatorname{mod} T)_0$,

$$F(L) := (L\varepsilon_1, L\varepsilon_2, f_L),$$

where $\varepsilon_1 := \begin{bmatrix} 1_R & 0 \\ 0 & 0 \end{bmatrix}, \varepsilon_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1_S \end{bmatrix}$ and $f_L : L\varepsilon_2 \otimes_S M \to L\varepsilon_1$ is defined by $f_L(l\varepsilon_2 \otimes m) := l \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ for all $l \in L$ and $m \in M$.

Morphisms: For each $\alpha \in \operatorname{Hom}_T(L, L')$,

$$F(\alpha) := (\alpha\,|_{L\varepsilon_1}, \alpha\,|_{L\varepsilon_2}).$$

Let A be a tilted algebra of type Δ, and set $B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$, $\mathcal{C} := \mathcal{C}(A, A, DA)$.

Then we have $\operatorname{mod} B \simeq \mathcal{C}$ by Proposition 2.2. By this equivalence, we identify $\operatorname{mod} B$ with $\mathcal{C}$.

Let $\{e_1, \dots, e_n\}$ be a complete set of orthogonal local idempotents of A. Then as is easily seen

$\{e_1^{[0]}, \dots, e_n^{[0]}, e_1^{[1]}, \dots, e_n^{[1]}\}$ is a complete set of orthogonal local idempotents of B, and $\{e_1^{[0]}B, \dots, e_n^{[0]}B, e_1^{[1]}B, \dots, e_n^{[1]}B\}$ is a complete set of isoclasses of projective indecomposable B-modules. The following is immediate.

Proposition 2.3. *For each $i = 1, \dots, n$, we have*

$$\begin{aligned} F(e_i^{[0]}B) &\cong (e_iA, 0, 0), \\ F(e_i^{[1]}B) &\cong (e_i(DA), e_iA, \mathrm{can}). \end{aligned}$$

In addition $\{D(Be_1^{[0]}), \dots, D(Be_n^{[0]}), D(Be_1^{[1]}), \dots, D(Be_n^{[1]})\}$ is a complete set of isoclasses of injective indecomposable B-modules. The following two statements are obvious.

Lemma 2.4. *For each $i = 1, \dots, n$, we have*

(1) $D\begin{bmatrix} Ae_i & 0 \\ (DA)e_i & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ D(Ae_i) & e_iA \end{bmatrix}$, *and*

(2) $D\begin{bmatrix} 0 & 0 \\ 0 & Ae_i \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ 0 & D(Ae_i) \end{bmatrix}$.

Proposition 2.5. *For each $i = 1, \dots, n$, we have*

$$\begin{aligned} F(D(Be_i^{[0]})) &\cong (e_i(DA), e_iA, \mathrm{can}) \cong e_i^{[1]}B, \\ F(D(Be_i^{[1]})) &\cong (0, e_i(DA), 0). \end{aligned}$$

3. Configurations

Definition 3.1. Let Λ be a standard representation-finite self-injective algebra. Then we set

$$\mathcal{C}_\Lambda := \{[\operatorname{rad} P] \in \Gamma_\Lambda \mid P : \text{projective(-injective) } \Lambda\text{-module}\},$$

which is called a *configuration* of Λ.

Definition 3.2. Let Γ be a stable translation quiver, and $\mathcal{C}$ be a subset of Γ_0. Then we define a translation quiver $\Gamma_{\mathcal{C}}$ by

$$\begin{aligned} (\Gamma_{\mathcal{C}})_0 &:= \Gamma_0 \sqcup \{p_x \mid x \in \mathcal{C}\}, \\ (\Gamma_{\mathcal{C}})_1 &:= \Gamma_1 \sqcup \{x \to p_x,\ p_x \to \tau^{-1}x\}, \end{aligned}$$

where the translation of $\Gamma_{\mathcal{C}}$ is the same as that of Γ. In particular, p_x are projective-injective[4] vertices for all $x \in \mathcal{C}$.

