ON THE DECOMPOSITION OF THE HOCHSCHILD COHOMOLOGY GROUP OF A MONOMIAL ALGEBRA SATISFYING A SEPARABILITY CONDITION

AYAKO ITABA, TAKAHIKO FURUYA AND KATSUNORI SANADA

Abstract. This paper is based on [14]. In this paper, we consider the finite connected quiver Q having two subquivers $Q^{(1)}$ and $Q^{(2)}$ with $Q = Q^{(1)} \cup Q^{(2)} = (Q_0^{(1)} \cup Q_0^{(2)}, Q_1^{(1)} \cup Q_1^{(2)})$. Suppose that $Q^{(i)}$ is not a subquiver of $Q^{(j)}$ where $\{i, j\} = \{1, 2\}$. For a monomial algebra $\Lambda = kQ/I$ obtained by the quiver Q, when the set $AP(n)$ $(n \geq 0)$ of overlaps constructed inductively by linking generators of I satisfies a certain separability condition, we propose the method so that we easily construct a minimal projective resolution of Λ as a right Λ^{e}-module and calculate the Hochschild cohomology group of Λ.

Key Words: Monomial algebra, associated sequence of path, Hochschild cohomology, path algebra.

2010 *Mathematics Subject Classification*: 16E40, 16G20.

1. Introduction

For a finite-dimensional algebra A over a field k, the Hochschild cohomology groups $\mathrm{HH}^n(A)$ of A is defined by

$$\mathrm{HH}^n(A) := \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, A)\,(n \geq 0),$$

where $A^{\mathrm{e}}{:=}A^{\mathrm{op}} \otimes_k A$ is the enveloping algebra of A. Note that there is a natural one to one correspondence between the family of A-A-bimodules and that of right A^{e}-modules. Moreover, the Hochschild cohomology rings $\mathrm{HH}^*(A)$ of A is the graded algebra defined by

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^{\mathrm{e}}}(A, A) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A)$$

with the Yoneda product.

The low-dimensional Hochschild cohomology groups are described as follows:

- $\mathrm{HH}^0(A) = Z(A)$ is the center of A.
- $\mathrm{HH}^1(A)$ is the space of derivations modulo the inner derivations. A derivation is a k-linear map $f : A \to A$ such that $f(ab) = af(b) + f(a)b$ for all $a, b \in A$. A derivation $f : A \to A$ is an inner derivation if there is some $x \in A$ such that $f(a) = ax - xa$ for all $a \in A$.

One important property of Hochschild cohomology is its invariance under Morita equivalence, stable equivalence of Morita type and derived equivalence.

The detailed version of this paper has been submitted for publication elsewhere.

Let k be an algebraically closed field and Q a finite connected quiver. Then kQ denotes the path algebra of Q over k in this paper. Let I be an admissible ideal of kQ. If I is generated by a finite number of paths in Q, then I is called a monomial ideal and $\Lambda := kQ/I$ a monomial algebra. For a finite-dimensional monomial algebra $\Lambda = kQ/I$, using a certain set $AP(n)$ of overlaps constructed inductively by linking generators of I, Bardzell gave a minimal projective Λ^{e}-resolution $(P_\bullet, \phi_\bullet)$ of Λ in [3] (so called Bardzell's resolution). By using Bardzell's resolution, the Hochschild cohomology of monomial algebras are studied in the following papers [11], [12], [9], etc.

In general, it is not easy to calculate the Hochschild cohomology of a finite-dimensional algebra. In order to calculate the Hochschild cohomology groups of a quiver algebra, can we use calculations of the Hochschild cohomology groups of quiver algebras obtained by subquivers of the original quiver?

