DUALITIES IN STABLE CATEGORIES

MITSUO HOSHINO, NORITSUGU KAMEYAMA AND HIROTAKA KOGA

Abstract. We provide a sufficient condition for a left and right noetherian ring A to have finite selfinjective dimension on one side and, as a corollary to it, we also provide a necessary and sufficient condition for A to have finite selfinjective dimension on both sides.

Let A be a left and right coherent ring. We denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely presented right A-modules. We consider left A-modules as right A^{op}-modules, where A^{op} denotes the opposite ring of A. For each $n > 0$ we denote by $\mathcal{G}_A^n$ the full subcategory of mod-A consisting of $X \in$ mod-A with $\mathrm{Ext}_A^i(X, A) = 0$ for $1 \leq i \leq n$ and, for convenience's sake, we set $\mathcal{G}_A^0 =$ mod-A. We set $C_A = \oplus\, E_A(S)$, where S runs over the non-isomorphic simple modules in Mod-A. Such a module C_A is unique up to isomorphism and called a minimal cogenerator for Mod-A. Extending [9, Lemma A] to coherent rings, we showed in [5] that if flat dim $C_{A^{\mathrm{op}}} < \infty$ and flat dim $C_A < \infty$ then flat dim $C_{A^{\mathrm{op}}} =$ flat dim C_A.

In this note, we first show that for any $n \geq 0$ we have flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \leq n$ if and only if for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with Y Gorenstein projective and proj dim $Z \leq n - 1$.

Next, we provide a condition which implies flat dim $C_{A^{\mathrm{op}}} \leq n$. It is obvious that $C_{A^{\mathrm{op}}}$ is flat if and only if $\mathcal{G}_A^0 = \mathcal{G}_A^1$. Since for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^1$ and Z projective, it follows that flat dim $C_{A^{\mathrm{op}}} \leq 1$ if and only if $\mathcal{G}_A^1 = \mathcal{G}_A^2$. So, in the following, we assume $n \geq 2$.

We denote by $D(-)$ both $\mathbf{R}\mathrm{Hom}_A^\bullet(-, A)$ and $\mathbf{R}\mathrm{Hom}_{A^{\mathrm{op}}}^\bullet(-, A)$. Our main theorem states that flat dim $C_{A^{\mathrm{op}}} \leq n$ if the following three conditions are satisfied: (a) $\mathcal{G}_A^n = \mathcal{G}_A^{n+1}$; (b) $\mathrm{H}^i(D\sigma'_{\geq n-1}(\sigma_{\leq n}(DX))) = 0$ for all $X \in \mathcal{G}_A^{n-2}$ and $i \leq -2$; (c) for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^{n-2}$ and proj dim $Z \leq n - 1$. In the above, the condition (c) is always satisfied for $n = 2$ and 3. Also, as a corollary to this theorem, we show that flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \leq n$ if and only if the following three conditions are satisfied: (a) $\mathcal{G}_A^n$ consists only of Gorenstein projectives; (b) $\mathrm{H}^i(D\sigma'_{\geq n-1}(\sigma_{\leq n}(DX))) = 0$ for all $X \in \mathcal{G}_A^{n-2}$ and $i \leq -2$; (c) for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^{n-2}$ and proj dim $Z \leq n - 1$.

The detailed version of this paper will be submitted for publication elsewhere.

1. Stable module theory

For a ring A, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. In particular, we denote by $\mathrm{Hom}_A(-,-)$ (resp., $\mathrm{Hom}_{A^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-A (resp., Mod-A^{op}).

In this note, complexes are cochain complexes and modules are considered as complexes concentrated in degree zero. We denote by $\mathcal{K}(\text{Mod-}A)$ the homotopy category of complexes over Mod-A, by $\mathcal{K}^-(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}(\text{Mod-}A)$ consisting of bounded above complexes over $\mathcal{P}_A$ and by $\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}^-(\mathcal{P}_A)$ consisting of complexes with bounded cohomology. We denote by $\mathcal{D}(\text{Mod-}A)$ the derived category of complexes over Mod-A. Also, we denote by $\mathrm{Hom}^\bullet_A(-,-)$ the associated single complex of the double hom complex and by $\mathbf{R}\mathrm{Hom}^\bullet_A(-,A)$ the right derived functor of $\mathrm{Hom}^\bullet_A(-,A)$. We refer to [2], [4] and [8] for basic results in the theory of derived categories.