[4]The word "projective-injective" stands for projective and injective.

Remark 3.3. The quiver of $\underline{\mathrm{mod}}\,\Lambda$ is the full subquiver ${}_s\Gamma_\Lambda$ of Γ_Λ with

$$({}_s\Gamma_\Lambda)_0 := \{x \mid x \text{ is a stable vertex of } \Gamma_\Lambda\}$$

(namely ${}_s\Gamma_\Lambda$ is obtained from Γ_Λ by removing all projective vertices), which is a stable translation quiver. Then it holds that $\mathcal{C}_\Lambda \subseteq ({}_s\Gamma_\Lambda)_0$, and we have

$$({}_s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong \Gamma_\Lambda. \tag{3.1}$$

Theorem 3.4. *Let Λ be a standard representation-finite self-injective algebra and Δ the Dynkin type of Λ. Then the following hold.*

(1) (Waschbüsch [5, 8]) *There exist a tilted algebra A of type Δ and an automorphism ϕ of $\hat{A}$ without fixed vertices such that $\Lambda \cong \hat{A}/\langle\phi\rangle$.*
(2) (Riedtmann [6]) *There is an isomorphism $f : {}_s\Gamma_{\hat{A}} \to \mathbb{Z}\Delta$. Denote also by ϕ the automorphism of ${}_s\Gamma_{\hat{A}}$ induced from ϕ canonically, and define an automorphism ϕ' of $\mathbb{Z}\Delta$ by the following commutative diagram:*

$$\begin{tikzcd}
{}_s\Gamma_{\hat{A}} \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z}\Delta \arrow[d, "\phi'"] \\
{}_s\Gamma_{\hat{A}} \arrow[r, "f"'] & \mathbb{Z}\Delta.
\end{tikzcd}$$

Then we have ${}_s\Gamma_\Lambda \cong {}_s\Gamma_{\hat{A}}/\langle\phi\rangle \cong \mathbb{Z}\Delta/\langle\phi'\rangle$.

By the formula (3.1) to compute Γ_Λ, it is enough to solve the following problem.

Problem 1. Let Λ be a standard representation-finite self-injective algebra, which has the form $\hat{A}/\langle\phi\rangle$ for some tilted algebra A of Dynkin type and an automorphism ϕ of $\hat{A}$ by Theorem 3.4. Then compute $\mathcal{C}_\Lambda$ from A.

Remark 3.5. Let $f' : {}_s\Gamma_\Lambda \to \mathbb{Z}\Delta/\langle\phi'\rangle$ be an isomorphism, and set $\mathcal{C} := f'(\mathcal{C}_\Lambda)$. Then we have

$$\Gamma_\Lambda \cong ({}_s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong (\mathbb{Z}\Delta/\langle\phi'\rangle)_{\mathcal{C}}.$$

Thus we can compute Γ_Λ by Theorem 3.4(2) if we can obtain the set $\mathcal{C}$.

On the other hand, the following holds by [2, Theorem 3.6].

Theorem 3.6 (Gabriel)**.** *Let R be a locally representation-finite and locally bounded $\Bbbk$-category, and G be a group consisting of automorphisms of R that acts freely on R. Then the AR-quiver Γ_R of R has an induced G-action, and we have $\Gamma_R/G \cong \Gamma_{R/G}$.*

Definition 3.7. Let A be a tilted algebra of Dynkin type. Then we set

$$\mathcal{C}_{\hat{A}} := \{[\mathrm{rad}\,P] \in \Gamma_{\hat{A}} \mid P : \text{projective(-injective) } \hat{A}\text{-module}\},$$

which is called the *configuration* of $\hat{A}$.

