NOTES ON THE HOCHSCHILD HOMOLOGY DIMENSION AND TRUNCATED CYCLES

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Abstract. In this paper, we show that if an algebra KQ/I with an ideal I of KQ contained in R_Q^m for an integer $m \geq 2$ has an m-truncated cycle, then this algebra has infinitely many nonzero Hochschild homology groups, where R_Q denotes the arrow ideal. Consequently, such an algebra of finite global dimension has no m-truncated cycles and satisfies an m-truncated cycles version of the no loops conjecture.

1. Introduction

In [8], Happel remarks that if all the higher Hochschild cohomology groups vanish for a finite dimensional algebra, then does the algebra have finite global dimension? This is called "Happel's question". It is shown in [3] that this does not hold in general.

On the other hand, in [7], Han conjectures the homology version of Happel's question, that is, if all the higher Hochschild homology groups of a finite dimensional algebra vanish, then is the algebra of finite global dimension? Moreover, he shows that the counter example of Happel's question in [3] satisfies Han's conjecture in [7].

In [4], Han's conjecture is approached with focusing on the combinatorics of quivers of algebras. Specifically, it is shown that all algebras having a 2-truncated cycle in which the product of two consecutive arrows is always zero, have infinitely many nonzero Hochschild homology groups. Consequently, 2-truncated cycles version of the well-known "no loops conjecture"holds: algebras of finite global dimension have no 2-truncated cycles. In addition, for arbitrary integer $m \geq 2$, an m-truncated cycles version of the "no loops conjecture"is conjectured. In particular, it is shown that monomial algebras satisfy an m-truncated cycles version of the "no loops conjecture". For finite dimensional elementary algebras, in [9], it is shown that the no loops conjecture can be derived from an earlier result of Lenzing in [12] (cf. [10]).

In this paper, we show the following assertion: Let K be a field, Q a finite quiver, R_Q the arrow ideal of KQ and $m \geq 2$ a positive integer. If an algebra KQ/I with an ideal $I \subset KQ$ contained in R_Q^m has an m-truncated cycle, then KQ/I has infinitely many nonzero Hochschild homology groups (Theorem 6). Consequently, in the case I is an admissible ideal of KQ which is contained in R_Q^m, then KQ/I satisfies an m-truncated cycles version of the "no loops conjecture". That is, if KQ/I has finite global dimension, then it contains no m-truncated cycles (Corollary 7). This result generalizes the result [4, Corollary 3.3].

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2. Preliminaries

Let K be a commutative ring and A a unital K-algebra. Thus, there exists a nonzero ring homomorphism $K \to A$, whose image is contained in the center of A. We assume that A is finitely generated as a K-module. Throughout the paper, $\otimes$ denotes $\otimes_K$ for the sake of simplicity.

For each $n \geq 1$, we denote the n-fold tensor product $A \otimes \cdots \otimes A$ of A over K by $A^{\otimes n}$ and the enveloping algebra of A by A^e.

Definition 1 ([13])**.** The Hochschild complex is the following complex:

$$\cdots \to M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M,$$

where M is a left A^e-module, the module $M \otimes A^{\otimes n}$ is in degree n, and the map $b : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ is given by the formula

$$\begin{aligned} & b(x \otimes a_1 \otimes \cdots \otimes a_n) := xa_1 \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i (x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n a_n x \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

The n-th Hochschild homology group $HH_n(A, M)$ of A with coefficients in the left A^e-module M is defined by the n-th homology group of the Hochschild complex above. In particular, $HH_n(A, A)$ is simply called the n-th Hochschild homology group of A, which is denoted by $HH_n(A)$.

It is well known that if the unital K-algebra A is a projective K-module, then the n-th Hochschild homology group $HH_n(A)$ is given by $\mathrm{Tor}_n^{A^e}(A, A)$. Now we recall the definition of the bar resolution of A.

Definition 2 ([13])**.** Let A be a unital K-algebra. The following resolution of the left A^e-module A denoted by C^{bar} is called the *bar resolution*:

$$C^{\mathrm{bar}} : \longrightarrow A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \longrightarrow \cdots \longrightarrow A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0,$$

where μ is multiplication and b' is defined by $b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$.

