BRAUER INDECOMPOSABILITY OF SCOTT MODULES

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Abstract. Let $p$ be a prime number and $k$ an algebraically closed field of characteristic $p$. Let $G$ be a finite group and $P$ a $p$-subgroup of $G$. In this article, we study the relationship between the fusion system $\mathcal{F}_P(G)$ and the Brauer indecomposability of the Scott $kG$-module in the case that $P$ is not necessarily abelian. We give an equivalent condition for the Scott $kG$-module with vertex $P$ to be Brauer indecomposable.

1. Introduction

Let $p$ be a prime number, $G$ a finite group, and $k$ an algebraically closed field of prime characteristic $p$. For a $kG$-module $M$ and a $p$-subgroup $Q$ of $G$, we denote by $M(Q)$ the Brauer quotient of $M$ with respect to $Q$. The Brauer quotient $M(Q)$ has naturally the structure of a $kN_G(Q)$-module.

Definition 1. A $kG$-module $M$ is said to be Brauer indecomposable if $M(Q)$ is indecomposable or zero as a $kQC_G(Q)$-module for any $p$-subgroup $Q$ of $G$.

Brauer indecomposability of $p$-permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2]).

Let $P$ be a $p$-subgroup of $G$. We denote by $\mathcal{F}_P(G)$ the fusion system of $G$ over $P$. In [1], a relationship between fusion system $\mathcal{F}_P(G)$ and Brauer indecomposability of $p$-permutation modules with vertex $P$ was given. One of the main result in [1] is the following.

Theorem 2 ([1, Theorem 1.1]). Let $P$ be a $p$-subgroup of $G$ and $M$ an indecomposable $p$-permutation $kG$-module with vertex $P$. If $M$ is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.

In the special case that $P$ is abelian and $M$ is the Scott $kG$-module $S(G, P)$, the converse of the above theorem holds.

Theorem 3 ([1, Theorem 1.2]). Let $P$ be an abelian $p$-subgroup of $G$. If $\mathcal{F}_P(G)$ is saturated, then $S(G, P)$ is Brauer indecomposable.

In general, the above theorem does not hold for non-abelian $P$. However, there are some cases in which the Scott $kG$-module $S(G, P)$ is Brauer indecomposable, even if $P$ is not necessarily abelian.

We study the condition that $S(G, P)$ to be Brauer indecomposable where $P$ is not necessarily abelian. The following result gives an equivalent condition for Scott $kG$-module with vertex $P$ to be Brauer indecomposable.

The detailed version of this paper will be submitted for publication elsewhere.
Theorem 4. Let $G$ be a finite group and $P$ a $p$-subgroup of $G$. Suppose that $M = S(G, P)$ and that $F_P(G)$ is saturated. Then the following are equivalent.

(i) $M$ is Brauer indecomposable.

(ii) For each fully normalized subgroup $Q$ of $P$, the module $\text{Res}^{N_G(Q)}_{QG}(N_G(Q), N_P(Q))$ is indecomposable.

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

The following theorem shows that $\text{Res}^{N_G(Q)}_{QG}(N_G(Q), N_P(Q))$ is indecomposable if $Q$ satisfies some conditions.

Theorem 5. Let $G$ be a finite group, $P$ a $p$-subgroup of $G$ and $Q$ a fully normalized subgroup of $P$. Suppose that $F_P(G)$ is saturated. Moreover, we assume that there is a subgroup $H_Q$ of $N_G(Q)$ satisfying following two conditions:

(i) $N_P(Q) \in \text{Syl}_p(H_Q)$

(ii) $|N_G(Q) : H_Q| = p^a$ ($a \geq 0$)

Then $\text{Res}^{N_G(Q)}_{QG}(N_G(Q), N_P(Q))$ is indecomposable.

The following is a consequence of above two theorems.

Corollary 6. Let $G$ be a finite group and $P$ a $p$-subgroup of $G$. Suppose that $F_P(G)$ is saturated. If for every fully normalized subgroup $Q$ of $P$ there is a subgroup $H_Q$ of $N_G(Q)$ satisfying the conditions of 5, then $S(G, P)$ is Brauer indecomposable.

Throughout this article, we denote by $L \cap_G H$ the set $\{gL \cap H \mid g \in G\}$ for subgroups $L$ and $K$ of $G$.

2. Preliminaries

2.1. Scott modules. First, We recall the definition of Scott modules and some of its properties:

Definition 7. For a subgroup $H$ of $G$, the Scott $kG$-module $S(G, H)$ with respect to $H$ is the unique indecomposable summand of $\text{Ind}_{H}^{G}k_H$ that contains the trivial $kG$-module.

If $P$ is a Sylow $p$-subgroup of $H$, then $S(G, H)$ is isomorphic to $S(G, P)$. By definition, the Scott $kG$-module $S(G, P)$ is a $p$-permutation $kG$-module.

By Green’s indecomposability criterion, the following result holds.

Lemma 8. Let $H$ be a subgroup of $G$ such that $|G : H| = p^a$ (for some $a \geq 0$). Then $\text{Ind}_{H}^{G}k_H$ is indecomposable. In particular, we have that $S(G, H) \cong \text{Ind}_{H}^{G}k_H$.

Hence, for $p$-subgroup $P$ of $G$, if there is a subgroup $H$ of $G$ such that $P$ is a Sylow $p$-subgroup of $H$ and $|G : H| = p^a$, then we have that $S(G, P) \cong \text{Ind}_{H}^{G}k_H$.

The following theorem gives us information of restrictions of Scott modules.
Theorem 9 ([3, Theorem 1.7]). Let $H$ be a subgroup of $G$ and $P$ a $p$-subgroup of $G$. If $Q$ is a maximal element of $P \cap_G H$, then $S(H, Q)$ is a direct summand of $\text{Res}^G_H S(G, P)$.

