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Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium and committee members are listed in the web page

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The Symposium in 2015 will be held at Nagoya University for Sept. 7 (Mon.)–10 (Thu.) and the program will be arranged by T. Nishinaka (Okayama Shoka Univ.).

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http://fuji.cec.yamanashi.ac.jp/˜ring/ (in Japanese)
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http://fuji.cec.yamanashi.ac.jp/˜ring/japan/ (in English)

Shûichi Ikehata
Okayama Japan
February, 2015

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Preface

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Akihiko Hida
Saitama, Japan
February, 2015

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Brauer indecomposability of Scott modules

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A characterization of the class of Harada rings

τ-RIGID-FINITE ALGEBRAS WITH RADICAL SQUARE ZERO

TAKAHIDE ADACHI

Abstract. In this note, we study τ-rigid-finite algebras with radical square zero.

Throughout this note, by an algebra we mean a basic connected finite dimensional algebra over an algebraically closed field K. By a module we mean a finite dimensional right module. Let Λ be an algebra. For a Λ-module M with a minimal projective presentation $P^{-1} \xrightarrow{p} P^0 \to M \to 0$, we define a Λ-module τM by an exact sequence

$$0 \to \tau M \to \nu P^{-1} \xrightarrow{\nu p} \nu P^0,$$

where $\nu := \mathrm{Hom}_K(\mathrm{Hom}_\Lambda(-, \Lambda), K)$ is the Nakayama functor.

The following module plays an important role in this note.

Definition 1. A Λ-module M is *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$. We denote by τ-rigidΛ the set of isomorphism classes of indecomposable τ-rigid Λ-modules.

In 1980's, Auslander-Smalo [4] have already studied τ-rigid modules from the viewpoint of torsion theory. Recently, from the perspective of tilting mutation theory, the authors in [2] introduced the notion of (support) τ-tilting modules as a special class of τ-rigid modules. They correspond bijectively with many important objects in representation theory, *i.e.*, functorially finite torsion classes, two-term silting complexes and cluster-tilting objects in a special cases. By the following proposition, finiteness of these objects is induced by that of τ-rigidΛ.

Proposition 2. [5] *Let Λ be an algebra. The following are equivalent:*

(1) *The set τ-rigidΛ is finite.*
(2) *There are finitely many isomorphism classes of basic support τ-tilting Λ-modules.*

Definition 3. An algebra Λ is called *τ-rigid-finite* if it satisfies the equivalent conditions in Proposition 2.

Our aim of this note is to study τ-rigid-finite algebras with radical square zero. In the rest of this note, let Λ be an algebra with radical square zero and $Q = (Q_0, Q_1)$ the quiver of Λ, where Q_0 is the vertex set and Q_1 is the arrow set. Namely, $\Lambda = \Lambda_Q$ is the path algebra of a quiver Q modulo the ideal generated by all paths of length 2. In representation theory of algebras with radical square zero, the notion of the separated quiver play a central role. For a quiver $Q = (Q_0, Q_1)$, we define a new quiver $Q^s = (Q_0^s, Q_1^s)$, called the

The detailed version of this paper will be submitted for publication elsewhere.

separated quiver of Q, as follows:

$$Q^s_0 := \{i^+, i^- \mid i \in Q_0\}, \quad Q^s_1 := \{i^+ \to j^- \mid (i \to j) \in Q_1\}.$$

Note that the separated quiver Q^s is bipartite and not connected even if Q is connected.

$$Q: \begin{tikzcd}[row sep=small] & 2 \arrow[dl, shift left] \arrow[d, shift left] \\ 1 \arrow[ur, shift left] \arrow[dr, shift left] & \\ & 3 \arrow[ul, shift left] \arrow[u, shift left] \end{tikzcd} \quad Q^s: \begin{tikzcd}[row sep=small] & 2^- & 3^+ \arrow[l] \arrow[dr] & \\ 1^+ \arrow[ur] \arrow[dr] & & & 1^- \\ & 3^- & 2^+ \arrow[l] \arrow[ur] & \end{tikzcd} \quad Q: \begin{tikzcd} 1 \arrow[r, shift left] \arrow[d, shift left] & 4 \arrow[l, shift left] \arrow[d, shift left] \\ 2 \arrow[r, shift left] \arrow[u, shift left] & 3 \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd} \quad Q^s: \begin{tikzcd} 1^+ \arrow[r] \arrow[d] & 4^- & 1^- & 4^+ \arrow[l] \arrow[d] \\ 2^- & 3^+ \arrow[l] \arrow[u] & 2^+ \arrow[u] \arrow[r] & 3^- \end{tikzcd}$$

The following proposition is well-known result.

Proposition 4. [3, X.2.4] *Let Λ be an algebra with radical square zero and KQ^s the path algebra of the separated quiver of the quiver of Λ. Then two algebras Λ and KQ^s are stably equivalent, that is, there is an equivalent between the associated module categories modulo projectives.*

We have the following famous theorem characterizing representation-finiteness.

Theorem 5. [6] *Let Λ be an algebra with radical square zero and Q the quiver of Λ. The following are equivalent:*

(1) *Λ is representation-finite.*
(2) *The separated quiver Q^s is a disjoint union of Dynkin quivers.*

The following theorem is an analog of Theorem 5 for τ-rigid-finiteness. A full subquiver Q' of Q^s is called a *single subquiver* if, for any $i \in Q_0$, the vertex set Q'_0 contains at most one of i^+ or i^-.

Theorem 6. [1] *Let Λ be an algebra with radical square zero and Q the quiver of Λ. The following are equivalent:*

(1) *Λ is τ-rigid-finite.*
(2) *Each single subquiver of Q^s is a disjoint union of Dynkin quivers.*

We give some comment for loops of a quiver.

Remark 7. Let $Q = (Q_0, Q_1)$ be a quiver with a loop ℓ, and $Q' = (Q'_0, Q'_1)$ the quiver with $Q'_0 = Q_0$ and $Q'_1 = Q_1 \setminus \{\ell\}$. Then there is a natural bijection between the set of single subquiver of Q^s and those of Q'^s. Hence Λ_Q is τ-rigid-finite if and only if $\Lambda_{Q'}$ is τ-rigid-finite.

$$Q: \begin{tikzcd} 1 \arrow[loop right] \end{tikzcd} \quad Q^s: 1^+ \longrightarrow 1^-$$

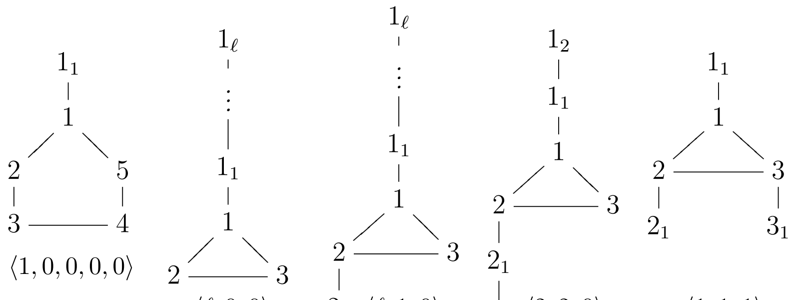
We give a main result of this note. Let $G = (V, E)$ be a connected graph, where V is the vertex set and E is the edge set. We define a quiver $Q_G = ((Q_G)_0, (Q_G)_1)$, called *the double quiver of* G, as follows:

$$(Q_G)_0 := V, \quad (Q_G)_1 := \{i \to j, \ i \leftarrow j \mid (i - j) \in E\}.$$

For non-negative integers $\ell_1, \ell_2, \ldots, \ell_n$, we define a graph $G := \langle \ell_1, \ldots, \ell_n \rangle$ as follows. G is an n-cycle such that each vertex v_i in the n-cycle is attached to a Dynkin graph A_{l_i} and the degree of v_i is at most three.

Theorem 8. *Let G be a connected graph with no loop. Then the following are equivalent:*

(1) Λ_{Q_G} *is* τ*-rigid-finite.*
(2) G *is one of the following graphs:*
  (a) *Dynkin graphs of type* A*,* D*, and* E*,*
  (b) *odd-cycles,*
  (c) $\langle 1,0,0,0,0\rangle$,
  (d) $\langle \ell,0,0\rangle$ $(1 \le \ell)$,
  (e) $\langle \ell,1,0\rangle$ $(1 \le \ell \le 4)$,
  (f) $\langle 2,2,0\rangle$,
  (g) $\langle 1,1,1\rangle$.



$\langle 1,0,0,0,0\rangle$ $\langle \ell,0,0\rangle$ $\langle \ell,1,0\rangle$ $\langle 2,2,0\rangle$ $\langle 1,1,1\rangle$

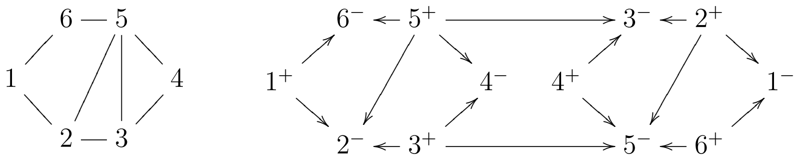
We can extend our theorem to the case of quivers/graphs with loops.

Remark 9. Assume that the quiver Q of Λ has a loop. By Remark 7, if there exists a graph G in Theorem 8 (2) such that Q_G is isomorphic to Q up to all loops, then Λ_Q is also τ-rigid-finite.

In the rest of this section, we give a proof of Theorem 8 by removing extended Dynkin graphs from connected single subquivers of the separated quiver. First we remove extended Dynkin graphs of type $\tilde{A}$ from the separated quiver. A graph is called an *n-cycle* if it is a cycle with exactly n vertices. In particular, it is called an *odd-cycle* if n is odd, and an *even-cycle* if n even. We write by $\overline{Q}$ the underlying graph of a quiver Q.

Lemma 10. *A graph G contains an even-cycle as a subgraph if and only if there exists a single subquiver Q' of Q_G^s such that $\overline{Q'}$ is an extended Dynkin graph of type $\tilde{A}$.*

Proof. Since Q_G^s is bipartite, all cycles as a subgraph in Q_G^s are even-cycles. Hence G contains an even-cycle as a subgraph. Conversely, assume that G contains an even-cycle as a subgraph. By taking a minimal even-cycle G' in G as a subgraph, $\overline{Q_G^s}$ includes G' as a full subgraph. Hence the assertion follows. □



By Lemma 10, we may assume that G contains no even-cycle as a subgraph. Since G is also bipartite, we have the following connection between G and Q_G^s. A *spanning tree of*

G is a subgraph of G that includes all of the vertices of G and is a tree. A *subtree of G* is a connected full subgraph of a spanning tree of G.

Proposition 11. *Let G be a graph with no even-cycle as a subgraph. Let G' be a graph. Then G' is a subtree of G if and only if there exists a connected single subquiver Q' of Q^s_G such that $\overline{Q'} = G'$. In particular, there is a naturally one-to-two correspondence between the set of subtrees of G and the set of connected single subquivers of Q^s_G.*

Proof. If G' is a subtree of G, then there exists a connected subquiver Q' of Q^s_G with $\overline{Q'} = G'$. By Lemma 10, Q' is clearly a full subquiver, and hence it is a single subquiver. Conversely, assume that Q' is a single subquiver Q' of Q^s_G with $\overline{Q'} = G'$. By Lemma 10, $\overline{Q'}$ is a tree. Since Q' is a full subquiver, $\overline{Q'}$ is a subtree of G by the definition of separated quivers. □

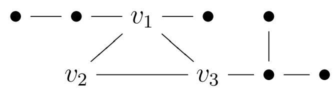
By Proposition 11, to remove non-Dynkin quivers from single subquivers of the separated quiver, we have only to concentrate on observing subtrees of graphs. For a tree, we have the following result.

Corollary 12. *Let G be a tree. Then the following are equivalent:*

(1) *Λ_{Q_G} is τ-rigid-finite.*
(2) *G is a Dynkin graph.*

Proof. Assume that G is a tree. G is Dynkin if and only if all subtrees of G are Dynkin. Thus the assertion follows from Theorem 6 and Proposition 11. □

By Corollary 12, we may assume that G contains exactly one odd-cycle and no even-cycles. Namely, G is an odd-cycle such that each vertex v in the odd-cycle is attached to a tree T_v.



We remove extended Dynkin graphs of type $\tilde{D}$ from the separated quiver Q^s_G.