Corollary 3.8. *Let A be a tilted algebra of Dynkin type, and ϕ be an automorphism of $\hat{A}$ without fixed vertices. Then we have*

$$\mathcal{C}_{\hat{A}}/\langle\phi\rangle \cong \mathcal{C}_\Lambda.$$

Therefore to solve Problem 1, it is enough to consider the following.

Problem 2. In the same setting as in Problem 1, compute $\mathcal{C}_{\hat{A}}$ from A.

Throughout the rest of their section

(1) let A be a tilted algebra of Dynkin type Δ, and set
(2) $B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$.

By (1), Γ_A has a section $\mathcal{S}$ whose underlying graph is isomorphic to Δ.

Definition 3.9. We call the following set the *configuration* of B:

$$\mathcal{C}_B := \{[\operatorname{rad} P] \in \Gamma_B \mid P : \text{projective-injective } B\text{-module}\}.$$

3.1. **Relationship among $\hat{A}$, B and A.** We set as follows:

$$\begin{aligned} I_{0,1} &= \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{0,1\}, j \in \{1,\dots,n\}\rangle, \\ I_0 &= \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{0\}, j \in \{1,\dots,n\}\rangle, \\ I_1 &= \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{1\}, j \in \{1,\dots,n\}\rangle. \end{aligned}$$

Then $\hat{A}/I_{0,1} \cong B$, $\hat{A}/I_0 \cong A^{[0]} (\cong A)$ and $\hat{A}/I_1 \cong A^{[1]} (\cong A)$. We also have

$$B \Big/ \begin{bmatrix} 0 & 0 \\ DA & 0 \end{bmatrix} \cong A^{[0]} \times A^{[1]}.$$

We have the following surjective algebra homomorphisms

$$\begin{tikzcd} & & A^{[0]} \\ \hat{A} \arrow[r, two heads] \arrow[urr, two heads] \arrow[drr, two heads] & B \arrow[r, two heads] & A^{[0]} \times A^{[1]} \arrow[u, two heads] \arrow[d, two heads] \\ & & A^{[1]}, \end{tikzcd}$$

which induce the following embeddings of categories

$$\begin{tikzcd} & & \operatorname{mod} A^{[0]} \arrow[dl, hook'] \\ \operatorname{mod} \hat{A} & \operatorname{mod} B \arrow[l, hook', "\sigma"'] & \\ & & \operatorname{mod} A^{[1]}. \arrow[ul, hook] \end{tikzcd}$$

We regard $\operatorname{mod} A \subseteq \operatorname{mod} B$ by the embedding $\operatorname{mod} A = \operatorname{mod} A^{[0]} \hookrightarrow \operatorname{mod} B$. The embeddings above give us the following embeddings of vertex sets of AR-quivers:

$$\begin{tikzcd}
& & (\Gamma_{A^{[0]}})_0 = (\Gamma_A)_0 \arrow[dl, hook', "\sigma_0"'] \\
(\Gamma_{\hat{A}})_0 & (\Gamma_B)_0 \arrow[l, hook', "\sigma"'] & \\
& & (\Gamma_{A^{[1]}})_0 . \arrow[ul, hook]
\end{tikzcd}$$

We define an ideal $\Bbbk(\mathbb{Z}\Delta)^+$ of the mesh category $\Bbbk(\mathbb{Z}\Delta)$ as follows:

$$\Bbbk(\mathbb{Z}\Delta)^+ := \langle (\mathbb{Z}\Delta)_1 + I_{\mathbb{Z}\Delta} \rangle.$$

Then the values of $m_\Delta := \min\{m \in \mathbb{N} \mid (\Bbbk(\mathbb{Z}\Delta)^+)^i = 0, \forall i \geq m\}$ are known as follows:

$$m_\Delta = \begin{cases} n & (\Delta = A_n) \\ 2n-3 & (\Delta = D_n) \\ 11 & (\Delta = E_6) \\ 17 & (\Delta = E_7) \\ 29 & (\Delta = E_8) \end{cases}.$$

We see the following by [1].