Definition 1. For a complex $X^\bullet$ and an integer $n \in \mathbb{Z}$, we denote by $\mathrm{Z}^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the nth cycle, the nth cocycle and the nth cohomology, respectively, and define the following truncations:

$$\sigma_{\le n}(X^\bullet) : \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Z}^n(X^\bullet) \to 0 \to \cdots ,$$
$$\sigma'_{\ge n}(X^\bullet) : \cdots \to 0 \to \mathrm{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots .$$

Note that for each $n \in \mathbb{Z}$ we have additive functors

$$\sigma_{\le n}(-), \sigma'_{\ge n}(-) : \mathcal{D}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$$

which are not exact.

Definition 2 ([3])**.** A module $X \in \text{Mod-}A$ is said to be coherent if it is finitely generated and every finitely generated submodule of it is finitely presented. A ring A is said to be left (resp., right) coherent if it is coherent as a left (resp., right) A-module.

Throughout the rest of this note, A is assumed to be a left and right coherent ring. Note that mod-A consists of coherent modules and is a thick abelian subcategory of Mod-A in the sense of [4].

We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ the full triangulated subcategory of $\mathcal{D}(\text{Mod-}A)$ consisting of complexes over mod-A with bounded cohomology. Note that the canonical functor $\mathcal{K}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$ gives rise to an equivalence of triangulated categories $\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$.

We denote by $D(-)$ both $\mathbf{R}\mathrm{Hom}^\bullet_A(-,A)$ and $\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-,A)$. There exists a bifunctorial isomorphism

$$\theta_{M^\bullet,X^\bullet} : \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A^{\mathrm{op}})}(M^\bullet, DX^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, DM^\bullet)$$

for $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ and $M^\bullet \in \mathcal{D}(\text{Mod-}A^{\mathrm{op}})$. For each $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ we set

$$\eta_{X^\bullet} = \theta_{DX^\bullet,X^\bullet}(\mathrm{id}_{DX^\bullet}) : X^\bullet \to D^2X^\bullet = D(DX^\bullet).$$

Definition 3 ([1] and [7]). A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have finite Gorenstein dimension if $DX^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})$ and if $\eta_{X^\bullet}$ is an isomorphism. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes of finite Gorenstein dimension.

For a module $X \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$, we set

$$\text{G-dim } X = \sup\{\ i \geq 0 \mid \mathrm{Ext}^i_A(X, A) \neq 0\}$$

if $X \neq 0$, and G-dim $X = 0$ if $X = 0$. Also, we set G-dim $X = \infty$ for a module $X \in \text{mod-}A$ with $X \notin \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$. Then G-dim X is called the Gorenstein dimension of $X \in \text{mod-}A$. A module $X \in \text{mod-}A$ is said to be Gorenstein projective if it has Gorenstein dimension zero.

Note that a module $X \in \text{mod-}A$ is Gorenstein projective if and only if it is reflexive, i.e., the canonical homomorphism

$$X \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A), x \mapsto (f \mapsto f(x))$$

is an isomorphism and $\mathrm{Ext}^i_A(X, A) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A) = 0$ for $i \neq 0$.

Definition 4. For each $X \in \text{mod-}A$, taking a projective resolution $P^\bullet \to X$ in mod-A, we set $\Omega^n X = \mathrm{Z}'^{-n}(P^\bullet)$ for $n \geq 0$ and $\mathrm{Tr}X = \mathrm{Z}'^{1}(\mathrm{Hom}^\bullet_A(P^\bullet, A))$.

We denote by $\underline{\text{mod}}$-A the residue category mod-$A/\mathcal{P}_A$ and by $\underline{\mathrm{Hom}}_A(-,-)$ the morphism set in $\underline{\text{mod}}$-A. Then we have additive functors

$$\mathrm{Tr} : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A^{\mathrm{op}} \quad \text{and} \quad \Omega^n : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A$$

for $n \geq 0$. We set $\Omega = \Omega^1$. Then Ω^n is the nth power of Ω for $n \geq 0$.

Proposition 5. *For any $n \geq 0$ there exists a bifunctorial isomorphism*

$$\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{Tr}(\Omega^n X), M) \xrightarrow{\sim} \underline{\mathrm{Hom}}_A(\mathrm{Tr}(\Omega^n M), X)$$

for $X \in \text{mod-}A$ and $M \in \text{mod-}A^{\mathrm{op}}$.

For each $n > 0$ we denote by $\mathcal{G}^n_A$ the full subcategory of mod-A consisting of $X \in \text{mod-}A$ with $\mathrm{Ext}^i_A(X, A) = 0$ for $1 \leq i \leq n$ and, for convenience's sake, we set $\mathcal{G}^0_A = \text{mod-}A$.