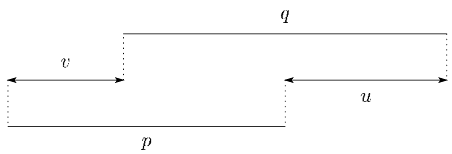
In this paper, for a finite-dimensional monomial algebra Λ, we propose a method so that we easily calculate the Hochschild cohomology groups of Λ under some conditions. Let Q be a finite connected quiver and $Q^{(i)}$ $(i = 1, 2)$ a subquiver of Q such that $Q = Q^{(1)} \cup Q^{(2)} = (Q_0^{(1)} \cup Q_0^{(2)}, Q_1^{(1)} \cup Q_1^{(2)})$. Let $I^{(1)} = \langle X \rangle$ (resp. $I^{(2)} = \langle Y \rangle$) be a monomial ideal of $kQ^{(1)}$ (resp. $kQ^{(2)}$) for X (resp. Y) a set of paths of $kQ^{(1)}$ (resp. $kQ^{(2)}$) and $I = \langle X, Y \rangle$ a monomial ideal of kQ. We assume that I and $I^{(i)}$ $(i = 1, 2)$ are admissible ideals. Then we define $\Lambda = kQ/I$, $\Lambda_{(1)} = kQ^{(1)}/I^{(1)}$ and $\Lambda_{(2)} = kQ^{(2)}/I^{(2)}$. Hence Λ and $\Lambda_{(i)}$ are finite-dimensional monomial algebras for $i = 1, 2$. For the monomial algebra Λ, under a *separability condition* (i.e. $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$), we investigate the minimal projective Λ^{e}-module resolution of Λ given by Bardzell ([3]). Moreover, under an additional condition, we show that, for $n \geq 2$, the Hochschild cohomology group $\mathrm{HH}^n(\Lambda)$ of Λ is isomorphic to the direct sum of the Hochschild cohomology groups $\mathrm{HH}^n(\Lambda_{(1)})$ and $\mathrm{HH}^n(\Lambda_{(2)})$.

Throughout this paper, for all arrows a of Q, we denote the origin of a by $o(a)$ and the terminus of a by $t(a)$. Also, for simplicity, we denote $\otimes_k$ by $\otimes$.

2. The set $AP(n)$ of overlaps and Bardzell's resolution

2.1. **The set $AP(n)$ of overlaps.** In this section, following [3] and [11], we will summarize the definition of the set $AP(n)$ $(n \geq 0)$ of overlaps.

Definition 1. A path $q \in kQ$ overlaps a path $p \in kQ$ with overlap pu if there exist u, v such that $pu = vq$ and $1 \leq l(u) \leq l(q)$, where $l(x)$ denotes the length of a path $x \in kQ$. Note that we allow $l(x) = 0$ here.



A path q properly overlaps a path p with overlap pu if q overlaps p and $l(v) \geq 1$.

Let $\Lambda = kQ/I$ be a finite-dimensional monomial algebra where $I = \langle \rho \rangle$ has a minimal set of generators ρ of paths of length at least 2.

Definition 2. For $n = 0, 1, 2$, we set

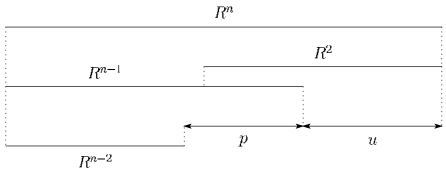
- $AP(0) := Q_0$ =(the set of all vertices of Q);

- $AP(1) := Q_1$ =(the set of all arrows of Q);
- $AP(2) := \rho$.

For $n \geq 3$, we define the set $AP(n)$ of all overlaps R^n formed in the following way: We say that $R^2 \in AP(2)$ maximally overlaps $R^{n-1} \in AP(n-1)$ with overlap $R^n = R^{n-1}u$ if

(1) $R^{n-1} = R^{n-2}p$ for some path p and $R^{n-2} \in AP(n-2)$;
(2) R^2 overlap p with overlap pu;
(3) there is no element of $AP(2)$ which overlaps p with overlap being a proper prefix of pu.

The construction of the paths in $AP(n)$ may be illustrated with the following picture of R^n:



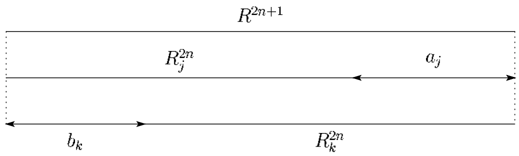
In short, overlaps are constructed by linking generators of an admissible monomial ideal I. A sequence of those generators of I is called the associated sequence of paths ([10]).

2.2. **Bardzell's resolution.** For a monomial algebra $\Lambda = kQ/I$, by using the set $AP(n)$, Bardzell determined a minimal projective Λ^{e}-resolution $(P_\bullet, \phi_\bullet)$ of Λ in [3].

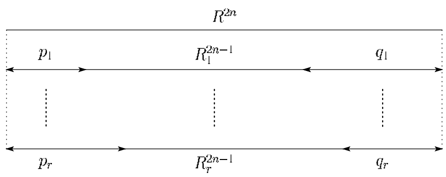
Definition 3. Let $(P_\bullet, \phi_\bullet)$ be the minimal projective Λ^{e}- resolution of Λ in [3]. Then, for $n \geq 0$, we set

$$P_n = \coprod_{R^n \in AP(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda.$$

From [3], if $R^{2n+1} \in AP(2n+1)$, then there uniquely exist R_j^{2n}, $R_k^{2n} \in AP(2n)$ and some paths a_j, b_k such that $R^{2n+1} = R_j^{2n} a_j = b_k R_k^{2n}$.



