BRAUER INDECOMPOSABILITY OF SCOTT MODULES

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Abstract. Let p be a prime number and k an algebraically closed field of characteristic p. Let G be a finite group and P a p-subgroup of G. In this article, we study the relationship between the fusion system $\mathcal{F}_P(G)$ and the Brauer indecomposability of the Scott kG-module in the case that P is not necessarily abelian. We give an equivalent condition for the Scott kG-module with vertex P to be Brauer indecomposable.

1. Introduction

Let p be a prime number, G a finite group, and k an algebraically closed field of prime characteristic p. For a kG-module M and a p-subgroup Q of G, we denote by $M(Q)$ the Brauer quotient of M with respect to Q. The Brauer quotient $M(Q)$ has naturally the structure of a $kN_G(Q)$-module.

Definition 1. A kG-module M is said to be Brauer indecomposable if $M(Q)$ is indecomposable or zero as a $kQC_G(Q)$-module for any p-subgroup Q of G.

Brauer indecomposability of p-permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2]).

Let P be a p-subgroup of G. We denote by $\mathcal{F}_P(G)$ the fusion system of G over P. In [1], a relationship between fusion system $\mathcal{F}_P(G)$ and Brauer indecomposability of p-permutation modules with vertex P was given. One of the main result in [1] is the following.

Theorem 2 ([1, Theorem 1.1]). *Let P be a p-subgroup of G and M an indecomposable p-permutation kG-module with vertex P. If M is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.*

In the special case that P is abelian and M is the Scott kG-module $S(G, P)$, the converse of the above theorem holds.

Theorem 3 ([1, Theorem 1.2]). *Let P be an abelian p-subgroup of G. If $\mathcal{F}_P(G)$ is saturated, then $S(G, P)$ is Brauer indecomposable.*

In general, the above theorem does not hold for non-abelian P. However, there are some cases in which the Scott kG-module $S(G, P)$ is Brauer indecomposable, even if P is not necessarily abelian.

We study the condition that $S(G, P)$ to be Brauer indecomposable where P is not necessarily abelian. The following result gives an equivalent condition for Scott kG-module with vertex P to be Brauer indecomposable.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 4. *Let G be a finite group and P a p-subgroup of G. Suppose that $M = S(G, P)$ and that $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent.*

(i) *M is Brauer indecomposable.*

(ii) *For each fully normalized subgroup Q of P, the module $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable.*

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

The following theorem shows that $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable if Q satisfies some conditions.

Theorem 5. *Let G be a finite group, P a p-subgroup of G and Q a fully normalized subgroup of P. Suppose that $\mathcal{F}_P(G)$ is saturated. Moreover, we assume that there is a subgroup H_Q of $N_G(Q)$ satisfying following two conditions:*

(i) $N_P(Q) \in Syl_p(H_Q)$

(ii) $|N_G(Q) : H_Q| = p^a$ $(a \geq 0)$

Then $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable.

The following is a consequence of above two theorems.

Corollary 6. *Let G be a finite group and P a p-subgroup of G. Suppose that $\mathcal{F}_P(G)$ is saturated. If for every fully normalized subgroup Q of P there is a subgroup H_Q of $N_G(Q)$ satisfies the conditions of 5, then $S(G, P)$ is Brauer indecomposable.*

Throughout this article, we denote by $L \cap_G H$ the set $\{{}^g L \cap H \mid g \in G\}$ for subgroups L and K of G.

2. Preliminaries

2.1. **Scott modules.** First, We recall the definition of Scott modules and some of its properties:

Definition 7. For a subgroup H of G, the Scott kG-module $S(G, H)$ with respect to H is the unique indecomposable summand of $\mathrm{Ind}^G_H k_H$ that contains the trivial kG-module.

If P is a Sylow p-subgroup of H, then $S(G, H)$ is isomorphic to $S(G, P)$. By definition, the Scott kG-module $S(G, P)$ is a p-permutation kG-module.

By Green's indecomposability criterion, the following result holds.

Lemma 8. *Let H be a subgroup of G such that $|G : H| = p^a$(for some $a \geq 0$). Then $\mathrm{Ind}^G_H k_H$ is indecomposable. In particular, we have that*

$$S(G, H) \cong \mathrm{Ind}^G_H.$$

Hence, for p-subgroup P of G, if there is a subgroup H of G such that P is a Sylow p-subgroup of H and $|G : H| = p^a$, then we have that

$$S(G, P) \cong \mathrm{Ind}^G_H k_H.$$

The following theorem gives us information of restrictions of Scott modules.

Theorem 9 ([3, Theorem 1.7]). *Let H be a subgroup of G and P a p-subgroup of G. If Q is a maximal element of $P \cap_G H$, then $S(H, Q)$ is a direct summand of $\mathrm{Res}^G_H S(G, P)$.*

2.2. **Brauer quotients.** Let M be a kG-module and H a subgroup of G. Let M^H be the set of H-fixed elements in M. For subgroups L of H, we denote by Tr^G_H the trace map $\mathrm{Tr}^H_L : M^L \longrightarrow M^H$. Brauer quotients are defined as follows.