Proposition 3.10. *Let $i = 0, 1$.*

(1) *The full subquiver $\mathcal{S}_B^{[i]}$ of Γ_B with the vertex set $\sigma_i(\mathcal{S}_0)$ forms a section of ${}_s\Gamma_B$.*
(2) *The full subquiver $\mathcal{S}_{\hat{A}}^{[i]}$ of $\Gamma_{\hat{A}}$ with the vertex set $\sigma\sigma_i(\mathcal{S}_0)$ forms a section of ${}_s\Gamma_{\hat{A}}$.*

Remark 3.11. A quiver Q without oriented cycles will be regarded as a poset by the order defined as follows:

For each $x, y \in Q_0, x \preceq y$:$\Leftrightarrow$there is a path in Q from x to y.

Definition 3.12. (1) We set $\mathcal{H}_B$ to be the full subquiver of Γ_B defined by the set

$$(\mathcal{H}_B)_0 := \{x \in (\Gamma_B)_0 \mid a \preceq x \preceq b \text{ for some } a \in (\mathcal{S}_B^{[0]})_0, b \in (\mathcal{S}_B^{[1]})_0\}$$

of vertices.

(2) We set $\mathcal{H}_{\hat{A}}^{[0,1]}$ to be the full subquiver of $\Gamma_{\hat{A}}$ defined by the set

$$(\mathcal{H}_{\hat{A}}^{[0,1]})_0 := \{x \in (\Gamma_{\hat{A}})_0 \mid a \preceq x \preceq b \text{ for some } a \in (\mathcal{S}_{\hat{A}}^{[0]})_0, b \in (\mathcal{S}_{\hat{A}}^{[1]})_0\}$$

of vertices.

Proposition 3.13. (1) *The map $\sigma : (\Gamma_B)_0 \to (\Gamma_{\hat{A}})_0$ is uniquely extended to a quiver isomorphism $\mathcal{H}_B \to \mathcal{H}_{\hat{A}}^{[0,1]}$.*
(2) *We have $\mathcal{S}_{\hat{A}}^{[1]} = \tau^{-m_\Delta}\mathcal{S}_{\hat{A}}^{[0]}$. We set $\mathcal{S}_{\hat{A}}^{[n]} := \tau^{-nm_\Delta}\mathcal{S}_{\hat{A}}^{[0]}$ for all $n \in \mathbb{Z}$.*

(3) *Set* $\mathcal{H}_{\hat{A}}^{[n,n+1]} := \tau^{-nm_{\triangle}}(\mathcal{H}_{\hat{A}}^{[0,1]})$ *for all* $n \in \mathbb{Z}$. *Then for each* $i = 0, 1$

$$
\begin{aligned}
(\Gamma_{\hat{A}})_i &= \bigcup_{n\in\mathbb{Z}} (\mathcal{H}_{\hat{A}}^{[n,n+1]})_i \\
(\mathcal{S}_{\hat{A}}^{[n+1]})_i &= (\mathcal{H}_{\hat{A}}^{[n,n+1]})_i \cap (\mathcal{H}_{\hat{A}}^{[n+1,n+2]})_i
\end{aligned}
$$

Roughly speaking, $\Gamma_{\hat{A}}$ is obtained by connecting infinite copies of $\mathcal{H}_B$ on both sides.

Example 3.14. Let A be the path algebra of the following quiver.