Corollary 6 (cf. [6, Proposition 1.1.1]). *For any $n \geq 0$ the pair of functors*

$$\mathrm{Tr} \circ \Omega^n : \mathcal{G}^n_A/\mathcal{P}_A \to \mathcal{G}^n_{A^{\mathrm{op}}}/\mathcal{P}_{A^{\mathrm{op}}} \quad \textit{and} \quad \mathrm{Tr} \circ \Omega^n : \mathcal{G}^n_{A^{\mathrm{op}}}/\mathcal{P}_{A^{\mathrm{op}}} \to \mathcal{G}^n_A/\mathcal{P}_A$$

defines a duality.

Lemma 7. *For any $n \geq 0$ the following are equivalent.*

(1) $\mathcal{G}^n_A = \mathcal{G}^{n+1}_A$.
(2) $\mathcal{G}^n_{A^{\mathrm{op}}}$ *consists only of torsionless modules.*

Lemma 8. *For any $n \geq 1$ and $X \in \text{mod-}A$ the following are equivalent.*

(1) G-dim $X \leq n$.
(2) *There exists an exact sequence $0 \to Z \to Y \to X \to 0$ in* mod-A *with Y Gorenstein projective and* proj dim $Z \leq n - 1$.

Lemma 9. *For any $X \in \text{mod-}A$ there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in* mod-A *with $Y \in \mathcal{G}^1_A$ and $Z \in \mathcal{P}_A$.*

2. Applications

In the following, we denote by $E_A(-)$ an injective envelope of a module in Mod-A and set $C_A = \oplus\ E_A(S)$, where S runs over the non-isomorphic simple modules in Mod-A. Such a module C_A is unique up to isomorphism and called a minimal cogenerator for Mod-A. We have seen in [5] that if flat dim $C_{A^{\mathrm{op}}} < \infty$ and flat dim $C_A < \infty$ then flat dim $C_{A^{\mathrm{op}}} =$ flat dim C_A.

According to Lemma 8, [5, Theorem 3.6] implies the following.

Proposition 10. *For any $n \geq 0$ the following are equivalent.*

(1) flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \leq n$.

(2) *For any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in* mod-A *with Y Gorenstein projective and* proj dim $Z \leq n-1$.

Remark 11. For any $n \geq 0$, flat dim $C_{A^{\mathrm{op}}} \leq n$ if and only if $\mathrm{Ext}_A^{n+1}(-, A)$ vanishes on mod-A. In particular, $C_{A^{\mathrm{op}}}$ is flat if and only if $\mathcal{G}_A^0 = \mathcal{G}_A^1$. Also, Lemma 9 implies that flat dim $C_{A^{\mathrm{op}}} \leq 1$ if and only if $\mathcal{G}_A^1 = \mathcal{G}_A^2$.

Throughout the rest of this note, we fix an integer $n \geq 2$.

Theorem 12. *We have* flat dim $C_{A^{\mathrm{op}}} \leq n$ *if the following three conditions are satisfied:*

(a) $\mathcal{G}_A^n = \mathcal{G}_A^{n+1}$;

(b) $\mathrm{H}^i(D\sigma'_{\geq n-1}(\sigma_{\leq n}(DX))) = 0$ *for all $X \in \mathcal{G}_A^{n-2}$ and $i \leq -2$;*

(c) *for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in* mod-A *with $Y \in \mathcal{G}_A^{n-2}$ and* pd $Z \leq n-1$.

In the above, the condition (c) is trivially satisfied if $n = 2$. Also, it follows by Lemma 9 that the condition (c) is satisfied for $n = 3$.

Corollary 13. *We have* flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \leq n$ *if and only if the following three conditions are satisfied:*

(a) $\mathcal{G}_A^n$ *consists only of Gorenstein projectives;*

(b) $\mathrm{H}^i(D\sigma'_{\geq n-1}(\sigma_{\leq n}(DX))) = 0$ *for all $X \in \mathcal{G}_A^{n-2}$ and $i \leq -2$;*

(c) *for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in* mod-A *with $Y \in \mathcal{G}_A^{n-2}$ and* pd $Z \leq n-1$.

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571, Japan
E-mail address: hoshino@math.tsukuba.ac.jp

Interdisciplinary Graduate School of Science and Technology
Shinshu University
Nagano, 390-8621, Japan
E-mail address: kameyama@math.shinshu-u.ac.jp

School of Information Environment
Tokyo Denki University
Chiba, 270-1382, Japan
E-mail address: koga@mail.dendai.ac.jp