Lemma 13. *Fix a positive integer k and $n := 2k+1$. Let G be an n-cycle such that each vertex v in the n-cycle is attached to a tree T_v. Then G contains an extended Dynkin graph of type $\tilde{D}$ as a subgraph if and only if it satisfies one of the following conditions:*

(a) *There is a vertex v in the n-cycle such that the degree is at least four.*
(b) *There is a vertex v in the n-cycle such that the degree is exactly three and T_v is not Dynkin graph of type A.*
(c) *$k > 1$ and there are at least two vertices in the n-cycle such that the degrees are at least three.*

Proof. Clearly, if G satisfies one of the conditions (a), (b), and (c), then it contains an extended Dynkin graph of type $\tilde{D}$. Conversely, assume that G contains an extended Dynkin graph of type $\tilde{D}$. Then $\tilde{D}_4$ has exactly one vertex whose degree is exactly four and $\tilde{D}_l$ has exactly two vertices whose degree is exactly three for any integer $\ell > 4$. We can check that G satisfies one of (a), (b), and (c). □

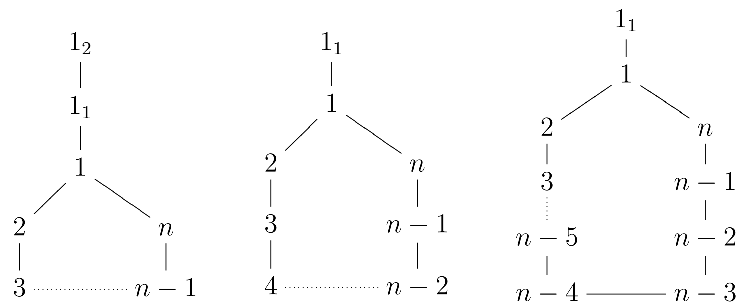
Fix a positive integer k and $n := 2k+1$. By Lemma 13, we may assume that G is one of the following graphs:

(a) $\langle \ell_1, 0 \ldots, 0\rangle$ if $k \geq 2$.
(b) $\langle \ell_1, \ell_2, \ell_3\rangle$ with $\ell_1 \geq \ell_2 \geq \ell_3$ if $k = 1$.

Finally, we remove extended Dynkin graphs of type $\tilde{E}$ from the separated quiver Q_G^s.

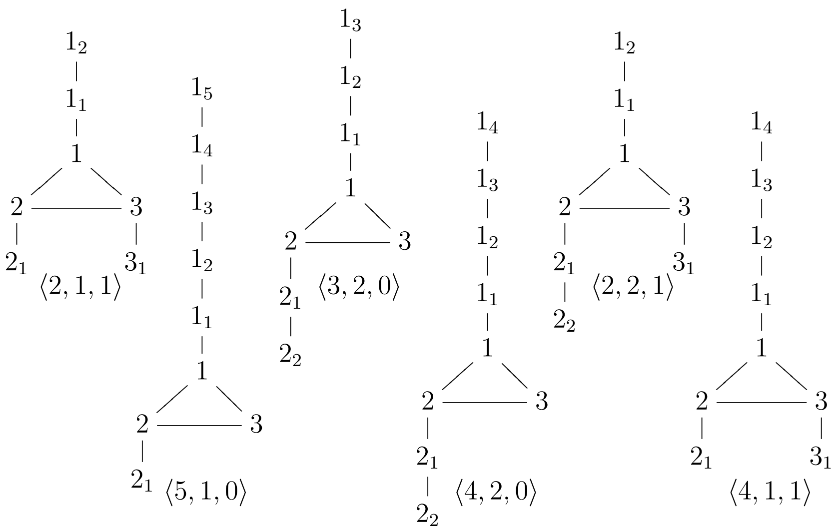
Lemma 14. *Fix a positive integer k and $n := 2k+1$. Assume that $G = \langle \ell_1, \ell_2, \cdots, \ell_n\rangle$.*

(1) *Assume that $k \geq 2$. The following graphs* (a), (b) *and* (c) *are the minimal graphs containing extended Dynkin graphs $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ respectively in the forms $\langle \ell_1, 0, \ldots, 0\rangle$.*
   (a) $\langle 2, 0, \ldots, 0\rangle$ $(k \geq 2)$
   (b) $\langle 1, 0, \ldots, 0\rangle$ $(k \geq 3)$
   (c) $\langle 1, 0, \ldots, 0\rangle$ $(k \geq 4)$



(a) $\ell_1 = 2$ *(b)* $\ell_1 = 1$ *if* $k \geq 3$ *(c)* $\ell_1 = 1$ *if* $k \geq 4$

(2) *Assume that $k = 1$. The following graphs* (d), (e) *and* (f) *are the minimal graphs containing extended Dynkin graphs $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ respectively in the forms $\langle \ell_1, \ell_2, \ell_3\rangle$.*
   (d) $\langle 2, 1, 1\rangle$.
   (e) $\langle 3, 2, 0\rangle$, $\langle 2, 2, 1\rangle$.
   (f) $\langle 5, 1, 0\rangle$, $\langle 4, 2, 0\rangle$, $\langle 4, 1, 1\rangle$.



Proof. We can check from the pictures above. $\square$

Now we are ready to prove Theorem 8.

Proof of Theorem 8. If G is a tree, then the assertion follows from Corollary 12. We assume that G is not a tree. By the argument above, we have the minimal set of graphs including extended Dynkin graphs of type $\tilde{A}$, $\tilde{D}$, or $\tilde{E}$. Thus Λ_{Q_G} is τ-rigid-finite if and only if G is one of nontrivial full subgraphs with the n-cycle of graphs in Lemma 14. The assertion follows from that G is the desired graph. $\square$

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ON SILTING-DISCRETE TRIANGULATED CATEGORIES

TAKUMA AIHARA

Abstract. The aim of this paper is to study silting-discrete triangulated categories. We establish a simple criterion for silting-discreteness in terms of 2-term silting objects. This gives a powerful tool to prove silting-discreteness of finite dimensional algebras. Moreover, we will show Bongartz-type Lemma for silting-discrete triangulated categories.

1. Introduction

In the study of triangulated categories, the class of tilting objects is one of the most important classes of objects, and tilting mutation for tilting objects often plays a crucial role, e.g. categorification of cluster algebras [8, 10] and Broué's conjecture in modular representation theory of finite groups [11]. From viewpoint of mutation, it was pointed out in [5] that one should deal with a more general class of silting objects than tilting objects, and silting mutation for silting objects were introduced. Moreover, the set of silting objects naturally has the structure of a partially orderd set which is closely related with silting mutation [5]. When a silting object is fixed, the partial order yields the notion of lengths of objects [3].

A problem is to understand the whole context of silting objects; e.g. to give a combinatorial description of silting objects. A triangulated category is called *silting-connected* provided all silting objects are reachable each other by iterated silting mutation. In this case, we can describe the combinatorial structure of the triangulated category in terms of silting objects and the relationship given by silting mutation. The *silting-discrete* triangulated categories are in some sense the simplest kinds of silting-connected triangulated categories [3], that is, the triangulated category admits a silting object A such that for any positive integer $\ell > 0$, there exist only finitely many silting objects of the length ℓ with respect to A: a finite dimensional algebra is also said to be *silting-discrete* if the perfect derived category of the algebra is silting-discrete. For example, we know that local algebras, path algebras of Dynkin type and representation-finite symmetric algebras are silting-discrete [5, 3].

We investigate silting-discrete triangulated categories and study the following question:

Question 1. *When is a triangulated category silting-discrete?*

The first aim of this pape is to give an answer to this question. A triangulated category is said to be *2-silting-finite* if for every silting object T, there exist only finitely many silting objects of the length 2 with respect to T.

A main result of this paper is the following theorem.

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Theorem 2 (Theorem 16)**.** *A triangulated category is silting-discrete if and only if it is 2-silting-finite.*

A great advantage of this theorem is that we can let Question 1 come down to the question of the finiteness of certain modules for algebras: For a silting object A, there is a one-to-one correspondence between silting objects of the length 2 with respect to A and support τ-tilting modules for the endomorphism algebra of A [2, 9].

Therefore, Theorem 2 gives a powerful tool to prove that a given finite dimensional algebra is silting-discrete. In fact, Theorem 2 will be applied in [1] and [4] to show that the following algebras are silting-discrete:

- Brauer graph algebras of type odd;
- Preprojective algebras of Dynkin type D_{2n}, E_7, E_8.

The second aim of this paper is to study a generalization of famous Bongartz's Lemma [6], which says that every (classical) pretilting module is partial tilting. On the other hand, a naive generalization of Bongartz's Lemma for tilting objects in a triangulated category fails: an easy example [12] shows that a pretilting object in a triangulated category is not necessarily partial tilting. In the previous paper [3], we observed that it is reasonable to consider Bongartz-type Lemma for silting objects in a triangulated category. Thus, we discuss the following question:

Question 3. *Is any presilting object partial silting?*

In this paper, we give a positive answer to Question 3 for silting-discrete triangulated categories.

Theorem 4 (Theorem 17)**.** *Any presilting object of a silting-discrete triangulated category is partial silting.*

A point for the proofs of Theorem 2 and Theorem 17 is to use a kind of induction on the length ℓ of a (pre)silting object T. To do this, we introduce the notion of "minimal silting objects" for T, which is a minimal element in a poset consisting of certain silting objects (see Definition 10 for details). The key result for the proofs of Theorem 2 and Theorem 17 is the following theorem.

Theorem 5 (Theorem 11)**.** *Let A be a silting object and T a presilting object of the length ℓ with respect to A. If there exists a minimal silting object P for T, then the length of T with respect to P is at most $\ell - 1$.*

This paper is organized as follows. In section 2, we introduce the notion of minimal silting objects and state a main theorem of this paper (Theorem 11). In section 3, we study silting-discrete triangulated categories and give the theorems on equivalent conditions of and Bongartz-type Lemma for silting-discrete triangulated categories (Theorem 16 and Theorem 17). In section 4, we give several examples of silting-discrete triangulated categories. Furthermore, we will know from the final example (Example 23) that the finiteness of silting objects of length 2 is not derived invariant.

Notation. Throughout this paper, let $\mathcal{T}$ be a Krull-Schmidt triangulated category and assume that it satisfies the following property:

(F) For any object X of $\mathcal{T}$, the additive closure $\mathsf{add}\, X$ is functorially finite in $\mathcal{T}$.

For example, let R be a complete local Noetherian ring and $\mathcal{T}$ an R-linear idempotent-complete triangulated category such that $\mathrm{Hom}_{\mathcal{T}}(X,Y)$ is a finitely generated R-module for any object X and Y of $\mathcal{T}$. Then $\mathcal{T}$ is a Krull-Schmidt triangulated category satisfying the property (F).

2. Minimal silting objects

In this section, we study silting mutation and a main theorem of this paper is stated.

Let us start with recalling the definition of silting objects.

Definition 6. (1) We say that an object T in $\mathcal{T}$ is *presilting* (*pretilting*) if it satisfies $\mathrm{Hom}_{\mathcal{T}}(T,T[i])=0$ for any $i>0$ ($i\neq 0$).
(2) An object T is said to be *silting* (*tilting*) if it is presilting (pretilting) and generates $\mathcal{T}$ by taking direct summands, mapping cones and shifts.
(3) A presilting object T is called *partial silting* provided it is a direct summand of some silting object.

We denote by $\mathsf{silt}\,\mathcal{T}$ the set of non-isomorphic basic silting objects in $\mathcal{T}$.

In the rest of this paper, we assume that $\mathcal{T}$ has a silting object.

It is known that the number of non-isomorphic indecomposable summands of any silting object does not depend on the choice of silting objects.

Proposition 7. [5] *Let T and U be silting objects of $\mathcal{T}$. Then the number of non-isomorphic indecomposable summands of T coincides with that of U.*

For objects M and N of $\mathcal{T}$, we write $M\geq N$ if $\mathrm{Hom}_{\mathcal{T}}(M,N[n])=0$ for any $n>0$. Note that $\geq$ is not a partial order on $\mathcal{T}$. According to [5], we have that $\geq$ gives a partial order on $\mathsf{silt}\,\mathcal{T}$.

We also recall silting mutation for silting objects.

Definition 8. Let T be a basic silting object of $\mathcal{T}$. For a decomposition $T:=X\oplus M$, we take a triangle

$$X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left $\mathsf{add}\,M$-approximation f of X. Then $\mu_X^-(T):=Y\oplus M$ is again silting, and we call it the *left mutation* of T with respect to X. Dually, define the right mutation $\mu_X^+(T)$. *(Silting) mutation* will mean either left or right mutation. Mutation is said to be *irreducible* if X is indecomposable.

We get basic properties of silting mutation.

Proposition 9. [5, 3] *With the notations as in Definition 8, the following hold:*

(1) We have the inequality $T>\mu_X^-(T)$.
(2) The right mutation $\mu_Y^+(\mu_X^-(T))$ of $\mu_X^-(T)$ with respect to Y is isomorphic to T.
(3) If X is indecomposable, then there is no silting object U satisfying $T>U>\mu_X^-(T)$.