$$1^{[0]} \longrightarrow 2^{[0]} \longrightarrow 3^{[0]}$$

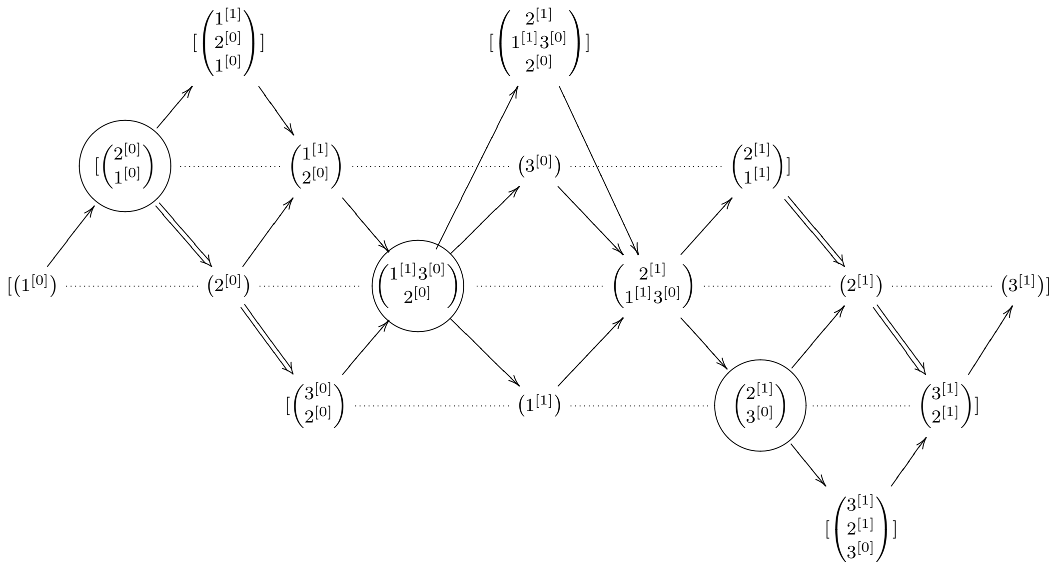
Then Γ_A is given as follows (double arrows present a section).

$$
\begin{tikzcd}
 & [\begin{pmatrix} 2^{[0]} \\ 1^{[0]} \end{pmatrix}] \arrow[dr, Rightarrow] & & & \\
{[(1^{[0]})} \arrow[ur] \arrow[rr, dotted, no head] & & (2^{[0]}) \arrow[dr, Rightarrow] \arrow[rr, dotted, no head] & & {(3^{[0]})]} \\
 & & & [\begin{pmatrix} 3^{[0]} \\ 2^{[0]} \end{pmatrix}] \arrow[ur] &
\end{tikzcd}
$$

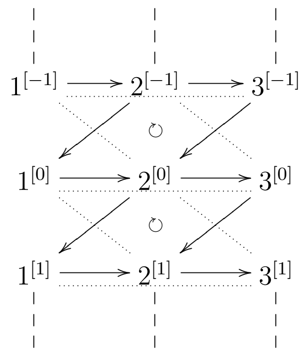
Therefore A is a tilted algebra of type A_3. Moreover $B = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} = \begin{bmatrix} A^{[0]} & 0 \\ (DA)^{[0]} & A^{[1]} \end{bmatrix}$ is an algebra given by following quiver with relations.

$$
\begin{tikzcd}
1^{[0]} \arrow[r] & 2^{[0]} \arrow[r] \arrow[dl] \arrow[d, phantom, "\circlearrowleft"] & 3^{[0]} \arrow[dl] \\
1^{[1]} \arrow[r] & 2^{[1]} \arrow[r] & 3^{[1]}
\end{tikzcd}
$$