2.2. Brauer quotients. Let $M$ be a $kG$-module and $H$ a subgroup of $G$. Let $M^H$ be the set of $H$-fixed elements in $M$. For subgroups $L$ of $H$, we denote by $\text{Tr}^H_L$ the trace map $\text{Tr}^H_L : M^L \rightarrow M^H$. Brauer quotients are defined as follows.

Definition 10. Let $M$ be a $kG$-module. For a $p$-subgroup $Q$ of $G$, the Brauer quotient of $M$ with respect to $Q$ is the $k$-vector space

$$M(Q) := M^Q / (\sum_{R \leq Q} \text{Tr}^Q_R(M^R)).$$

This $k$-vector space has a natural structure of $kN_G(Q)$-module.

Proposition 11. Let $P$ be a $p$-subgroup of $G$ and $M = S(G, P)$. Then $M(P) \cong S(N_G(P), P)$.

Proposition 12. Let $M$ be an indecomposable $p$-permutation $kG$-module with vertex $P$. Let $Q$ be a $p$-subgroup of $G$. Then $Q \leq_G P$ if and only if $M(Q) \neq 0$.

2.3. Fusion systems. For a $p$-subgroup $P$ of $G$, the fusion system $\mathcal{F}_P(G)$ of $G$ over $P$ is the category whose objects are the subgroups of $P$, and whose morphisms are the group homomorphisms induced by conjugation in $G$.

Definition 13. Let $P$ be a $p$-subgroup of $G$

(i) A subgroup $Q$ of $P$ is said to be fully normalized in $\mathcal{F}_P(G)$ if $|N_P(\sigma Q)| \leq |N_P(Q)|$ for all $x \in G$ such that $\sigma Q \leq P$.

(ii) A subgroup $Q$ of $P$ is said to be fully automized in $\mathcal{F}_P(G)$ if $p \nmid |N_G(Q) : N_P(Q)C_G(Q)|$.

(iii) A subgroup $Q$ of $P$ is said to be receptive in $\mathcal{F}_P(G)$ if it has the following property:

for each $R \leq P$ and $\varphi \in \text{Iso}_{\mathcal{F}_P(G)}(R, Q)$, if we set

$$N_\varphi := \{ g \in N_P(Q) \mid \exists h \in N_P(R), c_g \circ \varphi = \varphi \circ c_h \},$$

then there is $\overline{\varphi} \in \text{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\overline{\varphi} |_R = \varphi$.

Saturated fusion systems are defined as follows.

Definition 14. Let $P$ be a $p$-subgroup of $G$. The fusion system $\mathcal{F}_P(G)$ is saturated if the following two conditions are satisfied:

(i) $P$ is fully normalized in $\mathcal{F}_P(G)$.

(ii) For each subgroup $Q$ of $P$, if $Q$ is fully normalized in $\mathcal{F}_P(G)$, then $Q$ is receptive in $\mathcal{F}_P(G)$.

For example, if $P$ is a Sylow $p$-subgroup of $G$, then $\mathcal{F}_P(G)$ is saturated.

3. Sketch of Proof

In this section, let $P$ be a $p$-subgroup of $G$ and $M$ the Scott module $S(G, P)$.

Lemma 15. If $Q \leq P$ is fully normalized in $\mathcal{F}_P(G)$, then $N_P(Q)$ is a maximal element of $P \cap_G N_G(Q)$.
By above lemma, we can show that $S(N_G(Q), N_P(Q))$ is a direct summand of $M(Q)$ for each fully normalized subgroup $Q$ of $P$. Therefore, we have that Theorem 4 (i) implies 4 (ii).

Assume that Theorem 4 (ii) holds. We prove that $\text{Res}^{N_G(Q)}_{QCG(Q)}(M(Q))$ is indecomposable for each $Q \leq P$ by induction on $|P : Q|$. Without loss of generality, we can assume that $Q$ is fully normalized. If $M(Q)$ is decomposable, then by the following lemma, we can show that there is a subgroup $R$ such that $Q < R \leq P$ and $\text{Res}^{N_G(R)}_{RCG(R)}$ is decomposable, this contradicts the induction hypothesis.

**Lemma 16.** Suppose that a subgroup $Q$ of $P$ is fully automatized and receptive. Then for any $g \in G$ such that $Q \leq gP$, we have that $N_{gP}(Q) \leq _{NG} N_{P}(Q)$.

Hence, $M(Q)$ is indecomposable, and isomorphic to $S(N_G(Q), N_P(Q))$. Consequently, Theorem 4 (ii) implies 4 (i).

Theorem 5 is proved by using properties of Scott modules and the following lemma.

**Lemma 17.** If $Q$ is fully automatized subgroup of $P$, and there is a subgroup $H_Q \leq N_G(Q)$ containing $N_P(Q)$ such that $|N_G(Q) : H_Q| = p^n$, then $C_G(Q)H_Q = N_G(Q)$.

### 4. Example

We set $p = 2$ and

$$G := \langle a, x, y \mid a^4 = x^2 = e, a^2 = y^2, xax = a^{-1}, ay = ya, xy = yx \rangle,$$

$$P := \langle a, xy \rangle.$$  

Then $G$ is a finite group of order 16, and $P$ is isomorphic to the quaternion group of order 8. Hence, $P$ is a non-abelian $p$-subgroup of $G$. One can easily show that $G$ and $P$ satisfy the hypothesis of the Corollary 6. Therefore, $S(G, P)$ is Brauer indecomposable.

**References**


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