(4) *Let U be a presilting object with $T \geq U$ which does not belong to $\mathsf{add}\, T$. For $U_0 := U$, take triangles*

$$\begin{array}{l} U_1 \longrightarrow T_0 \xrightarrow{f_0} U_0 \longrightarrow U_1[1] \\ \cdots \\ U_\ell \longrightarrow T_{\ell-1} \xrightarrow{f_{\ell-1}} U_{\ell-1} \longrightarrow U_\ell[1] \\ 0 \longrightarrow T_\ell \xrightarrow{f_\ell} U_\ell \longrightarrow 0 \end{array}$$

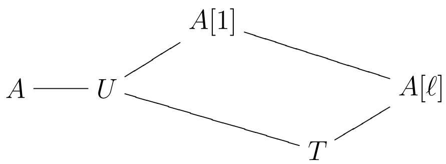
where f_i is a minimal right $\mathsf{add}\, T$-approximation of U_i for $0 \leq i \leq \ell$. Let X be an indecomposable summand of T. If X belongs to $\mathsf{add}\, T_\ell$, then we have $\mu_X^-(T) \geq U$.

We always use the following terminology.

Definition 10. We define a subset of $\mathsf{silt}\, \mathcal{T}$ as follows:

$$\nabla(A;T) := \{U \in \mathsf{silt}\, \mathcal{T} \mid A \geq U \geq A[1] \text{ and } U \geq T\},$$

where A is a silting object and T is a presilting object with $A \geq T$. We can take a non-negative interger ℓ such that $T \geq A[\ell]$. Thus, one visualize such a U as follows:



Now we state the main theorem of this paper.

Theorem 11. *If there exists a minimal element P in the poset $\nabla(A;T)$, then we have $T \geq P[\ell-1]$.*

We can inductively get silting objects.

Corollary 12. *With the notation as in Definition 10, assume that for any silting object B with $A \geq B \geq T$, the poset $\nabla(B;T)$ admits a minimal element. Then there exists a silting object P in $\mathcal{T}$ satisfying $P \geq T \geq P[1]$.*

Proof. We may assume $\ell \geq 2$. Since we have a minimal element A_1 in $\nabla(A;T)$, by Theorem 11 it is obtained that $A_1 \geq T \geq A_1[\ell-1]$. As our assumption, we can repeat this argument and have a sequence

$$A \geq A_1 \geq \cdots \geq A_{\ell-1} \geq T \geq A_{\ell-1}[1] \geq \cdots \geq A_1[\ell-1] \geq A[\ell]$$

of silting objects with A_{i+1} minimal in $\nabla(A_i;T)$ for $0 \leq i \leq \ell-2$. Thus, we get the desired silting object $P := A_{\ell-1}$. □

From Corollary 12 and [3, Proposition 2.16], we immediately obtain the following corollary.

Corollary 13. *Under the assumption as in Corollary 12, T is a partial silting object.*

3. Silting-discrete triangulated categories

In this section, we discuss silting-discrete triangulated categories.

We begin with recalling the definition of silting-discrete triangulated categories.

Definition 14. A triangulated category $\mathcal{T}$ is said to be *silting-discrete* if there exists a silting object A such that for any $\ell > 0$, the subset $\{T \in \mathsf{silt}\,\mathcal{T} \mid A \geq T \geq A[\ell]\}$ of $\mathsf{silt}\,\mathcal{T}$ is a finite set.

For a silting object A of $\mathcal{T}$, we denote by $\mathsf{2silt}_A\,\mathcal{T}$ the subset of $\mathsf{silt}\,\mathcal{T}$ consisting of all basic silting objects T with $A \geq T \geq A[1]$.

We can easily check the following lemma.

Lemma 15. *Let A be a silting object of $\mathcal{T}$. If $\mathsf{2silt}_A\,\mathcal{T}$ is a finite set, then for every presilting object T of $\mathcal{T}$ with $A \geq T$, the poset $\nabla(A;T)$ has a minimal element.*

We say that $\mathcal{T}$ is *2-silting-finite* if $\mathsf{2silt}_T\,\mathcal{T}$ is a finite set for any silting object T of $\mathcal{T}$.

Now the first main theorem of this section is stated.

Theorem 16. *The following are equivalent:*

(1) *$\mathcal{T}$ is silting-discrete.*
(2) *It is 2-silting-finite.*
(3) *It admits a silting object A such that $\mathsf{2silt}_P\,\mathcal{T}$ is a finite set for any iterated irreducible left mutation P of A.*

Proof. It is obvious that the implications (1)⇒(2)⇒(3) hold.

We show that the implication (3)⇒(1) holds. Let T be a silting object with $A \geq T \geq A[\ell]$ for some $\ell > 0$. Since $\mathsf{2silt}_A\,\mathcal{T}$ is a finite set, we observe that the poset $\nabla(A;T)$ has a minimal element P by Lemma 15. It follows from Theorem 11 that the inequalities $P \geq T \geq P[\ell-1]$ hold, whence one has

$$\{T \in \mathsf{silt}\,\mathcal{T} \mid A \geq T \geq A[\ell]\} \subseteq \bigcup_{P \in \mathsf{2silt}_A\,\mathcal{T}} \{U \in \mathsf{silt}\,\mathcal{T} \mid P \geq U \geq P[\ell-1]\}.$$

By [3, Theorem 3.5], the finiteness of $\mathsf{2silt}_A\,\mathcal{T}$ leads to the conclusion that P can be obtained from A by iterated irreducible left mutation. Therefore, our assumption yields that $\mathsf{2silt}_P\,\mathcal{T}$ is also a finite set. Repeating this argument leads to the assertion. □

We remark that the finiteness of $\mathsf{2silt}_P\,\mathcal{T}$ depends on the choice of silting objects P: For a left mutation P of a silting object A, the set $\mathsf{2silt}_P\,\mathcal{T}$ is not necessarily a finite set even if $\mathsf{2silt}_A\,\mathcal{T}$ is finite (see Example 23).

Finally, we have the second main theorem of this section, which is a direct consequence of Corollary 13.

Theorem 17. *If $\mathcal{T}$ is silting-discrete, then every presilting object is partial silting.*

4. Examples

This section is devoted to giving several examples of silting-discrete triangulated categories.

The first example is an observation from the viewpoint of triangle dimensions in the sense of Rouquier [13]: a triangulated category $\mathcal{T}$ has *triangle dimension 0* ($\dim \mathcal{T} = 0$) if $\mathcal{T} = \mathsf{add}\{M[i] \mid i \in \mathbb{Z}\}$ for some object M of $\mathcal{T}$.

Example 18. If $\dim \mathcal{T} = 0$, then $\mathcal{T}$ is silting-discrete.

In the rest of this paper, let Λ be a finite dimensional algebra over an algebraically closed field k which is indecomposable and basic. We denote by $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ the bounded homotopy category of finitely generated projective Λ-modules. Then it is a Krull-Schmidt triangulated category satisfying the property (F).

An algebra Λ is said to be *silting-discrete* if $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ is silting-discrete.

We give several examples of silting-discrete algebras. The most easiest example of silting-discrete algebras is the class of local algebras [5].

We characterize silting-discrete hereditary algebras.

Example 19. Assume that Λ is hereditary. Then the following are equivalent:

(1) Λ is silting-discrete;
(2) It is of Dynkin type A, D, E;
(3) $\mathsf{2silt}_\Lambda(\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda))$ is a finite set.

Proof. We can easily show the implications $(2)\overset{\text{Ex.18}}{\Longrightarrow}(1)\overset{\text{Def.}}{\Longrightarrow}(3)\overset{\text{Easy}}{\Longrightarrow}(2)$. □

A concept of derived-discrete algebras was introduced in [14]: an algebra Λ is said to be *derived-discrete* if for every positive element x of $K_0(A)^{(\mathbb{Z})}$, there exist only finitely many isomorphism classes of indecomposable objects X of the bounded derived category $\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ such that $(\underline{\dim} H^i(X))_{i\in\mathbb{Z}} = x$ where $K_0(A), \underline{\dim} M$ and H^i stand for the Grothendieck group of $\operatorname{mod}\Lambda$, the dimension vecter of a module M and the i-th cohomological functor.

Recently, the following result was proved by Broomhead-Pauksztello-Ploog.

Example 20. [7]Any derived-discrete algebra with finite global dimension is silting-discrete.

We know two classes of silting-discrete symmetric algebras.

Example 21. [3, 1] An algebra Λ is silting-discrete if it is either

(1) a representation-finite symmetric algebra or
(2) a Brauer graph algebra of type odd.

The following example was shown by a joint work with Y. Mizuno.

Example 22. [4] The preprojective algebra of Dynkin type $D_{2n} (n \geq 2), E_7, E_8$ is silting-discrete.

We close this paper by giving an example which says that the finiteness of $\mathsf{2silt}_P \mathcal{T}$ depends on the choice of silting objects P.

Example 23. Let Λ be the algebra presented by the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "x_2"] & \\ 1 \arrow[ur, "x_1"] \arrow[dr, "y_1"'] & & 4 \\ & 3 \arrow[ur, "y_2"'] & \end{tikzcd}$$

with relations $x_1x_2 = 0 = y_1y_2$. Then $\mathsf{2silt}_\Lambda(\mathsf{K}^\mathrm{b}(\operatorname{proj}\Lambda))$ is a finite set. Now, let $T := \mu^-_{P_2}\mu^-_{P_3}\mu^-_{P_4}(\Lambda)$, which is isomorphic to a tilting module whose endomorphism algebra Γ is the path algebra obtained by the quiver

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 4 \\ & 3 \arrow[ur] & \end{tikzcd} .$$

We conclude from Example 19 that $\mathsf{2silt}_\Gamma(\mathsf{K}^\mathrm{b}(\operatorname{proj}\Gamma))$, hence $\mathsf{2silt}_T(\mathsf{K}^\mathrm{b}(\operatorname{proj}\Lambda))$, is not a finite set.

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TILTING COMPLEXES OVER PREPROJECTIVE ALGEBRAS OF DYNKIN TYPE

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Abstract. In this note, we explain a connection between braid groups and tilting complexes over preprojective algebras of Dynkin (A,D,E) type. More precisely, we classify all tilting complexes by giving a bijection with elements of the braid groups.

1. Introduction

Derived categories are nowadays considered as a fundamental object in many branches of mathematics including representation theory and algebraic geometry. One of the important problems is to study their equivalences. By Rickard's Morita theorem for derived categories, it is known that derived equivalences are controlled by tilting complexes [28]. Tilting theory provides several useful methods for studying tilting complexes and, in particular, mutation plays a significant role. Roughly speaking, mutation is an operation, for a certain class of objects, to obtain a new object from a given one by replacing a summand. In the case of tilting modules, their mutation was formulated by Riedtmann-Schofield and Happel-Unger [30, 16, 32]. For example, APR (Auslander-Platzeck-Reiten) tilting modules [5] and Okuyama-Rickard complexes [29, 27, 18] can be regarded as a special case of tilting mutation. One of the negative aspects of tilting mutation is that some summands of a tilting complex can not be replaced to get a new one and hence we can not repeat tilting mutation. To remove this disadvantage, Aihara-Iyama studied a wider class of mutation, called silting mutation and it is shown that silting mutation is always possible and it admits a combinatorial description [4].

We give a further development of tilting (silting) theory and we determine all tilting complexes over preprojective algebras of Dynkin type.

2. Main results

2.1. **Preprojective algebras.** Preprojective algebras was first introduced by Gelfand-Ponomarev [15], and later formulated and developed in [14, 7]. Since then, they are one of the fundamental objects in the representation theory (refer to a survey paper [31]).

Let K be an algebraically closed field and Q a finite connected acyclic quiver. We denote by $\overline{Q}$ the double quiver of Q, which is obtained by adding an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q_1. The *preprojective algebra* $\Lambda_Q = \Lambda$ associated to Q is the algebra $K\overline{Q}/I$, where I is the ideal in the path algebra $K\overline{Q}$ generated by the relations of

The detailed version of this paper will be submitted for publication elsewhere.

the form:

$$\sum_{a\in Q_1}(aa^* - a^*a).$$

Let Q be a Dynkin quiver and e_i the primitive idempotent of Λ associated with $i \in Q_0$. Then the preprojective algebra of Q is finite dimensional and selfinjective [11, Theorem 4.8]. We denote the Nakayama permutation of Λ by $\sigma : Q_0 \to Q_0$ (i.e. $D(\Lambda e_{\sigma(i)}) \cong e_i\Lambda$, where $D := \mathrm{Hom}_K(-, K)$).

Note that Λ_Q does not depend on the orientation of Q.