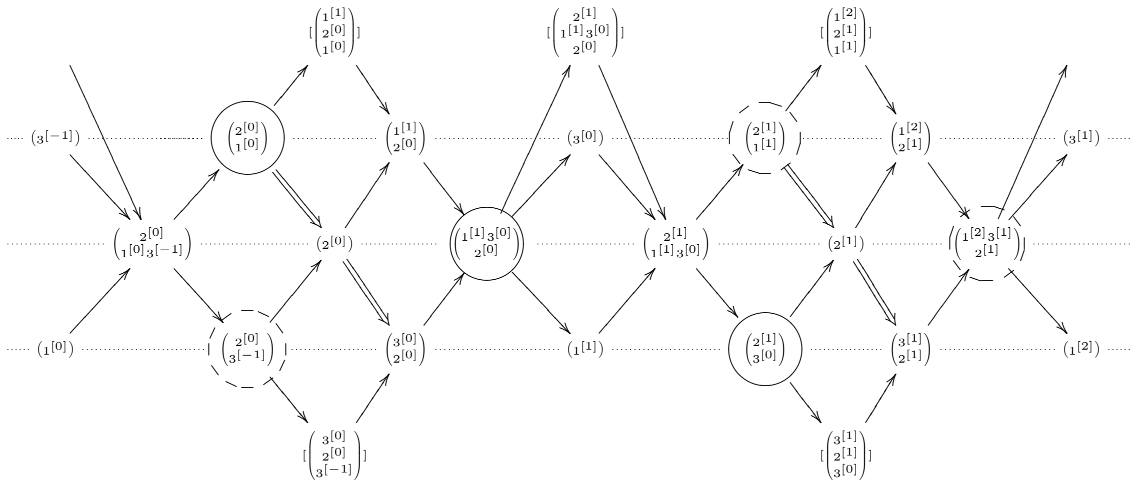
Then Γ_B is given as follows (elements of $\mathcal{C}_B$ are encircled).



In the above, $\mathcal{H}_B$ is given by the full subquiver consisting of vertices between the left section and the right section. $\hat{A}$ is given by the following quiver with relations.



Then $\Gamma_{\hat{A}}$ is follows (each element of $\mathcal{C}_{\hat{A}}$ is encircled by a broken or solid line, in particular solid circles present elements of $\mathcal{C}_B$). In this case we have $m_\Delta = 3$.



The following is immediate from Proposition 3.13.

Corollary 3.15. *We have* $\mathcal{C}_{\hat{A}} = \tau^{\mathbb{Z}m_\Delta}\sigma(\mathcal{C}_B)$.

By this corollary, Problem 2 is reduced to the following.

Problem 3. Let A be a tilted algebra of Dynkin type Δ, and B as above. Then give the configuration $\mathcal{C}_B$ from A.

The purpose of this section is to solve Problem 3.

Definition 3.16. (1) We define an ideal $\mathcal{PI}$ of $\operatorname{mod} B$ as follows and set $\widetilde{\operatorname{mod}}\, B := (\operatorname{mod} B)/\mathcal{PI}$. For each $X, Y \in (\operatorname{mod} B)_0$

$$\mathcal{PI}(X,Y) := \{f \in \operatorname{Hom}_B(X,Y) \,|\, f \text{ factors through a projective-injective } B\text{-module}\}$$

Let $\tilde{(?)}\colon \operatorname{mod} B \to \widetilde{\operatorname{mod}}\, B$ be the canonical functor and set

$$\widetilde{\operatorname{Hom}}_B(\tilde{X}, \tilde{Y}) := (\widetilde{\operatorname{mod}}\, B)(\tilde{X}, \tilde{Y})$$

for all $X, Y \in \operatorname{mod} B$. Thus $\tilde{X} = X$ for all $X \in (\operatorname{mod} B)_0$ and $\tilde{f} = f + \mathcal{PI}(X,Y)$ for all $f \in \operatorname{Hom}_B(X,Y)$.

(2) We denote by $\operatorname{mod}_{\mathcal{PI}} B$ the full subcategory of $\operatorname{mod} B$ consisting of B-modules without projective-injective direct summands.

(3) Let X and $Y \in \operatorname{mod}_{\mathcal{PI}} B$. Then it is well known that $\mathcal{PI}(X,Y) \subseteq \operatorname{rad}_B(X,Y)$. We set $\widetilde{\operatorname{rad}}_B(X,Y) := \operatorname{rad}_B(X,Y)/\mathcal{PI}(X,Y)$.