Definition 10. Let M be a kG-module. For a p-subgroup Q of G, the Brauer quotient of M with respect to Q is the k-vector space

$$M(Q) := M^Q/(\sum_{R<Q} \mathrm{Tr}^Q_R(M^R)).$$

This k-vector space has a natural structure of $kN_G(Q)$-module.

Proposition 11. *Let P be a p-subgroup of G and $M = S(G, P)$. Then $M(P) \cong S(N_G(P), P)$.*

Proposition 12. *Let M be an indecomposable p-permutation kG-module with vertex P. Let Q be a p-subgroup of G. Then $Q \leq_G P$ if and only if $M(Q) \neq 0$.*

2.3. **Fusion systems.** For a p-subgroup P of G, the fusion system $\mathcal{F}_P(G)$ of G over P is the category whose objects are the subgroups of P, and whose morphisms are the group homomorphisms induced by conjugation in G.

Definition 13. Let P be a p-subgroup of G

(i) A subgroup Q of P is said to be fully normalized in $\mathcal{F}_P(G)$ if $|N_P({}^xQ)| \leq |N_P(Q)|$ for all $x \in G$ such that ${}^xQ \leq P$.

(ii) A subgroup Q of P is said to be fully automized in $\mathcal{F}_P(G)$ if $p \nmid |N_G(Q) : N_P(Q)C_G(Q)|$.

(iii) A subgroup Q of P is said to be receptive in $\mathcal{F}_P(G)$ if it has the following property: for each $R \leq P$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}_P(G)}(R, Q)$, if we set

$$N_\varphi := \{g \in N_P(Q) \mid \exists h \in N_P(R), c_g \circ \varphi = \varphi \circ c_h\},$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\overline{\varphi}\,|_R = \varphi$.

Saturated fusion systems are defined as follows.

Definition 14. Let P be a p-subgroup of G. The fusion system $\mathcal{F}_P(G)$ is saturated if the following two conditions are satisfied:

(i) P is fully normalized in $\mathcal{F}_P(G)$.

(ii) For each subgroup Q of P, if Q is fully normalized in $\mathcal{F}_P(G)$, then Q is receptive in $\mathcal{F}_P(G)$.

For example, if P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G)$ is saturated.

3. Sketch of Proof

In this section, let P be a p-subgroup of G and M the Scott module $S(G, P)$.

Lemma 15. *If $Q \leq P$ is fully normalized in $\mathcal{F}_P(G)$, then $N_P(Q)$ is a maximal element of $P \cap_G N_G(Q)$.*

By above lemma, we can show that $S(N_G(Q), N_P(Q))$ is a direct summand of $M(Q)$ for each fully normalized subgroup Q of P. Therefore, we have that Theorem 4 (i) implies 4 (ii).

Assume that Theorem 4 (ii) holds. We prove that $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)}(M(Q))$ is indecomposable for each $Q \leq P$ by induction on $|P : Q|$. Without loss of generality, we can assume that Q is fully normalized. If $M(Q)$ is decomposable, then by the following lemma, we can show that there is a subgroup R such that $Q < R \leq P$ and $\mathrm{Res}^{N_G(R)}_{RC_G(R)}$ is decomposable, this contradicts the induction hypothesis.

Lemma 16. *Suppose that a subgroup Q of P is fully automized and receptive. Then for any $g \in G$ such that $Q \leq {}^gP$, we have that $N_{^gP}(Q) \leq_{N_G(Q)} N_P(Q)$.*

Hence, $M(Q)$ is indecomposable, and isomorphic to $S(N_G(Q), N_P(Q))$. Consequently, Theorem 4 (ii) implies 4 (i).

Theorem 5 is proved by using properties of Scott modules and the following lemma.

Lemma 17. *If Q is fully automized subgroup of P, and there is a subgroup $H_Q \leq N_G(Q)$ containing $N_P(Q)$ such that $|N_G(Q) : H_Q| = p^a$, then $C_G(Q)H_Q = N_G(Q)$.*

4. Example

We set $p = 2$ and

$$
\begin{aligned}
G &:= \langle a, x, y \mid a^4 = x^2 = e, a^2 = y^2, \\
&\qquad\qquad xax = a^{-1}, ay = ya, xy = yx \rangle, \\
P &:= \langle a, xy \rangle.
\end{aligned}
$$

Then G is a finite group of order 16, and P is isomorphic to the quaternion group of order 8. Hence, P is a non-abelian p-subgroup of G. One can easily show that G and P satisfy the hypothesis of the Corollary 6. Therefore, $S(G, P)$ is Brauer indecomposable.

References

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