2.2. **Weyl group.** We refer to [8, 19] for basic properties of the Weyl (Coxeter) group. Let Q be a quiver of type $A, B(C), D, E$ and F. The *Weyl group* W_Q associated to Q is defined by the generators s_i ($i \in Q_0$) and relations $(s_is_j)^{m(i,j)} = 1$, where

$$m(i,j) := \begin{cases} 1 & \text{if } i = j; \\ 2 & \text{if no edge between } i \text{ and } j; \\ 3 & \text{if there is an edge } i \text{ --- } j, \\ 4 & \text{if there is an edge } i \overset{4}{\text{---}} j. \end{cases}$$

Each element $w \in W_Q$ can be written in the form $w = s_{i_1} \cdots s_{i_k}$. If k is minimal among all such expressions for w, then k is called the *length* of w and we denote by $l(w) = k$. In this case, we call $s_{i_1} \cdots s_{i_k}$ a *reduced expression* of w.

Let σ be the Nakayama permutation of Λ. Then σ acts on an element of the Weyl group W_Q by $\sigma(w) := s_{\sigma(i_1)}s_{\sigma(i_2)} \cdots s_{\sigma(i_\ell)}$ for $w = s_{i_1}s_{i_2} \cdots s_{i_\ell} \in W_Q$. We define the subgroup W_Q^σ of W_Q by

$$W_Q^\sigma := \{w \in W \mid \sigma(w) = w\}.$$

Then we have the following result. (See [13, Chapter 13]).

Theorem 1. *Let Q be a Dynkin (A,D,E) quiver and W_Q the Weyl group of Q. Let $Q' = Q$ if Q is type D_{2n}, E_7 and E_8. Otherwise, let Q' be a quiver, respectively, given by the following type.*

Q	A_{2n-1}, A_{2n}	D_{2n+1}	E_6
Q'	B_n	B_{2n}	F_4

Then W_Q^σ is isomorphic to $W_{Q'}$.

We call the quiver Q' given in Theorem 1 the *folding quiver* of Q.

Example 2. Let Q be a quiver of type A_5. Then one can check that W_Q^σ is given by $\langle s_1, (s_2s_4), (s_3s_5)\rangle$ and this group is isomorphic to $W_{Q'}$, where Q' is a quiver of type B_3.

2.3. **Support τ-tilting modules.** The notion of support τ-tilting modules was introduced in [2], as a generalization of tilting modules. We refer to [2, 21] for several nice properties of support τ-tilting modules.

Let Λ be a finite dimensional algebra and we denote by τ the AR translation [6].

Definition 3. We call a Λ-module X *τ-tilting* if X is $\mathrm{Hom}_\Lambda(X, \tau X) = 0$ and $|X| = |\Lambda|$, where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X.

Moreover, we call a Λ-module X *support τ-tilting* if there exists an idempotent e of Λ such that X is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

We denote by $\mathrm{s}\tau\text{-tilt}\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules.

Remark 4. We note that support τ-tilting modules can be described as pairs. These definition are essentially same.

Now let Q be a Dynkin quiver with $Q_0 = \{1, \ldots, n\}$ and Λ the preprojective algebra of Q. We denote by $I_i := \Lambda(1 - e_i)\Lambda$ for $i \in Q_0$. We denote by $\langle I_1, \ldots, I_n\rangle$ the set of ideals of Λ which can be written as

$$I_{i_1} I_{i_2} \cdots I_{i_k}$$

for some $k \geq 0$ and $i_1, \ldots, i_k \in Q_0$.

Then following result plays an important role in this note.

Theorem 5. [9, 25] *Under the above notation,*

(a) *There exists a bijection $W_Q \to \langle I_1, \ldots, I_n\rangle$, which is given by $w \mapsto I_w = I_{i_1} I_{i_2} \cdots I_{i_k}$ for any reduced expression $w = s_{i_1} \cdots s_{i_k}$.*

(b) *It gives a bijection between the elements of the Weyl group W_Q and the set $\mathrm{s}\tau\text{-tilt}\Lambda$ of isomorphism classes of basic support τ-tilting Λ-modules.*

We remark that the above ideals I_w are tilting modules in the case of non-Dynkin type in [20, 9].

2.4. **Silting complexes.** Silting complexes are a generalization of tilting complexes, which were introduced by Keller-Vossieck [23]. They were originally invented as a tool for studying tilting complexes. Nonetheless, silting complexes have turned out to have deep connections with several important complexes such as t-structures [10, 24, 12, 22].

We recall the definition of silting complexes as follows.

Definition 6. Let Λ be a finite dimensional algebra and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ the bounded homotopy category of the finitely generated projective Λ-modules.

(a) We call a complex P in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is *presilting* (respectively, *pretilting*) if it satisfies $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, P[i]) = 0$ for any $i > 0$ (respectively, $i \neq 0$).

(b) We call a complex P in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ *silting* (respectively, *tilting*) if it is presilting (respectively, pretilting) and the smallest thick subcategory containing P is $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.

We denote by $\mathrm{silt}\,\Lambda$ (respectively, $\mathrm{tilt}\,\Lambda$) the set of non-isomorphic basic silting (respectively, tilting) complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.

For complexes P and Q of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$, we write $P \geq Q$ if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, Q[i]) = 0$ for any $i > 0$. Then the relation $\geq$ gives a partial order on $\mathrm{silt}\,\Lambda$ [4, Theorem 2.11] (cf. [17]).

Moreover, a complex $T \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is called *2-term* provided it is concerned in the degree 0 and -1. We denote by $2\text{-silt}\,\Lambda$ (respectively, $2\text{-tilt}\,\Lambda$) the subset of $\mathrm{silt}\,\Lambda$ (respectively, $\mathrm{tilt}\,\Lambda$) consisting of 2-term complexes. Note that a complex T is 2-term if and only if $\Lambda \geq T \geq \Lambda[1]$.

Then we have the following nice correspondence.

Theorem 7. [2, Theorem 3.2] *Let Λ be a finite dimensional algebra. There exists a bijection*

$$\mathrm{s}\tau\text{-tilt}\Lambda \leftrightarrow 2\text{-silt}\,\Lambda.$$

By the above correspondence, we can give a description of 2-term silting complexes by calculating support τ-tilting modules, which is much simpler than calculations of silting complexes.

From now on, let Q be a Dynkin quiver and Λ the preprojective algebra of Q. Then, as a corollary of Theorem 5 and 7, we have the following corollary.

Corollary 8. *We have a bijection*

$$W_Q \leftrightarrow 2\text{-silt}\,\Lambda.$$

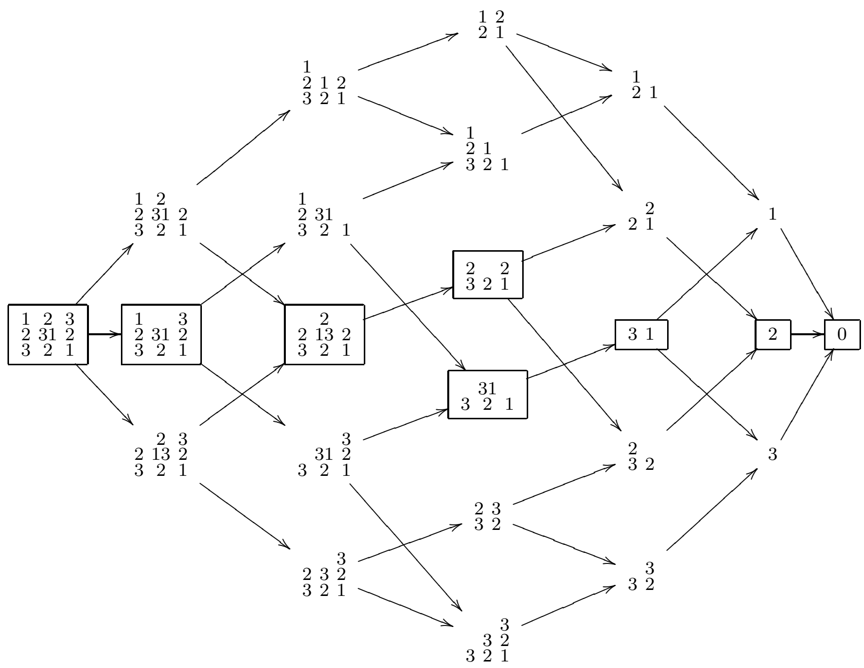
Thus we can parameterize 2-term silting complexes by the Weyl group. Moreover, we can describe 2-term tilting complexes in terms of the Weyl group by the following proposition.

Proposition 9. *Let $\nu := D\operatorname{Hom}_\Lambda(-,\Lambda)$ the Nakayama functor of Λ and $\sigma : Q_0 \to Q_0$ the Nakayama permutation of Λ. Then $\nu(I_w) \cong I_w$ if and only if $\sigma(w) = w$. In particular, We have a bijection*

$$W_Q^\sigma \leftrightarrow 2\text{-tilt}\,\Lambda.$$

Then by Theorem 1, we can understand W_Q^σ as another type of the Weyl group.

Example 10. Let Q be a quiver of type A_3 and Λ the preprojective algebra of Q. Then the *support τ-tilting quiver* of Λ ([2, Definition 2.29]) is given as follows.



The framed modules indicate *ν-stable* modules [26] (i.e. $I_w \cong \nu(I_w)$), which is equivalent to say that $\sigma(w) = w$ by Proposition 9. Hence Theorem 1 implies that these modules correspond to the subgroup $W_Q^\sigma = \langle (s_1 s_3), s_2 \rangle$ and it is isomorphic to the Weyl group of type B_2.

Next we use (silting) mutation. Let $\Lambda = X \oplus Y$. We denote by $\mu_X(\Lambda)$ the left mutation of Λ with respect to X. It is not necessarily tilting in general (cf.[1]). However, if it is tilting, then we have the following nice result.

Proposition 11. *Assume that $\mu_X(\Lambda)$ is a tilting complex, then we have an isomorphism*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(\mu_X(\Lambda)) \cong \Lambda.$$

Togher with this proposition, the finiteness of $2\text{-silt}\,\Lambda$ implies that *tilting-discreteness* of Λ and we conclude that any tilting complex is obtained from Λ by iterated mutation (see [3]). Then we extend Proposition 9 and obtain the following consequence.

Theorem 12. *Let Q be a Dynkin quiver, Λ the preprojective algebra of Q and Q' the folding quiver of Q. We denote the braid group by $B_{Q'}$. Then we have a bijection*

$$B_{Q'} \leftrightarrow \mathrm{tilt}\,\Lambda.$$

Thus we can parametrize any tilting complex by the braid group.

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GORENSTEINNESS ON THE PUNCTURED SPECTRUM

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Abstract. In this article, we shall characterize torsionfreeness of modules with respect to a semidualizing module in terms of the Serre's condition (S_n). As an application we give a characterization of Cohen-Macaulay rings R such that $R_\mathfrak{p}$ is Gorenstein for all prime ideals $\mathfrak{p}$ with height less than n.

1. Introduction

Auslander and Bridger introduce a notion of *n-torsionfree* as generalization of reflexive [1]. Evans and Griffith give a characterization of n-torsionfree modules [3].

The notion of *n-torsionfree* with respect to a semidualizing module has been introduced by Takahashi [6]. In this article, we study an n-torsionfreeness of modules with respect to a semidualizing module in terms of the Serre's condition (S_n). Recently, Dibaei and Sadeghi [2] give a similar property independently.

Proposition 1. *Let n be a non-negative integer. Assume that R satisfies the conditions (G^C_{n-1}) and (S_n). Then the following statements are equivalent for an R-module M:*

(1) *M is n-C-torsionfree,*
(2) *There exists a exact sequence $0 \to M \to P^1_C \to \cdots \to P^n_C$ such that each P^i_C is a direct summand of direct sum of finite copies of C and that C-dual sequence ${P^n_C}^\dagger \to \cdots \to {P^1_C}^\dagger \to M^\dagger \to 0$ is exact. Here, $(-)^\dagger = \mathrm{Hom}(-, C)$.*
(3) *M is n-C-syzygy,*
(4) *M satisfies the condition (S_n).*

The following throrem is a main theorem of this article.

Theorem 2. *Let R be a Cohen-Macaulay local ring with a dualizing module ω. For non-negative integer n, the following conditions are equivalent:*

(1) *$C_\mathfrak{p}$ is dualizing $R_\mathfrak{p}$-module for all prime ideal $\mathfrak{p}$ of hight at most n,*
(2) *$(S_{n+1})(R) = \Omega^{n+1}_C(\mathrm{mod} R)$,*
(3) *$\omega \in \Omega^{n+1}_C(\mathrm{mod} R)$.*

This theorem recovers a result of Leuschke and Wiegand [5] which gives a characterization of Cohen-Macaulay rings R such that $R_\mathfrak{p}$ is Gorenstein for all prime ideals $\mathfrak{p}$ with height less than n.