Definition 3.17. For AR-quiver Γ_B of B, we define the full translation subquiver $\tilde{\Gamma}_B$ as follows.

$$(\tilde{\Gamma}_B)_0 := \{X \in (\Gamma_B)_0 \mid X \text{ is } \textit{not} \text{ projective-injective. }\}$$

Moreover we set

$$\operatorname{supp}(s_X) := \{Y \in (\tilde{\Gamma}_B)_0 \mid s_X(Y) \neq 0\},$$

where the map $s_X : (\tilde{\Gamma}_B)_0 \to \mathbb{Z}_{\geq 0}$ is defined by $s_X(Y) := \dim \widetilde{\mathrm{Hom}}_B(\tilde{X}, \tilde{Y})$ $(Y \in (\tilde{\Gamma}_B)_0)$ for all $X \in (\tilde{\Gamma}_B)_0$.

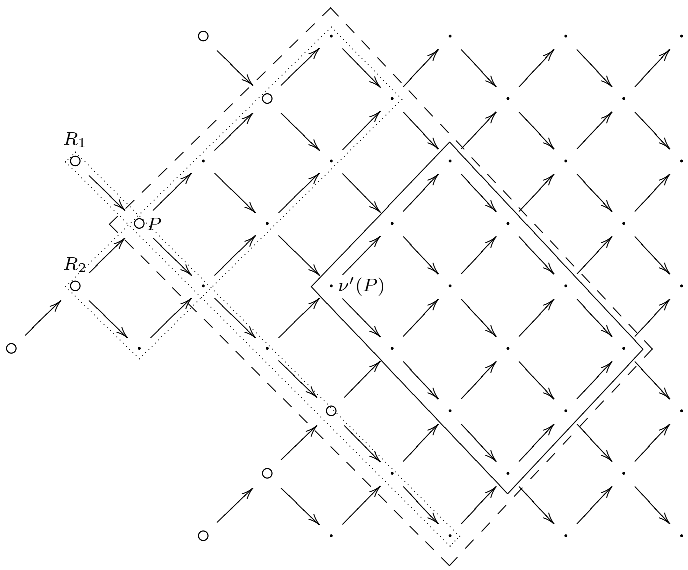
Definition 3.18. Let P be a projective indecomposable A-module, and $\operatorname{rad} P = \bigoplus_{i=1}^r R_i$ with R_i indecomposable for all i. Then we define a full subquiver $\mathcal{R}_P$ of $\tilde{\Gamma}_B$ by

$$(\mathcal{R}_P)_0 := \operatorname{supp}(s_P) \setminus \left(\bigcup_{i=1}^r \operatorname{supp}(s_{R_i}) \right).$$

Definition 3.19. We regard the subquiver $\mathcal{R}_P$ as a poset by Remark 3.11. For a projective indecomposable A-module P, we set

$$\nu'(P) := \min \mathcal{R}_P.$$

Example 3.20. In the following figure, the vertices inside broken lines form $\operatorname{supp}(s_P)$ and those inside doted lines form $(\bigcup_{i=1}^r \operatorname{supp}(s_{R_i}))$. Therefore the subquiver $\mathcal{R}_P$ consists of the vertices inside solid lines, and $\nu'(P)$ is the minimum element of $\mathcal{R}_P$. Projective vertices are presented by white circles $\circ$.



We have the following the proof of which is omitted.

Proposition 3.21. *Let P be a projective indecomposable A-module. then $\nu'(P) \cong \operatorname{top} P$.*

We will give an alternative definition of the map ν' below, which is easier to compute than the first one.

Definition 3.22. Let $P \in \operatorname{mod} B$ be projective.

(1) Let $\mathcal{P}_P$ be the full subcategory of $\operatorname{mod} B$ consisting of projective modules Q such that P is not a direct summand of Q.