Let A and B be two K-algebras and suppose that $f : A \to B$ is a K-algebra homomorphism. Then f is a homomorphism of rings, the composition map of f and the map $K \to A$ giving the K-algebra structure of A is equal to the map $K \to B$ giving the K-algebra structure of B. This implies that $bf^{\otimes(n+1)} = f^{\otimes n} b$, therefore $\{f^{\otimes n}\}_{n \in \mathbb{N}}$ is a chain map between the Hochschild complex of A and the one of B. For each $n \geq 0$, this map of Hochschild complexes induces a map $f^{\otimes(n+1)} : HH_n(A) \to HH_n(B)$ of Hochschild homology groups. The following fact is the key of the main theorem in [4]: if we can show that the image of $HH_n(A) \to HH_n(B)$ is nonzero, then this forces $HH_n(A)$ to be nonzero. This fact is also important for our main theorem.

Finally, in [4], the Hochschild homology dimension of the algebra A is defined by

$$\mathrm{HHdim}\, A = \sup\{n \in \mathbb{Z} \mid HH_n(A) \neq 0\},$$

which is treated in the main theorem.

3. The Hochschild homology of truncated quiver algebras

In this section, for a truncated quiver algebra we give elements in the complex, induced by Sköldberg's projective resolution $\boldsymbol{P}$, which correspond to nonzero homology classes.

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. For an arrow $\alpha \in Q_1$, its source and target are denoted by $s(\alpha)$ and $t(\alpha)$, respectively. A path in Q is a sequence of arrows $\alpha_1\alpha_2\cdots\alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \dots, n-1$. The set of all paths of length n is denoted by Q_n.

For a path γ of Q, $|\gamma|$ denotes the length of γ. A path γ is said to be a cycle if $|\gamma| \geq 1$ and its source and target coincide. The period of a cycle γ is defined by the smallest integer i such that $\gamma = \delta^j$ $(j \geq 1)$ for a cycle δ of length i, which is denoted by $\operatorname{per}\gamma$. A cycle is said to be a basic cycle if the length of the cycle coincides with its period. It is also called a proper cycle [7]. Denote by Q_n^{c} (respectively Q_n^{b}) the set of cycles (respectively basic cycles) of length n. Let $G_n = \langle g \rangle$ be the cyclic group of order n and the path $\alpha_1\cdots\alpha_{n-1}\alpha_n$ a cycle where α_i is an arrow in Q. Then we define the action of G_n on Q_n^{c} by $g\cdot(\alpha_1\cdots\alpha_{n-1}\alpha_n) := \alpha_n\alpha_1\cdots\alpha_{n-1}$, and Q_n^{c}/G_n denotes the set of all G_n-orbits on Q_n^{c}. Similarly, G_n acts on Q_n^{b}, and Q_n^{b}/G_n denotes the set of all G_n-orbits on Q_n^{b}. For $\bar\gamma \in Q_n^{\mathrm{c}}/G_n$, we denote by $\operatorname{per}\bar\gamma$ the period of γ, that is $\operatorname{per}\bar\gamma := \operatorname{per}\gamma$. For convenience we use the notation Q_0^{c}/G_0 for the set of vertices Q_0.

Sköldberg gives an projective resolution $\boldsymbol{P}$ of a truncated quiver algebra A. Moreover, by means of the complex $\bigoplus_i \bigoplus_{\bar\gamma\in Q_i^{\mathrm{c}}/G_i} K_{\bar\gamma,n}$ given by the following isomorphism:

$$A\otimes_{A^e} P_n \xrightarrow{\varphi} A\otimes_{KQ_0^e} K\Gamma^{(n)} \xrightarrow{\sim} \bigoplus_i \bigoplus_{\bar\gamma\in Q_i^{\mathrm{c}}/G_i} K_{\bar\gamma,n},$$

he gives the module structure of $HH_n(A)$, where the set $\Gamma^{(*)}$ is given by

$$\Gamma^{(i)} = \begin{cases} Q_{cm} & \text{if } i = 2c\ (c \geq 0), \\ Q_{cm+1} & \text{if } i = 2c+1\ (c \geq 0). \end{cases}$$

In order to prove our main theorem, we investigate elements in $A\otimes_{KQ_0^e}\Gamma^{(*)}$ which correspond to nonzero homology classes.

Lemma 3. *Let K be a field and $A = KQ/R_Q^m$ a truncated quiver algebra. For an element $\bar\gamma \in Q_{cm}^{\mathrm{c}}/G_{cm}$ with $\gamma = \alpha_1\cdots\alpha_{cm}(\alpha_1,\dots,\alpha_{cm}\in Q_1)$, the following elements correspond to non-zero homology classes:*

$$\alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i} \in A\otimes_{KQ_0^e}\Gamma^{((c-1)m+1)},$$

where $d = \gcd(m, \operatorname{per}\bar\gamma)$ and $i = 1, 2, \dots, d-1$.