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2. Preliminaries

Throughout the rest of this article, let R be a commutative noetherian ring. All modules are assumed to be finitely generated. In this section, we give some notions and properties.

An R-module C is called *semidualizing* if the homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and if $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. A rank 1 free module R and a dualizing module ω over Cohen-Macaulay local rings are typical examples of semidualizing modules. From now on, we fix a semidualizing module C and put $(-)^\dagger = \mathrm{Hom}_R(-, C)$.

Let $\cdots \to P_1 \xrightarrow{\partial} P_0 \to M \to 0$ be a projective resolution of an R-module M. We define a *C-transpose* module $\mathrm{Tr}_C M$ of M the cokernel of $P_0^\dagger \xrightarrow{\partial^\dagger} P_1^\dagger$. We remark that $\mathrm{Tr}_C M$ is uniquely determined up to direct summands of finite direct sums of copy of C. Note that if C is isomorphic to R then C-transpose coincides with ordinary (Auslander) transpose. An R-module M is called *n-C-torsionfree* if $\mathrm{Ext}^i_R(\mathrm{Tr}_C M, C) = 0$ for all $1 \le i \le n$.

We denote by λ_M the natural map $M \to M^{\dagger\dagger}$. n-C-torsionfreeness has following properties similar to ordinary n-torsionfreeness [1]. One can show this by diagram chasing (c.f. [6]).

Proposition 3. *Let M be an R-module.*

(1) *M is 1-C-torsionfree if and only if λ_M is a monomorphism,*
(2) *M is 2-C-torsionfree if and only if λ_M is an isomorphism,*
(3) *Let $n \ge 3$. M is n-C-torsionfree if and only if λ_M is an isomorphism and if $\mathrm{Ext}^i_R(M^\dagger, C) = 0$ for all $1 \le i \le n-2$.*

An R-module M is called *n-C-syzygy* if there exists an exact sequence $0 \to M \to P_C^1 \to P_C^2 \to \cdots \to P_C^n$ such that each P_C^i is a direct summand of finite direct sums of copy of C. We set $\Omega^n_C(\mathrm{mod}R)$ the class of n-C-syzygy modules.

We say that an R-module M satisfies the Serre's condition (S_n) if $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \ge \min\{n, \dim R_\mathfrak{p}\}$ for each prime ideal $\mathfrak{p}$ of R. We denote by $(S_n)(R)$ the class of modules which satisfies (S_n)-condition.

We say that R satisfies the condition (G_n^C) if injective dimension of $C_\mathfrak{p}$ (as an $R_\mathfrak{p}$-module) is finite for all prime ideal $\mathfrak{p}$ of height at most n. In this case, $R_\mathfrak{p}$ is Cohen-Macaulay local ring with canonical module $C_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$ of height at most n. Note that R satisfies (G_n^R) if and only if $R_\mathfrak{p}$ is Gorenstein local ring for all prime ideal $\mathfrak{p}$ of height at most n.

3. Proofs

In this section, we give a proof of the Proposition 1 and the Theorem 2.

Proof of Proposition 1.

(1) $\Rightarrow$ (2). We prove by induction on n. We assume $n = 1$. Let $f : R^r \to M^\dagger$ be a left add$R$-approximation of M. Then f is epimorphism. Since M is 1-C-torsionfree, λ_M is monomorphism and so is $f^\dagger \lambda_M : M \to M^{\dagger\dagger} \to (R^r)^\dagger = C^r$. One can check $(f^\dagger \lambda_M)^\dagger = f$.

Assume $n \ge 2$. Since M is 1-C-torsionfree, there exists a short exact sequence $0 \to M \to P_C^1 \to N \to 0$ such that the daggar dual sequence $0 \to N^\dagger \to (P_C^1)^\dagger \to M^\dagger \to 0$ is exact. Then we have a following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & P_C^1 & \longrightarrow & N & \longrightarrow & 0 \\
 & & {\scriptstyle\lambda_M}\downarrow & & {\scriptstyle\lambda_{P_C^1}}\downarrow & & {\scriptstyle\lambda_N}\downarrow & & \\
0 & \longrightarrow & M^{\dagger\dagger} & \longrightarrow & {P_C^1}^{\dagger\dagger} & \longrightarrow & N^{\dagger\dagger} & \longrightarrow & \mathrm{Ext}_R^1(M^\dagger, C) & \longrightarrow & 0.
\end{array}$$

Since $\mathrm{Ext}_R^i(N^\dagger, C) \cong \mathrm{Ext}_R^{i+1}(M^\dagger, C)$ for each $i > 0$, N is $(n-1)$-C-torsionfree. By induction assumption, there exists a exact sequence $0 \to N \to P_C^2 \to \cdots \to P_C^n$ such that the daggar dual sequence $(P_C^n)^\dagger \to \cdots \to (P_C^2)^\dagger \to N^\dagger \to 0$ is exact. Conbining exact sequences, we get an exact sequence $0 \to M \to P_C^1 \to P_C^2 \to \cdots \to P_C^n$ such that the daggar dual sequence $(P_C^n)^\dagger \to \cdots \to (P_C^1)^\dagger \to M^\dagger \to 0$ is exact.

The implication (2) $\Rightarrow$ (3) is obvious by the definition.

Since $\mathrm{depth}_{R_\mathfrak{p}} C_\mathfrak{p} = \mathrm{depth}_{R_\mathfrak{p}} R_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$, C satisfies (S_n). Thus one can check the implication (3) $\Rightarrow$ (4) by using depth lemma.

We prove the implication (4) $\Rightarrow$ (1) by induction n. Assume $n = 1$. Let $\mathfrak{p}$ be an associated prime ideal of M. Since M satisfies the condition (S_1), we have $\dim R_\mathfrak{p} = 0$. Furthermore, the assumption that R satisfies (G_0^C) implies that $C_\mathfrak{p}$ is a dualizing module and that $\mathrm{Hom}_R(M, C)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) \neq 0$. In particular, $\mathrm{Hom}_R(M, C) \neq 0$.

Let $f_1, f_2, \ldots, f_m$ be a generating system of $\mathrm{Hom}(M, C)$ and put $f = {}^t(f_1, f_2, \ldots, f_m) : M \to C^{\oplus m}$. Suppose that $N = \ker f$ is not zero. Let $\mathfrak{q}$ be an associated prime ideal of N. Since $\mathfrak{q}$ is also an associated prime ideal of M, we have $\dim R_\mathfrak{q} = 0$. Noting that $C_\mathfrak{q}$ is dualizing module over $R_\mathfrak{q}$, we see that $f_\mathfrak{q}$ is a monomorphism. This yields that $N_\mathfrak{q} = 0$. This contradicts that $\mathfrak{q}$ is an associated prime ideal of N. Hence f is a monomorphism.

Since $f^{\dagger\dagger}\lambda_M = \lambda_{C^{\oplus m}} f$ is a monomorphism, we obtain that λ_M is a monomorphism. This means that M is 1-C-torsionfree by Proposition 3.

Assume $n \geq 2$. Since M satisfies the condition (S_1), M is 1-C-torsionfree. In particular, there exists a short exact sequence $0 \to M \to P_C \to N \to 0$ such that the daggar dual sequence $0 \to N^\dagger \to (P_C)^\dagger \to M^\dagger \to 0$ is exact. Then we get a following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & P_C & \longrightarrow & N & \longrightarrow & 0 \\
 & & {\scriptstyle\lambda_M}\downarrow & & {\scriptstyle\lambda_{P_C}}\downarrow & & {\scriptstyle\lambda_N}\downarrow & & \\
0 & \longrightarrow & M^{\dagger\dagger} & \longrightarrow & P_C^{\dagger\dagger} & \longrightarrow & N^{\dagger\dagger} & \longrightarrow & \mathrm{Ext}_R^1(M^\dagger, C) & \longrightarrow & 0.
\end{array}$$

Note that $\mathrm{Ext}_R^i(N^\dagger, C) \cong \mathrm{Ext}_R^{i+1}(M^\dagger, C)$ for each $i > 0$. It is enough to prove that N satisfies the condition (S_{n-1}). Indeed, if N satisfies the condition (S_{n-1}), N is $(n-1)$-C-torsionfree by induction assumption. Then we can show that M is n-C-torsionfree by the above commutative diagram.

From now on, we shall show that N satisfies the condition (S_{n-1}). Let $\mathfrak{p}$ be a prime ideal. If $\dim R_\mathfrak{p} \geq n$, we have $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min\{n, \dim R_\mathfrak{p}\} = n$. Therefore we obtain $\mathrm{depth}_{R_\mathfrak{p}} N_\mathfrak{p} \geq n-1$ by depth lemma.

Assume $\dim R_\mathfrak{p} \leq n-1$. Since R satisfies the condition (G_{n-1}^C), $R_\mathfrak{p}$ is Cohen-Macaulay with canonical module $C_\mathfrak{p}$. Inequalities $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min\{n, \dim R_\mathfrak{p}\} = \dim R_\mathfrak{p} = \mathrm{depth}_{R_\mathfrak{p}} R_\mathfrak{p}$ gives that $M_\mathfrak{p}$ is a maximal Cohen-Macaulay $R_\mathfrak{p}$-module. Thus so are $(M_\mathfrak{p})^{\dagger_\mathfrak{p}}$, $R_\mathfrak{p}$ and $(N_\mathfrak{p})^{\dagger_\mathfrak{p}}$.

It comes from a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\mathfrak{p}} & \longrightarrow & (P_C)_{\mathfrak{p}} & \longrightarrow & N_{\mathfrak{p}} & \longrightarrow & 0 \\
 & & {\scriptstyle \lambda_{M_{\mathfrak{p}}}}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \lambda_{(P_C)_{\mathfrak{p}}}}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \lambda_{N_{\mathfrak{p}}}}\Big\downarrow & & \\
0 & \longrightarrow & (M_{\mathfrak{p}})^{\dagger_{\mathfrak{p}}\dagger_{\mathfrak{p}}} & \longrightarrow & (P_C)_{\mathfrak{p}}^{\dagger_{\mathfrak{p}}\dagger_{\mathfrak{p}}} & \longrightarrow & (N_{\mathfrak{p}})^{\dagger_{\mathfrak{p}}\dagger_{\mathfrak{p}}} & \longrightarrow & 0,
\end{array}$$

we can see that $\lambda_{N_{\mathfrak{p}}}$ is an isomorphism and that $N_{\mathfrak{p}} \cong (N_{\mathfrak{p}})^{\dagger_{\mathfrak{p}}\dagger_{\mathfrak{p}}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module. Therefore we have $\operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} = \dim R_{\mathfrak{p}} \geq \min\{n-1, \dim R_{\mathfrak{p}}\}$. Thus N satisfies the condition (S_{n-1}). $\square$

Now, we can prove the Main theorem.

Proof of Theorem 2.

(1) $\Rightarrow$ (2) It is obvious by Proposition 1.

(2) $\Rightarrow$ (3) A dualizing module ω satisfies the Serre's condition (S_n), so we have $\omega \in \Omega_C^n($ mod $R)$.

(3) $\Rightarrow$ (1) There is an exact sequence

$$0 \to \omega \to P_C^1 \to P_C^2 \to \cdots \to P_C^n \to M \to 0$$

such that each P_C^i is a direct summand of direct sum of finite copy of C. For any prime ideal $\mathfrak{p}$ of height less than n, $(\Omega_C^{n-1} M)_{\mathfrak{p}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module. Then the exact sequence $0 \to \omega_{\mathfrak{p}} \to (P_C^1)_{\mathfrak{p}} \to (\Omega_C^{n-1} M)_{\mathfrak{p}} \to 0$ splits. This indicates $\omega_{\mathfrak{p}} \cong C_{\mathfrak{p}}$. Thus we have $\operatorname{id}_{R_{\mathfrak{p}}} C_{\mathfrak{p}} = \operatorname{id}_{R_{\mathfrak{p}}} \omega_{\mathfrak{p}} < \infty$. $\square$

4. EXAMPLE

Jorgensen, Leuschke and Sather-Wagstaff [4] have been determined the structure of rings which admits non-trivial semidualizing modules.

We give a class of Cohen-Macaulay local rings R which have a non-trivial semidualizing module C by using their result. Moreover, $C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$-module for all non-maximal prime ideal $\mathfrak{p}$ of R.