(2) We define an ideal $\mathcal{I}_P$ of $\operatorname{mod} B$ and the factor category $\underline{\operatorname{mod}}^P B := \operatorname{mod} B/\mathcal{I}_P$ of $\operatorname{mod} B$ by setting

$$\mathcal{I}_P(X,Y) := \{f \in \operatorname{Hom}_B(X,Y) \,|\, f \text{ factors through an object in } \mathcal{P}_P\},$$

and set

$$\underline{\operatorname{Hom}}_B^P(X,Y) := \operatorname{Hom}_B(X,Y)/\mathcal{I}_P(X,Y)$$

for all $X, Y \in \operatorname{mod} B$. Let $\underline{(?)}\colon \operatorname{mod} B \to \underline{\operatorname{mod}}^P B$ be the canonical functor. Thus $\underline{X} = X$ for all $X \in (\operatorname{mod} B)_0$ and $\underline{f} = f + \mathcal{I}_P(X,Y)$ for all $f \in \operatorname{Hom}_B(X,Y)$.

$$\operatorname{supp}(s'_P) := \{X \in (\tilde{\Gamma}_B)_0 \mid s'_P(X) \neq 0\} \subseteq (\tilde{\Gamma}_B)_0$$

where the map $s'_P : (\tilde{\Gamma}_B)_0 \to \mathbb{Z}_{\geq 0}$ is defined by $s_P(X) := \dim \underline{\operatorname{Hom}}_B^P(P,X)$ $(X \in (\tilde{\Gamma}_B)_0)$ for all $P \in (\tilde{\Gamma}_B)_0$.

The easier way to compute ν' is given by the following three statements, which we state without proofs.

Lemma 3.23. *Let Q and X be in $\operatorname{mod} B$. If Q is projective and there is an epimorphism $Q \to X$, then the projective cover of X is a direct summand of Q.*

Lemma 3.24. *If $f : X \to \operatorname{top} P$ is nonzero in $\operatorname{mod} B$, then $\underline{f} \neq 0$.*

Proposition 3.25. *Let P be a projective indecomposable A-module. Then we have*

$$\max \operatorname{supp}(s'_P) \cong \operatorname{top} P.$$

Thus $\nu'(P) = \max \operatorname{supp}(s'_P)$.

Next we define a map sending a simple A-module to an element of the configurations.

Lemma 3.26. *Let S be a simple A-module, and Q the injective hull of S in $\operatorname{mod} B$. Then the left $(\widetilde{\operatorname{mod}} B)$-module $\widetilde{\operatorname{Hom}}_B(S,\text{-})$ has a simple socle, and*

$$\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S,-) \cong \widetilde{\operatorname{Hom}}_B(\operatorname{rad} Q,-)/\widetilde{\operatorname{rad}}(\operatorname{rad} Q,-).$$

It follows by the lemma above that the poset $\operatorname{supp}(s_S)$ has the maximum element for each simple A-module S. We then set $\nu_B(S)$ to be the maximum element. The following is immediate.

Proposition 3.27. *Let S be a simple A-module, and Q the injective hull of S in $\operatorname{mod} B$. Then we have $\nu_B(S) \cong \operatorname{rad} Q$.*

We finally obtain the following by Propositions 3.25 and 3.27.

Theorem 3.28. *Let $\mathcal{P}$ be a complete set of representatives of isoclass of indecomposable projective A-modules. Then we have*

$$\mathcal{C}_B = \nu_B(\nu'(\mathcal{P})).$$

Hence as is stated before, $\mathcal{C}_\Lambda$ is obtained as follows.

Theorem 3.29.

$$\mathcal{C}_\Lambda = \mathcal{C}_{\hat{A}}/\langle\phi\rangle = (\tau^{\mathbb{Z}m_\Delta}\sigma(\mathcal{C}_B))/\langle\phi\rangle = (\tau^{\mathbb{Z}m_\Delta}\sigma\nu_B\nu'(\mathcal{P}))/\langle\phi\rangle.$$

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