For even degree elements $R^{2n} \in AP(2n)$, there exist $r \geq 1$, $R_l^{2n-1} \in AP(2n-1)$ and paths p_l, q_l for $l = 1, 2, \ldots, r$ such that $R^{2n} = p_1 R_1^{2n-1} q_1 = \cdots = p_r R_r^{2n-1} q_r$.



Remark 4. Note that $o(R_j^{2n}) \otimes a_j \in \Lambda o(R_j^{2n}) \otimes t(R_j^{2n})\Lambda$ and $b_k \otimes t(R_k^{2n}) \in \Lambda o(R_k^{2n}) \otimes t(R_k^{2n})\Lambda$. Also, note that $p_l \otimes q_l \in \Lambda o(R_l^{2n-1}) \otimes t(R_l^{2n-1})\Lambda$.

Definition 5. The map $\phi_{2n+1} : P_{2n+1} \longrightarrow P_{2n}$ is given as follows. If $R^{2n+1} = R^{2n}_j a_j = b_k R^{2n}_k \in AP(2n+1)$, then

$$o(R^{2n+1}) \otimes t(R^{2n+1}) \longmapsto o(R^{2n}_j) \otimes a_j - b_k \otimes t(R^{2n}_k).$$

The map $\phi_{2n} : P_{2n} \longrightarrow P_{2n-1}$ is given as follows. If $R^{2n} = p_1 R^{2n-1}_1 q_1 = \cdots = p_r R^{2n-1}_r q_r$, then

$$o(R^{2n}) \otimes t(R^{2n}) \longmapsto \sum_{l=1}^{r} p_l \otimes q_l.$$

The following result is the main theorem in [3].

Bardzell's Theorem ([3, Theorem 4.1]) Let Q be a finite quiver, and suppose that $\Lambda = kQ/I$ is a monomial algebra with an admissible ideal I. Then the sequence

$$\cdots \to P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \to 0$$

is a minimal projective resolution of Λ as a right Λ^{e}-module, where π is the multiplication map.

3. The decomposition of Hochschild cohomology groups

We recall our setting.

- $Q = Q^{(1)} \cup Q^{(2)}$,
- $I^{(1)} = \langle X \rangle$ be a monomial ideal generated by X a set of paths of $kQ^{(1)}$,
- $I^{(2)} = \langle Y \rangle$ a monomial ideal generated by Y a set of paths of $kQ^{(2)}$,
- $I = \langle X, Y \rangle$ a monomial ideal of kQ,
- $\Lambda = kQ/I$, $\Lambda_{(1)} = kQ^{(1)}/I^{(1)}$, $\Lambda_{(2)} = kQ^{(2)}/I^{(2)}$: finite-dimensional algebras,
- $AP(2) := X \cup Y$, $AP^{(1)}(2) := X$, $AP^{(2)}(2) := Y$.

Then, as in the definition of $AP(n)$ of overlaps, we define $AP^{(1)}(n)$, $AP^{(2)}(n)$. Moreover, we define projective Λ^{e}-modules as follows:

$$P^{(1)}_n = \coprod_{R^n \in AP^{(1)}(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda,$$

$$P^{(2)}_n = \coprod_{R^n \in AP^{(2)}(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda,$$

$$P_n = \coprod_{R^n \in AP(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda.$$

To prove our main result, we need the following lemma. As mentioned in Introduction, we consider the *separability condition* $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$.

Lemma 6. *Let $i \in \{1, 2\}$. If we assume $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$, then we have the following*:

(a) *For all $n \geq 1$, $AP(n) = AP^{(1)}(n) \cup AP^{(2)}(n)$.*

(b) *For all $n \geq 1$, $AP^{(1)}(n) \cap AP^{(2)}(n) = \varnothing$.*

(c) *Let $n \geq 1$ and $p^n \in AP(n)$. Then R^n is a path of $kQ^{(i)}$ if and only if $R^n \in AP^{(i)}(n)$.*

By Bardzell's Theorem and Lemma 6, we have the following proposition.

Proposition 7. ([14, Proposition 3.2]) *If the condition $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$ holds, then, in the following minimal projective resolution of Λ:*

$$\cdots \to P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} P_{n-1} \to \cdots \xrightarrow{\phi_3} P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0,$$

for any $n \geq 1$, P_n is isomorphic to $P_n^{(1)} \oplus P_n^{(2)}$ as right Λ^e-modules and $\phi_{n+1} = \phi_{n+1}^{(1)} \oplus \phi_{n+1}^{(2)}$, where $\phi_{n+1}^{(i)} : P_{n+1}^{(i)} \to P_n^{(i)}$ $(i = 1, 2)$ is the restriction of ϕ_{n+1}.