Lemma 4. *Let K be a field and $A = KQ/R_Q^m$ a truncated quiver algebra. For an element $\bar\gamma \in Q_{cm+e}^{\mathrm{c}}/G_{cm+e}(1 \leq e \leq m-1)$ with $\gamma = \alpha_1\cdots\alpha_{cm+e}(\alpha_1,\dots,\ \alpha_{cm+e}\in Q_1)$, the following element corresponds to a non-zero homology class:*

$$\alpha_{cm+1}\cdots\alpha_{cm+e}\otimes\alpha_1\cdots\alpha_{cm} \in A\otimes_{KQ_0^e}\Gamma^{(cm)}.$$

We note that there is the following chain map in [6], which we denote by θ. This chain map θ induces a quasi-isomorphism $\mathrm{id}_A\otimes\theta : A\otimes_{A^e} C^{\mathrm{bar}} \to A\otimes_{A^e}\boldsymbol{Q}$, which we denote by θ for the sake of simplicity.

A chain map π from Cibils' projective resolution $\boldsymbol{Q}$ to $\boldsymbol{P}$ given in [1] induces a quasi-isomorphism $\bar{\pi} = \mathrm{id}_{\mathrm{A}} \otimes \pi : A \otimes_{A^e} \boldsymbol{Q} \longrightarrow A \otimes_{A^e} \boldsymbol{P}$. We use the following composition map of chain maps from the Hochschild complex to Sköldberg's complex by Φ;

$$\begin{array}{ccccc} A \otimes_{A^e} Q_n & \xleftarrow{\theta} & A \otimes_{A^e} (C^{\mathrm{bar}})_n = A \otimes_{A^e} A^{\otimes(n+2)} & \xleftarrow{\psi} & A^{\otimes(n+1)} \\ \big\downarrow{\scriptstyle \bar{\pi}} & & & & \\ A \otimes_{A^e} P_n & \xrightarrow{\varphi} & A \otimes_{KQ_0^e} K\Gamma^{(n)} \xrightarrow{\sim} \bigoplus\limits_i \bigoplus\limits_{\bar{\gamma} \in Q_i^c/G_i} K_{\bar{\gamma},n}, & & \end{array}$$

where ψ is given by $\psi(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes_{A^e} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$.

4. The m-truncated cycles version of the "no loops conjecture"

Let K be a field, Q a finite quiver, R_Q the arrow ideal of KQ and $m \geq 2$ a positive integer. In this section, we show that if an algebra KQ/I with $I \subset R_Q^m$ has an m-truncated cycle (see Definition 5), then the algebra has infinite Hochschild homology dimension. Moreover, we show that the algebra satisfies an m-truncated cycles version of the "no loops conjecture".

If $I \subset R_Q^2$ is an ideal in the path algebra KQ, then a finite sequence $\alpha_1, \ldots, \alpha_u$ of arrows which satisfies the equations $t(\alpha_i) = s(\alpha_{i+1})$ $(i = 1, \ldots, u-1)$ and $t(\alpha_u) = s(\alpha_1)$ is called *a cycle* in KQ/I in [4].

Definition 5 ([4]). A cycle $\alpha_1, \ldots, \alpha_u$ in KQ/I is *m-truncated* for an integer $m \geq 2$ if

$$\alpha_i \cdots \alpha_{i+m-1} = 0 \quad \text{and} \quad \alpha_i \cdots \alpha_{i+m-2} \neq 0 \quad \text{in } KQ/I$$

for all i, where the indices are modulo u.

By means of composition map Φ, we have the following our main theorem by the Lemma 3 and 4.

Theorem 6. *Let K be a field, Q a finite quiver and $I \subset KQ$ an ideal contained in R_Q^m. Suppose that KQ/I contains an m-truncated cycle $\alpha_1, \ldots, \alpha_u$. Then the following holds:*

(i) *Assume that* $\gcd(m, \mathrm{per}(\alpha_1 \cdots \alpha_u)) \neq 1$. *For every $n \geq 1$ with $un \equiv 0 \pmod{m}$, the element*

$$\begin{aligned} &\alpha_{(c-1)m+2} \cdots \alpha_{cm} \otimes \alpha_1 \otimes \alpha_2 \cdots \alpha_m \otimes \alpha_{m+1} \\ &\quad \otimes \alpha_{m+2} \cdots \alpha_{2m} \otimes \alpha_{2m+1} \otimes \cdots \otimes \alpha_{(c-2)m+2} \cdots \alpha_{(c-1)m} \otimes \alpha_{(c-1)m+1}, \end{aligned}$$

where $c = un/m$, represents a nonzero element in $HH_{2c-1}(KQ/I)$.