Proposition 4. *Let k be a field and $S = k[[x_1, x_2, \dots, x_m, y_1, y_2]]$ be a formal power series ring. For $f_1, f_2, \dots, f_r \in k[[x_1, x_2, \dots, x_m]]$ and $\ell \geq 2$, we set ideals $I_1 = (f_1, f_2, \dots, f_r)S$ and $I_2 = (y_1, y_2)^\ell S$. Assume that $T = S/I_1$ is a $(d+2)$-dimensional Cohen-Macaulay ring which is not Gorenstein and that T satisfies the condition (G_{n+2}^T). Putting $R = T/I_2$ and $C = \operatorname{Ext}_T^2(R, T)$, then the followings hold:*

1. *R is d-dimensional Cohen-Macaulay ring,*
2. *C is neither R nor dualizing R-module,*
3. *R satisfies the condition (G_n^C).*

Proof. (1) is clear. (2) is comes from [4]. We show (3). Let $\mathfrak{p}$ be a prime ideal of R with height at most n. Since $P = \mathfrak{p}S$ is a prime ideal of S with height at most $n+2$, we have that $S_{\mathfrak{p}} = S_P$ is Gorenstein. Therefore $C_{\mathfrak{p}} = \operatorname{Ext}_{S_P}^2(R_{\mathfrak{p}}, S_P)$ is a canonical $R_{\mathfrak{p}}$-module. $\square$

In the end of this article, we give examples of 1-dimensional Cohen-Macaulay rings R and semidualizing module C such that R satisfies the condition (G_0^C) but not the condition (G_n^R) for all n.

Example 5. Let k be a field and let $S = k[[x_1, x_2, x_3, y_1, y_2]]/(x_2^2 - x_1x_3, x_2x_3, x_3^2)$ be a 3-dimensional Cohen-Macaulay local ring which is not Gorenstein. We set $R = S/(y_1^2, y_1y_2, y_2^2)$ which is a 1-dimensional Cohen-Macaulay local ring. Note that all the prime ideals of R are $\mathfrak{p} = (x_2, x_3, y_1, y_2)$ and $\mathfrak{m} = (x_1, x_2, x_3, y_1, y_2)$. It is easy to see that $S_\mathfrak{p}$ is Gorenstein but $R_\mathfrak{p}$ is not Gorenstein. In particular, R does not satisfy the condition (G_0^R). Putting $C = \mathrm{Ext}_S^2(R, S)$, one can check that C is a semidualizing R-module which is neither R nor canonical module. Since $S_\mathfrak{p}$ is Gorenstein, we can see that $C_\mathfrak{p}$ is a canonical module over $R_\mathfrak{p}$. This yield that R satisfy the condition (G_0^C).

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TILTED ALGEBRAS AND CONFIGURATIONS OF SELF-INJECTIVE ALGEBRAS OF DYNKIN TYPE

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Abstract. All algebras are assumed to be basic, connected finite-dimensional algebras over an algebraically closed field. We give an easier way to calculate a bijection from the set of isoclasses of tilted algebras of Dynkin type Δ to the set of configurations on the translation quiver $\mathbb{Z}\Delta$.

Introduction

This work is a generalization of Hironobu Suzuki's Master thesis [7] that dealt with representation-finite self-injective algebras of type A in a combinatorial way. Throughout this paper n is a positive integer and $\Bbbk$ is an algebraically closed field, and all algebras considered here are assumed to be basic, connected, finite-dimensional associative $\Bbbk$-algebras.

Let Δ be a Dynkin graph of type A, D, E with the set $\Delta_0 := \{1, \dots, n\}$ of vertices. We set $\mathbf{C}_n$ to be the set of configurations on the translation quiver $\mathbb{Z}\Delta$ (see Definition 1.6), and $\mathbf{T}_n$ to be the set of isoclasses of tilted algebras of type Δ. Then Bretscher, Läser and Riedtmann have given a bijection $c\colon \mathbf{T}_n \to \mathbf{C}_n$ in [1]. But the map c is not given in a direct way, it needs a long computation of a function on $\mathbb{Z}\Delta$. In this paper we will give an easier way to calculate the map c by giving a map sending each projective A-module over a tilted algebra A in $\mathbf{T}_n$ to an element of the configuration $c(A)$.

We fix an orientation of each Dynkin graph Δ to have a quiver $\vec{\Delta}$ as in the following table.

Δ	A_n $(n \geq 1)$	D_n $(n \geq 4)$	E_n $(n = 6, 7, 8)$
$\vec{\Delta}$	$\underset{1}{\circ} \to \underset{2}{\circ} \to \cdots \to \underset{n}{\circ}$	$\underset{1}{\circ} \to \cdots \to \underset{n-2}{\overset{\overset{\circ\, n}{\uparrow}}{\circ}} \to \underset{n-1}{\circ}$	$\underset{1}{\circ} \to \cdots \to \underset{n-3}{\overset{\overset{\circ\, n}{\uparrow}}{\circ}} \to \underset{n-2}{\circ} \to \underset{n-1}{\circ}$
m_Δ	n	$2n-3$	11, 17, 29, respectively

This orientation of Δ gives us a coordinate system on the set $(\mathbb{Z}\Delta)_0 := \mathbb{Z} \times \Delta_0$ of vertices of $\mathbb{Z}\Delta := \mathbb{Z}\vec{\Delta}$ as presented in [1, fig. 1] and in [3, Fig. 13], and by definition the full subquiver $\mathcal{S}$ of $\mathbb{Z}\Delta$ consisting of $\{(0, i) \mid i \in \Delta_0\}$ is isomorphic to $\vec{\Delta}$.

Let A be a tilted algebra of type Δ. Then by identify A with the $(0, 0)$-entry of the repetitive category $\hat{A}$, the vertex set of AR-quiver Γ_A is embedded into the vertex set of the stable AR-quiver ${}_s\Gamma_{\hat{A}}$ $(\cong \mathbb{Z}\Delta)$ of $\hat{A}$. Further the configuration $\mathcal{C} := c(A)$ of $\mathbb{Z}\Delta$ computed in [1] is given by the vertices of $\mathbb{Z}\Delta$ corresponding to radicals of projective

The detailed version of this paper will be submitted for publication elsewhere.

indecomposable $\hat{A}$-modules. Note that the configuration $\mathcal{C}$ has a period m_Δ listed in the table, thus $\mathcal{C} = \tau^{m_\Delta \mathbb{Z}}\mathcal{F}$ for some subset $\mathcal{F}$ of $\mathcal{C}$. By $\mathcal{P} = \{(p(i), i) \mid i \in \Delta_0\}$ we denote the set of images of the projective vertices of Γ_A in $\mathbb{Z}\Delta$ and set

$$\mathbb{N}\mathcal{P} := \{(m, i) \in (\mathbb{Z}\Delta)_0 \mid p(i) \leq m, i \in \Delta_0\}.$$

Since the mesh category $\Bbbk(\mathbb{Z}\Delta)$ is a Frobenius category, it has the Nakayama permutation $\hat{\nu}$ on $(\mathbb{Z}\Delta)_0$ that is defined by the isomorphism

$$\Bbbk(\mathbb{Z}\Delta)(x, \text{-}) \cong \operatorname{Hom}_{\Bbbk}(\Bbbk(\mathbb{Z}\Delta)(\text{-}, \hat{\nu}x), \Bbbk)$$

for all $x \in (\mathbb{Z}\Delta)_0$. The explicit formula of $\hat{\nu}$ is given in [3, pp. 48–50]. (Note that it should be corrected as $\hat{\nu}(p, q) = (p + q + 2, 6 - q)$ if $q \leq 5$ when $\Delta = E_6$ as pointed out in [1, 1.1]). In this paper we will define a map $\nu' \colon \mathcal{P} \to \mathbb{N}\mathcal{P}$ using the supports of starting functions $\dim_{\Bbbk} \Bbbk(\mathbb{Z}\Delta)(x, \text{-}) \colon \mathbb{N}\mathcal{P} \to \mathbb{Z}$ for $x \in \mathbb{N}\mathcal{P}$ (cf. [3, Fig. 15]). Then ν' has the following property.

Lemma 0.1. *Let $x \in \mathcal{P}$ and P be the projective indecomposable A-module corresponding to x. Then $\nu' x$ corresponds to the simple module* $\operatorname{top} P$.

In this paper, we make use of modules over the algebra

$$B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$$

to compute an $\mathcal{F}$ above (the configuration (see Definition 3.9) of B gives $\mathcal{F}$.) We will define a map $\nu := \nu_B$ from the set of isoclasses of simple A-modules to $\mathcal{C}$, which coincides with the restriction of the Nakayama permutation $\hat{\nu}$ if A is hereditary.

Lemma 0.2. *Assume that a vertex $x \in \mathbb{Z}\Delta$ corresponds to a simple A-module S and let Q be the injective hull of S over $\hat{A}$. Then $\nu(x)$ corresponds to* $\operatorname{rad} Q$, *and hence $\nu(x) \in \mathcal{C}$.*

Combining the lemmas above we obtain the following.

Proposition 0.3. *If $x \in \mathcal{P}$, then $\nu(\nu' x) \in \mathcal{C}$.*

This leads us to the following definition.

Definition 0.4. We define a map $c_A \colon \mathcal{P} \to \mathcal{C}$ by $c_A(x) := \nu(\nu' x)$ for all $x \in \mathcal{P}$.

The image of the map c_A gives us an $\mathcal{F}$ above, namely we have the following.

Theorem 0.5. *The map c_A is an injection, and we have $c(A) = \tau^{m_\Delta \mathbb{Z}} \operatorname{Im} c_A$.*

Corollary 0.6. *If A is hereditary, then $c_A = \hat{\nu}\nu'$ and we have $c(A) = \tau^{m_\Delta \mathbb{Z}} \operatorname{Im} \hat{\nu}\nu'$.*

Section 1 is devoted to preparations. In Section 2 we will give the complete list of indecomposable projectives and indecomposable injectives over the triangular matrix algebra B. In Section 3 we state our main results.

1. Preliminaries

1.1. **Algebras and categories.** A category $\mathcal{C}$ is called a $\Bbbk$-*category* if the morphism sets $\mathcal{C}(x,y)$ are $\Bbbk$-vector spaces, and the compositions $\mathcal{C}(y,z)\times\mathcal{C}(x,y)\to\mathcal{C}(x,z)$ are $\Bbbk$-bilinear for all $x,y,z\in\mathcal{C}_0$ ($\mathcal{C}_0$ is the class of objects of $\mathcal{C}$, we sometimes write $x\in\mathcal{C}$ for $x\in\mathcal{C}_0$). In the sequel *all categories are assumed to be* $\Bbbk$*-categories* unless otherwise stated.

To construct repetitive categories and to make use of a covering theory we need to extend the range of considerations from algebras to categories. First we regard an algebra as a special type of categories by constructing a category $\operatorname{cat} A$ from an algebra A as follows.

(1) We fix a decomposition $1=e_1+\cdots+e_n$ of the identity element 1 of A as a sum of orthogonal primitive idempotents.
(2) We set the object class of $\operatorname{cat} A$ to be the set $\{e_1,\dots,e_n\}$.
(3) For each pair (e_i,e_j) of objects, we set $(\operatorname{cat} A)(e_i,e_j):=e_jAe_i$.
(4) We define the composition of $\operatorname{cat} A$ by the multiplication of A.

The obtained category $\operatorname{cat} A$ is uniquely determined up to isomorphisms not depending on the decomposition of 1. The category $C=\operatorname{cat} A$ is a small category having the following three properties.

(1) Distinct objects are not isomorphic.
(2) For each object x of C the algebra $C(x,x)$ is local.
(3) For each pair (x,y) of objects of C the morphism space $C(x,y)$ is finite-dimensional.

A small category with these three properties is called a *spectroid*[1] and its objects are sometimes called *points*. A spectroid with only a finite number of points is called *finite*. The category $\operatorname{cat} A$ is a finite spectroid. Conversely we can construct a matrix algebra from a finite spectroid C as follows.

$$\operatorname{alg} C:=\{(m_{yx})_{x,y\in C}\mid m_{yx}\in C(x,y),\forall x,y\in C\}.$$

Here we have $\operatorname{alg}\operatorname{cat} A\cong A,\operatorname{cat}\operatorname{alg} C\cong C$. Therefore we can identify the class of algebras and the class of finite spectroids by using cat and alg.

A spectroid C is called *locally bounded* if for each point x the set $\{y\in C\mid C(x,y)\neq 0 \text{ or } C(y,x)\neq 0\}$ is a finite set. Of course algebras (= finite spectroids) are locally bounded. In the range of locally bounded spectroids we can freely construct repetitive categories or consider coverings.

Remark 1.1. We can construct the "path-category" $\Bbbk Q$ from a locally finite quiver Q by the same way as in the definition of the path-algebra. The only different part is in the following definition of compositions: For paths μ,ν with[2] $s(\mu)\neq t(\nu)$, it was defined as $\mu\nu=0$ in the path-algebra, but in contrast the composition $\mu\nu$ is not defined in the path-category.