Remark 8. For $i = 1, 2$, $b_k \in \Lambda_{(i)} o(R_k^{2n})$, $a_j \in t(R_j^{2n})\Lambda_{(i)}$, $p_l \in \Lambda_{(i)} o(R_j^{2n+1})$ and $q_l \in t(R_l^{2n+1})\Lambda_{(i)}$ actually hold. So, for $n \geq 1$, $\phi_{n+1}^{(i)}$ sends $\coprod_{R^{n+1} \in AP^{(i)}(n+1)} \Lambda_{(i)} o(R^{n+1}) \otimes t(R^{n+1})\Lambda_{(i)}$ to $\coprod_{R^n \in AP^{(i)}(n)} \Lambda_{(i)} o(R^n) \otimes t(R^n)\Lambda_{(i)}$ (not just to $\coprod_{R^n \in AP(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda$). Therefore, $(\coprod_{R^n \in AP^{(i)}(n)} \Lambda_{(i)} o(R^n) \otimes t(R^n)\Lambda_{(i)}; \phi_{n+1}^{(i)})_{n \geq 1}$ is exactly a part of degree $n \geq 1$ for the minimal projective resolution of $\Lambda_{(i)}$ $(i = 1, 2)$.

The following theorem is our main result.

Theorem 9. ([14, Theorem 3.3]) *If the condition $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$ holds and, for each $i = 1, 2$, $o(R^n)\Lambda t(R^n) = o(R^n)\Lambda_{(i)} t(R^n)$ holds for any $n \geq 1$ and any $R^n \in AP^{(i)}(n)$, then we have the direct sum decomposition of Hochschild cohomology groups*

$$\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$$

for any $n \geq 2$.

Remark 10. For $n = 0, 1$, the above equation fails in general (see Example 14 for the case $n = 1$).

If $Q_0^{(1)} \cap Q_0^{(2)} = \{v_0\}$ and $v_0 \Lambda v_0 = k v_0$, then we have $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$. Also, by Lemma 6 and Theorem 9, we have the following corollary.

Corollary 11. ([14, Corollary 3.4]) *In the case $Q_0^{(1)} \cap Q_0^{(2)} = \{v_0\}$ and $v_0 \Lambda v_0 = k v_0$, we have the direct sum decomposition of the Hochschild cohomology groups*

$$\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$$

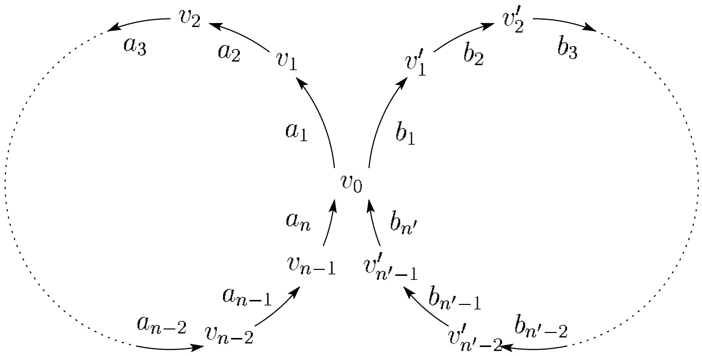
for any $n \geq 2$.

Remark 12. Hence, for a finite dimensional monomial algebra obtained by linking some quivers bound by monomial relations successively, we can also decompose the Hochschild cohomology groups as in Corollary 11.

4. Examples

In this section, we give two examples of monomial algebras satisfying the condition $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$.