(ii) *Let e be an integer with $1 \leq e \leq m-1$. For every $n \geq 1$ with $un \equiv e \pmod{m}$, the element*

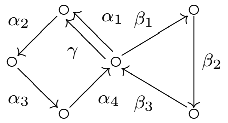
$$\begin{aligned} \sum_{0 \leq j_1, \ldots, j_c \leq m-2} &\alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{un} \\ &\otimes \alpha_1 \cdots \alpha_{1+j_1} \otimes \alpha_{2+j_1} \otimes \alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2} \otimes \alpha_{4+j_1+j_2} \otimes \cdots \\ &\quad \otimes \alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c} \otimes \alpha_{2c+j_1+\cdots+j_c}, \end{aligned}$$

where $c = (un-e)/m$, represents a nonzero element in $HH_{2c}(KQ/I)$.

In particular, the Hochschild homology dimension HHdim$(KQ/I) = \infty$.

Corollary 7. *Let K be a field, Q a finite quiver and I an admissible ideal in KQ with $I \subset R_Q^m$. If the algebra KQ/I has finite global dimension, then it contains no m-truncated cycles.*

Example 8. Let B be an algebra given by the quiver with relations:



$$\alpha_i\alpha_{i+1}\alpha_{i+2} = \beta_1\beta_2\beta_3 = \beta_3\gamma\alpha_2 = 0,$$
$$\beta_2\beta_3\alpha_1 = \beta_2\beta_3\gamma,$$

where the indices of α_i are modulo 4 ($1 \le i \le 4$). Then B has the 3-truncated cycle $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. By the Theorem 6, we have HHdim $B = \infty$. Therefore, the global dimension of B is infinite.

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ON A GENERALIZATION OF COMPLEXES AND THEIR DERIVED CATEGORIES.

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Abstract. When we want to understand the reason why the equation $d^2 = 0$ has the beautiful consequences, one way is to consider generalizations of it and research how its properties vary. One natural candidate of a generalization is the notion of N-complex, that is, gradeds object equipped with a morphism d of degree 1 such that $d^N = 0$. This was introduced by Kapranov [5] and Sarkaria [7] independently. Nowadays there is a vast collection of literatures on the subject.

For an N-complex X, there are several cohomology functors. More precisely, for $1 \leq r \leq N-1$, we define a cohomorogy functor to be

$$\mathrm{H}^i_{(r)}(X) := \frac{\mathrm{Ker}[d^r : X^i \to X^{i+r}]}{\mathrm{Im}[d^{N-r} : X^{i-N+r} \to X^i]}.$$

As a new feature, it is observed that there are several relations between these cohomology functors [5, 1].

On the other hands, Iyama-Kato-Miyachi [4] construct and study the homotopy category $\mathsf{K}_N(R)$, the derived category $\mathsf{D}_N(R)$ of N-complexes. They showed that the derived category $\mathsf{D}_N(R)$ is equivalent as triangulated categories to the derived category (in the ordinary sense) $\mathsf{D}(R \otimes_{\mathbf{k}} \mathbf{k}\overrightarrow{A}_{N-1})$. Inspired by their results, we introduce the notion of A-complexes for a graded self-injective algebra A. We construct and study the homotopy category, the derived category of and the cohomology functors. As a consequence, we see that the relations between various cohomology functors of N-complexes comes from representation theory of the graded algebra $\mathbf{k}[\delta]/(\delta^N)$ with $deg \mathbf{k} = 0, \deg \delta = 1$.

1. N-complexes (Kapranov, Sarkaria, G. Kato, Dubois-Violette, Hiramatsu-G. Kato, Iyama-K. Kato-Miyachi ...)

1.1. N-complexes. Our setup is the followings:

- $N \geq 2$ is an integer greater than 1.
- R is an algebra over a field $\mathbf{k}$.

For simplicity, in this note N-(A-)complexes are that of R-modules.

Definition 1. An *N-complex* X (of R-modules) is a graded R-module $\bigoplus_{i \in \mathbb{Z}} X^i$ equipped with an endomorphism d_X of degree 1 (the differential of X) such that $d_X^N = 0$.

$$d_X^N = d_X \circ d_X \circ \cdots d_X \qquad (N \text{ times }).$$

$$\cdots \to X^{i-1} \xrightarrow{d_X} X^i \xrightarrow{d_X} X^{i+1} \to \cdots$$

The detailed version of this paper will be submitted for publication elsewhere.