A locally bounded spectroid C is also presented as the form $\Bbbk Q/I$ for some locally finite quiver Q and for some ideal I of the path-category $\Bbbk Q$ such that I is included in the ideal

[1] a terminology used in [4]

[2] Here $s(\mu)$ and $t(\nu)$ stand for the source of μ and the target of ν and compositions are written from the right to the left.

of $\Bbbk Q$ generated by the set of paths of length 2. Here the quiver Q is uniquely determined by C up to isomorphisms. This Q is called *the quiver* of C.

A (right) *module* over a spectroid C is a contravariant functor $C \to \operatorname{Mod}\Bbbk$. From a usual (right) module over an algebra A we can construct a contravariant functor $\operatorname{cat} A \to \operatorname{Mod}\Bbbk$ by the correspondence $e_i \mapsto Me_i$ for each point e_i in $\operatorname{cat} A$, and $f \mapsto (\cdot f\colon Me_j \to Me_i)$ for each $f \in e_jAe_i = (\operatorname{cat} A)(e_i, e_j)$. Conversely, from a contravariant functor $F\colon \operatorname{cat} A \to \operatorname{Mod}\Bbbk$ we can construct an A-module $\bigoplus_{i=1}^n F(e_i)$; and these constructions are inverse to each other. In this way we can identify A-modules and modules over $\operatorname{cat} A$.

The set of projective indecomposable modules over a spectroid C is given by $\{C(\text{-}, x)\}_{x\in C}$ up to isomorphism, and finitely generated projective C-modules are nothing but finite direct sums of these. Using this we can define finitely generated modules or finitely presented modules over C by the same way as those over algebras.

The dimension of a C-module M is defined to be the dimension of $\bigoplus_{x\in C} M(x)$. When C is locally bounded, a C-module is finitely presented if and only if it is finitely generated if and only if it is finite-dimensional.

1.2. **Repetitive category.**

Definition 1.2. Let A be an algebra with a basic set of local idempotents $\{e_1, \dots, e_n\}$.

(1) The *repetitive category* $\hat{A}$ of A is a spectroid defined as follows.

Objects: $\hat{A}_0 := \{x^{[i]} := (x, i) \mid x \in \{e_1, \dots, e_n\}, i \in \mathbb{Z}\}$.

Morphisms: Let $x^{[i]}, y^{[j]} \in \hat{A}_0$. Then we set

$$\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} := (f, i) \mid f \in A(x, y)\} & (j = i) \\ \{\varphi^{[i]} := (\varphi, i) \mid \varphi \in DA(y, x)\} & (j = i + 1) \\ 0 & \text{otherwise.} \end{cases}$$

Compositions: The composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is defined as follows.

(i) If $j = i, k = j$, then we use the composition of A:

$$A(y, z) \times A(x, y) \to A(x, z).$$

(ii) If $j = i, k = j+1$, then we use the right A-module structure of $DA(\text{-}, ?)$:

$$DA(z, y) \times A(x, y) \to DA(z, x).$$

(iii) If $j = i+1, k = j$, then we use the left A-module structure of $DA(\text{-}, ?)$:

$$A(y, z) \times DA(y, x) \to DA(z, x).$$

(iv) Otherwise the composition is zero.

(2) For each $i \in \mathbb{Z}$, we denote by $A^{[i]}$ the full subcategory of $\hat{A}$ whose object class is $\{x^{[i]} \mid x \in \{e_1, \dots, e_n\}\}$.

(3) We define the *Nakayama automorphism* ν_A of $\hat{A}$ as follows: for each $i \in \mathbb{Z}, x, y \in A, f \in A(x, y)$ and $\phi \in DA(y, x)$,

$$\nu_A(x^{[i]}) := x^{[i+1]}, \nu_A(f^{[i]}) := f^{[i+1]}, \nu_A(\varphi^{[i]}) := \varphi^{[i+1]}.$$

Remark 1.3. (1) If a spectroid A is locally bounded, then so is $\hat{A}$.

(2) When A is an algebra, the set of all $\mathbb{Z} \times \mathbb{Z}$-matrices with only a finite number of nonzero entries whose diagonal entries belong to A, $(i+1, i)$ entries belong to DA for all $i \in \mathbb{Z}$, and other entries are zero forms an infinite-dimensional algebra without identity element, which is called the *repetitive algebra* of A. The repetitive category $\hat{A}$ is nothing but this repetitive algebra regarded as a spectroid in a similar way. This is not an algebra (= a finite spectroid) any more, but a locally bounded spectroid.

Definition 1.4 (Gabriel [2])**.** Let C be a locally bounded spectroid with a free[3] action of a group G. Then we define the *orbit category* C/G of C by G as follows.

(1) The objects of C/G are the G-orbits Gx of objects x of C.
(2) For each pair Gx, Gy of objects of C/G we set

$$(C/G)(Gx, Gy) := \left\{ ({}_bf_a)_{a,b} \in \prod_{(a,b) \in Gx \times Gy} C(a,b) \,\middle|\, {}_{gb}f_{ga} = g({}_bf_a),\ \text{for all } g \in G \right\}.$$

(3) The composition is defined by

$$({}_dh_c)_{c,d} \cdot ({}_bf_a)_{a,b} := \left(\sum_{b \in Gy} {}_dh_b \cdot {}_bf_a \right)_{a,d}.$$

for all $({}_bf_a)_{a,b} \in (C/G)(Gx, Gy), ({}_dh_c)_{c,d} \in (C/G)(Gy, Gz)$. Note that each entry of the right hand side is a finite sum because C is locally bounded.

A functor $F \colon C \to C'$ is called a *Galois covering* with group G if it is isomorphic to the canonical functor $\pi \colon C \to C/G$, namely if there exists an isomorphism $H \colon C/G \to C'$ such that $F = H\pi$.

Remark 1.5. If A is an algebra and a group G acts freely on the category $\hat{A}$, then $\hat{A}/G$ turns out to be a self-injective spectroid. In particular, when $\hat{A}/G$ is a finite spectroid, it becomes a self-injective algebra. In this way we can construct a great number of self-injective algebras.

Definition 1.6. From a quiver Q we can construct a translation quiver $\mathbb{Z}Q$ as follows.

- $(\mathbb{Z}Q)_0 := \mathbb{Z} \times Q_0$,
- $(\mathbb{Z}Q)_1 := \mathbb{Z} \times Q_1 \cup \{(i, \alpha') \mid i \in \mathbb{Z}, \alpha \in Q_1\}$,
- We define the sources and the targets of arrows by

$$(i, \alpha) \colon (i, s(\alpha)) \to (i, t(\alpha)),\ (i, \alpha') \colon (i, t(\alpha)) \to (i+1, s(\alpha))$$

  for all $(i, \alpha) \in \mathbb{Z} \times Q_1$.
- We take the bijection $\tau \colon (\mathbb{Z}Q)_0 \to (\mathbb{Z}Q)_0, (i, x) \mapsto (i-1, x)$ as the translation.

In addition, we can define a polarization by $(i+1, \alpha) \mapsto (i, \alpha'), (i, \alpha') \mapsto (i, \alpha)$. Note that by construction the translation quiver $\mathbb{Z}Q$ does not have any projective or injective vertices.

[3] $1 \neq g \in G$, $x \in C_0$ implies $gx \neq x$

For example,

$$
Q = \begin{tikzcd}[row sep=small]
 & 1 \\
2 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\
 & 3
\end{tikzcd}
\qquad \text{gives} \qquad
\mathbb{Z}Q = \begin{tikzcd}[row sep=small, column sep=small]
\cdots & & (-1,1) \arrow[dr, "(-1,\alpha')"] & & (0,1) \arrow[dr, "(0,\alpha')"] & & (1,1) & \cdots \\
\cdots & (-1,2) \arrow[ur, "(-1,\alpha)"] \arrow[dr, "(-1,\beta)"'] & & (0,2) \arrow[ur, "(0,\alpha)"] \arrow[dr, "(0,\beta)"'] & & (1,2) \arrow[ur, "(1,\alpha)"] \arrow[dr, "(1,\beta)"'] & & \cdots \\
\cdots & & (-1,3) \arrow[ur, "(-1,\beta')"'] & & (0,3) \arrow[ur, "(0,\beta')"'] & & (1,3) & \cdots
\end{tikzcd}.
$$

Remark 1.7. When Q is a Dynkin quiver with the underlying graph Δ, the isoclass of $\mathbb{Z}Q$ does not depend on orientations of Δ, therefore we set $\mathbb{Z}\Delta := \mathbb{Z}Q$.

2. Triangular Matrix Algebras

Definition 2.1. Let R and S be algebras, M be an S-R-bimodule. We define a category $\mathcal{C} = \mathcal{C}(R, S, M)$ as follows.

Objects: $\mathcal{C}_0 := \{(X, Y, f) \mid X_R \in \operatorname{mod} R, Y_S \in \operatorname{mod} S, f \in \operatorname{Hom}_A(Y \otimes_S M, X)\}$.

Morphisms: Let $(X, Y, f), (X', Y', f') \in \mathcal{C}_0$. Then we set

$$
\mathcal{C}((X,Y,f),(X',Y',f')) := \left\{ (\phi_0, \phi_1) \in \operatorname{Hom}_R(X, X') \times \operatorname{Hom}_S(Y, Y') \;\middle|\; \begin{tikzcd} Y \otimes_S M \arrow[r, "f"] \arrow[d, "\phi_1 \otimes 1_M"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\phi_0"] \\ Y' \otimes_S M \arrow[r, "f'"'] & X' \end{tikzcd} \right\}.
$$

Compositions: Let $(X, Y, f), (X', Y', f'), (X'', Y'', f'') \in \mathcal{C}_0$ and let

$$(\phi_0, \phi_1) \in \mathcal{C}((X, Y, f), (X', Y', f')), (\phi_0', \phi_1') \in \mathcal{C}((X', Y', f'), (X'', Y'', f'')).$$

Then we set

$$(\phi_0', \phi_1')(\phi_0, \phi_1) := (\phi_0'\phi_0, \phi_1'\phi_1) \in \mathcal{C}((X, Y, f), (X'', Y'', f'')).$$

Then the following is well known.

Proposition 2.2. *Let R and S be algebras, M be an S-R-bimodule. Then*

$$\operatorname{mod} \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \simeq \mathcal{C}(R, S, M).$$

Recall that an equivalence $F : \operatorname{mod} \begin{bmatrix} R & 0 \\ M & S \end{bmatrix} \to \mathcal{C}(R, S, M)$ is given as follows.

Objects: For each $L \in (\operatorname{mod} T)_0$,

$$F(L) := (L\varepsilon_1, L\varepsilon_2, f_L),$$

where $\varepsilon_1 := \begin{bmatrix} 1_R & 0 \\ 0 & 0 \end{bmatrix}, \varepsilon_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1_S \end{bmatrix}$ and $f_L : L\varepsilon_2 \otimes_S M \to L\varepsilon_1$ is defined by $f_L(l\varepsilon_2 \otimes m) := l \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ for all $l \in L$ and $m \in M$.

Morphisms: For each $\alpha \in \operatorname{Hom}_T(L, L')$,

$$F(\alpha) := (\alpha\,|_{L\varepsilon_1}, \alpha\,|_{L\varepsilon_2}).$$

Let A be a tilted algebra of type Δ, and set $B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$, $\mathcal{C} := \mathcal{C}(A, A, DA)$.

Then we have $\operatorname{mod} B \simeq \mathcal{C}$ by Proposition 2.2. By this equivalence, we identify $\operatorname{mod} B$ with $\mathcal{C}$.

Let $\{e_1, \dots, e_n\}$ be a complete set of orthogonal local idempotents of A. Then as is easily seen

$\{e_1^{[0]}, \dots, e_n^{[0]}, e_1^{[1]}, \dots, e_n^{[1]}\}$ is a complete set of orthogonal local idempotents of B, and $\{e_1^{[0]}B, \dots, e_n^{[0]}B, e_1^{[1]}B, \dots, e_n^{[1]}B\}$ is a complete set of isoclasses of projective indecomposable B-modules. The following is immediate.

Proposition 2.3. *For each $i = 1, \dots, n$, we have*

$$\begin{aligned} F(e_i^{[0]}B) &\cong (e_iA, 0, 0), \\ F(e_i^{[1]}B) &\cong (e_i(DA), e_iA, \mathrm{can}). \end{aligned}$$

In addition $\{D(Be_1^{[0]}), \dots, D(Be_n^{[0]}), D(Be_1^{[1]}), \dots, D(Be_n^{[1]})\}$ is a complete set of isoclasses of injective indecomposable B-modules. The following two statements are obvious.