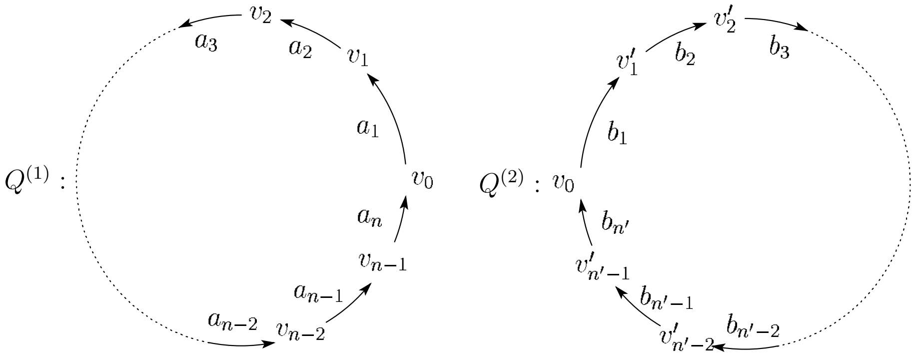
Example 13. Let Q be a quiver



bound by

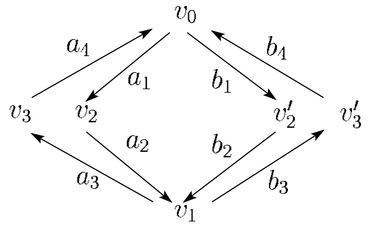
$$\begin{aligned} I =& \langle a_1a_2\cdots a_m, a_2a_3\cdots a_{m+1}, \dots, a_na_1\cdots a_{-n+m+1}, \\ & b_1b_2\cdots b_{m'}, b_2b_3\cdots b_{m'+1}, \dots, b_{n'}b_1\cdots b_{-n'+m'+1}\rangle \end{aligned}$$

for any integers m, $m' \geq 2$ with $m \leq n$ and $m' \leq n'$. We set the algebra $\Lambda = kQ/I$. Let $Q^{(1)}$ be the subquiver of Q bound by $I^{(1)} = \langle a_1a_2\cdots a_m, a_2a_3\cdots a_{m+1}, \dots, a_na_1\cdots a_{-n+m+1}\rangle$ and $Q^{(2)}$ be the subquiver of Q bound by $I^{(2)} = \langle b_1b_2\cdots b_{m'}, b_2b_3\cdots b_{m'+1}, \dots, b_{n'}b_1 \cdots b_{-n'+m'+1}\rangle$, where $Q_0^{(1)} \cap Q_0^{(1)} = \{v_0\}$ and $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$. We set $\Lambda_{(i)} = kQ^{(i)}/I^{(i)}$



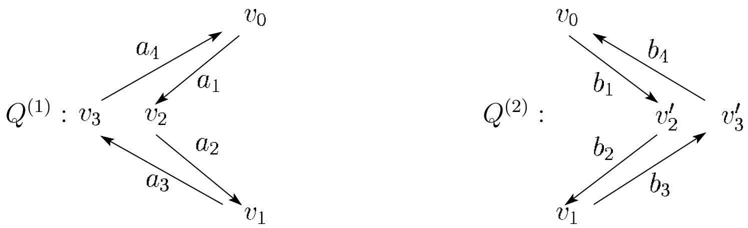
for $i = 1, 2$. Then the condition of Corollary 11 is satisfied. Applying Corollary 11, we obtain the direct sum decomposition of the Hochschild cohomology groups $\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$ for any $n \geq 2$. Also, since $\Lambda_{(i)}$ $(i = 1, 2)$ is a self-injective Nakayama algebra, we know the dimension of $\mathrm{HH}^n(\Lambda_{(i)})$ from [5, Propositions 4.4, 5.3] for $i = 1, 2$, and so we have the dimension of $\mathrm{HH}^n(\Lambda)$ by the decomposition above.

Example 14. Let Q be a quiver



bound by $I = \langle a_1a_2, a_2a_3, a_3a_4, a_4a_1, b_1b_2, b_2b_3, b_3b_4, b_4b_1\rangle$. We set the algebra $\Lambda = kQ/I$. Let $Q^{(1)}$ be the subquiver of Q bound by $I^{(1)} = \langle a_1a_2, a_2a_3,\ a_3a_4, a_4a_1\rangle$ and $Q^{(2)}$ be the subquiver of Q bound by $I^{(2)} = \langle b_1b_2, b_2b_3,\ b_3b_4, b_4b_1\rangle$, where $Q_0^{(1)} \cap Q_0^{(1)} = \{v_0, v_1\}$ and $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$.

We set $\Lambda_{(i)} = kQ^{(i)}/I^{(i)}$ for $i = 1, 2$. Then $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$ holds and for each $i = 1, 2$, $o(R^n)\Lambda t(R^n) = o(R^n)\Lambda_{(i)}t(R^n)$ holds for any $n \geq 1$ and any $R^n \in AP^{(i)}(n)$. Applying Theorem 9, we obtain the direct sum decomposition of the Hochschild cohomology groups $\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$ for any $n \geq 2$.



On the other hand, by direct computations, we have $\dim_k \mathrm{HH}^1(\Lambda) = 3$ and $\dim_k \mathrm{HH}^1(\Lambda_{(i)}) = 1$ $(i = 1, 2)$. Hence the above decomposition does not hold for $n = 1$.

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