Lemma 2.4. *For each $i = 1, \dots, n$, we have*

(1) $D \begin{bmatrix} Ae_i & 0 \\ (DA)e_i & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ D(Ae_i) & e_iA \end{bmatrix}$, *and*

(2) $D \begin{bmatrix} 0 & 0 \\ 0 & Ae_i \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ 0 & D(Ae_i) \end{bmatrix}$.

Proposition 2.5. *For each $i = 1, \dots, n$, we have*

$$\begin{aligned} F(D(Be_i^{[0]})) &\cong (e_i(DA), e_iA, \mathrm{can}) \cong e_i^{[1]}B, \\ F(D(Be_i^{[1]})) &\cong (0, e_i(DA), 0). \end{aligned}$$

3. Configurations

Definition 3.1. Let Λ be a standard representation-finite self-injective algebra. Then we set

$$\mathcal{C}_\Lambda := \{[\operatorname{rad} P] \in \Gamma_\Lambda \mid P : \text{projective(-injective) } \Lambda\text{-module}\},$$

which is called a *configuration* of Λ.

Definition 3.2. Let Γ be a stable translation quiver, and $\mathcal{C}$ be a subset of Γ_0. Then we define a translation quiver $\Gamma_{\mathcal{C}}$ by

$$\begin{aligned} (\Gamma_{\mathcal{C}})_0 &:= \Gamma_0 \sqcup \{p_x \mid x \in \mathcal{C}\}, \\ (\Gamma_{\mathcal{C}})_1 &:= \Gamma_1 \sqcup \{x \to p_x,\ p_x \to \tau^{-1}x\}, \end{aligned}$$

where the translation of $\Gamma_{\mathcal{C}}$ is the same as that of Γ. In particular, p_x are projective-injective[4] vertices for all $x \in \mathcal{C}$.

[4]The word "projective-injective" stands for projective and injective.

Remark 3.3. The quiver of $\underline{\mathrm{mod}}\,\Lambda$ is the full subquiver ${}_s\Gamma_\Lambda$ of Γ_Λ with

$$({}_s\Gamma_\Lambda)_0 := \{x \mid x \text{ is a stable vertex of } \Gamma_\Lambda\}$$

(namely ${}_s\Gamma_\Lambda$ is obtained from Γ_Λ by removing all projective vertices), which is a stable translation quiver. Then it holds that $\mathcal{C}_\Lambda \subseteq ({}_s\Gamma_\Lambda)_0$, and we have

$$({}_s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong \Gamma_\Lambda. \tag{3.1}$$

Theorem 3.4. *Let Λ be a standard representation-finite self-injective algebra and Δ the Dynkin type of Λ. Then the following hold.*

(1) (Waschbüsch [5, 8]) *There exist a tilted algebra A of type Δ and an automorphism ϕ of $\hat{A}$ without fixed vertices such that $\Lambda \cong \hat{A}/\langle\phi\rangle$.*
(2) (Riedtmann [6]) *There is an isomorphism $f : {}_s\Gamma_{\hat{A}} \to \mathbb{Z}\Delta$. Denote also by ϕ the automorphism of ${}_s\Gamma_{\hat{A}}$ induced from ϕ canonically, and define an automorphism ϕ' of $\mathbb{Z}\Delta$ by the following commutative diagram:*

$$\begin{tikzcd}
{}_s\Gamma_{\hat{A}} \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z}\Delta \arrow[d, "\phi'"] \\
{}_s\Gamma_{\hat{A}} \arrow[r, "f"'] & \mathbb{Z}\Delta.
\end{tikzcd}$$

Then we have ${}_s\Gamma_\Lambda \cong {}_s\Gamma_{\hat{A}}/\langle\phi\rangle \cong \mathbb{Z}\Delta/\langle\phi'\rangle$.

By the formula (3.1) to compute Γ_Λ, it is enough to solve the following problem.

Problem 1. Let Λ be a standard representation-finite self-injective algebra, which has the form $\hat{A}/\langle\phi\rangle$ for some tilted algebra A of Dynkin type and an automorphism ϕ of $\hat{A}$ by Theorem 3.4. Then compute $\mathcal{C}_\Lambda$ from A.

Remark 3.5. Let $f' : {}_s\Gamma_\Lambda \to \mathbb{Z}\Delta/\langle\phi'\rangle$ be an isomorphism, and set $\mathcal{C} := f'(\mathcal{C}_\Lambda)$. Then we have

$$\Gamma_\Lambda \cong ({}_s\Gamma_\Lambda)_{\mathcal{C}_\Lambda} \cong (\mathbb{Z}\Delta/\langle\phi'\rangle)_{\mathcal{C}}.$$

Thus we can compute Γ_Λ by Theorem 3.4(2) if we can obtain the set $\mathcal{C}$.

On the other hand, the following holds by [2, Theorem 3.6].

Theorem 3.6 (Gabriel)**.** *Let R be a locally representation-finite and locally bounded $\Bbbk$-category, and G be a group consisting of automorphisms of R that acts freely on R. Then the AR-quiver Γ_R of R has an induced G-action, and we have $\Gamma_R/G \cong \Gamma_{R/G}$.*

Definition 3.7. Let A be a tilted algebra of Dynkin type. Then we set

$$\mathcal{C}_{\hat{A}} := \{[\mathrm{rad}\,P] \in \Gamma_{\hat{A}} \mid P : \text{projective(-injective) } \hat{A}\text{-module}\},$$

which is called the *configuration* of $\hat{A}$.

Corollary 3.8. *Let A be a tilted algebra of Dynkin type, and ϕ be an automorphism of $\hat{A}$ without fixed vertices. Then we have*

$$\mathcal{C}_{\hat{A}}/\langle\phi\rangle \cong \mathcal{C}_\Lambda.$$

Therefore to solve Problem 1, it is enough to consider the following.

Problem 2. In the same setting as in Problem 1, compute $\mathcal{C}_{\hat{A}}$ from A.

Throughout the rest of their section

(1) let A be a tilted algebra of Dynkin type Δ, and set
(2) $B := \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$.

By (1), Γ_A has a section $\mathcal{S}$ whose underlying graph is isomorphic to Δ.

Definition 3.9. We call the following set the *configuration* of B:

$$\mathcal{C}_B := \{[\operatorname{rad} P] \in \Gamma_B \mid P : \text{projective-injective } B\text{-module}\}.$$

3.1. **Relationship among $\hat{A}$, B and A.** We set as follows:

$$\begin{aligned} I_{0,1} &= \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{0,1\}, j \in \{1, \dots, n\} \rangle, \\ I_0 &= \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{0\}, j \in \{1, \dots, n\} \rangle, \\ I_1 &= \langle e_j^{[i]} \mid i \in \mathbb{Z} \setminus \{1\}, j \in \{1, \dots, n\} \rangle. \end{aligned}$$

Then $\hat{A}/I_{0,1} \cong B$, $\hat{A}/I_0 \cong A^{[0]}(\cong A)$ and $\hat{A}/I_1 \cong A^{[1]}(\cong A)$. We also have

$$B \Big/ \begin{bmatrix} 0 & 0 \\ DA & 0 \end{bmatrix} \cong A^{[0]} \times A^{[1]}.$$

We have the following surjective algebra homomorphisms

$$\begin{tikzcd} & & & A^{[0]} \\ \hat{A} \arrow[r, two heads] \arrow[urrr, two heads] \arrow[drrr, two heads] & B \arrow[r, two heads] & A^{[0]} \times A^{[1]} \arrow[ur, two heads] \arrow[dr, two heads] & \\ & & & A^{[1]}, \end{tikzcd}$$

which induce the following embeddings of categories

$$\begin{tikzcd} & & \operatorname{mod} A^{[0]} \arrow[dl, hook'] \\ \operatorname{mod} \hat{A} & \operatorname{mod} B \arrow[l, hook', "\sigma"'] & \\ & & \operatorname{mod} A^{[1]}. \arrow[ul, hook] \end{tikzcd}$$

We regard $\operatorname{mod} A \subseteq \operatorname{mod} B$ by the embedding $\operatorname{mod} A = \operatorname{mod} A^{[0]} \hookrightarrow \operatorname{mod} B$. The embeddings above give us the following embeddings of vertex sets of AR-quivers:

$$\begin{tikzcd}
 & & (\Gamma_{A^{[0]}})_0 = (\Gamma_A)_0 \arrow[dl, hook', "\sigma_0"'] \\
(\Gamma_{\hat{A}})_0 & (\Gamma_B)_0 \arrow[l, hook', "\sigma"'] & \\
 & & (\Gamma_{A^{[1]}})_0 . \arrow[ul, hook]
\end{tikzcd}$$

We define an ideal $\Bbbk(\mathbb{Z}\Delta)^+$ of the mesh category $\Bbbk(\mathbb{Z}\Delta)$ as follows:

$$\Bbbk(\mathbb{Z}\Delta)^+ := \langle (\mathbb{Z}\Delta)_1 + I_{\mathbb{Z}\Delta} \rangle.$$

Then the values of $m_\Delta := \min\{m \in \mathbb{N} \mid (\Bbbk(\mathbb{Z}\Delta)^+)^i = 0, \forall i \geq m\}$ are known as follows:

$$m_\Delta = \begin{cases} n & (\Delta = A_n) \\ 2n-3 & (\Delta = D_n) \\ 11 & (\Delta = E_6) \\ 17 & (\Delta = E_7) \\ 29 & (\Delta = E_8) \end{cases}.$$

We see the following by [1].

Proposition 3.10. *Let $i = 0, 1$.*

(1) *The full subquiver $\mathcal{S}_B^{[i]}$ of Γ_B with the vertex set $\sigma_i(\mathcal{S}_0)$ forms a section of ${}_s\Gamma_B$.*
(2) *The full subquiver $\mathcal{S}_{\hat{A}}^{[i]}$ of $\Gamma_{\hat{A}}$ with the vertex set $\sigma\sigma_i(\mathcal{S}_0)$ forms a section of ${}_s\Gamma_{\hat{A}}$.*

Remark 3.11. A quiver Q without oriented cycles will be regarded as a poset by the order defined as follows:

For each $x, y \in Q_0, x \preceq y :\Leftrightarrow$there is a path in Q from x to y.

Definition 3.12. (1) We set $\mathcal{H}_B$ to be the full subquiver of Γ_B defined by the set

$$(\mathcal{H}_B)_0 := \{x \in (\Gamma_B)_0 \mid a \preceq x \preceq b \text{ for some } a \in (\mathcal{S}_B^{[0]})_0, b \in (\mathcal{S}_B^{[1]})_0\}$$

of vertices.
(2) We set $\mathcal{H}_{\hat{A}}^{[0,1]}$ to be the full subquiver of $\Gamma_{\hat{A}}$ defined by the set

$$(\mathcal{H}_{\hat{A}}^{[0,1]})_0 := \{x \in (\Gamma_{\hat{A}})_0 \mid a \preceq x \preceq b \text{ for some } a \in (\mathcal{S}_{\hat{A}}^{[0]})_0, b \in (\mathcal{S}_{\hat{A}}^{[1]})_0\}$$

of vertices.

Proposition 3.13. (1) *The map $\sigma : (\Gamma_B)_0 \to (\Gamma_{\hat{A}})_0$ is uniquely extended to a quiver isomorphism $\mathcal{H}_B \to \mathcal{H}_{\hat{A}}^{[0,1]}$.*
(2) *We have $\mathcal{S}_{\hat{A}}^{[1]} = \tau^{-m_\Delta}\mathcal{S}_{\hat{A}}^{[0]}$. We set $\mathcal{S}_{\hat{A}}^{[n]} := \tau^{-nm_\Delta}\mathcal{S}_{\hat{A}}^{[0]}$ for all $n \in \mathbb{Z}$.*

(3) *Set* $\mathcal{H}_{\hat{A}}^{[n,n+1]} := \tau^{-nm_{\triangle}}(\mathcal{H}_{\hat{A}}^{[0,1]})$ *for all* $n \in \mathbb{Z}$. *Then for each* $i = 0, 1$

$$(\Gamma_{\hat{A}})_i = \bigcup_{n\in\mathbb{Z}} (\mathcal{H}_{\hat{A}}^{[n,n+1]})_i$$
$$(\mathcal{S}_{\hat{A}}^{[n+1]})_i = (\mathcal{H}_{\hat{A}}^{[n,n+1]})_i \cap (\mathcal{H}_{\hat{A}}^{[n+1,n+2]})_i$$

Roughly speaking, $\Gamma_{\hat{A}}$ is obtained by connecting infinite copies of $\mathcal{H}_B$ on both sides.

Example 3.14. Let A be the path algebra of the following quiver.

$$\begin{tikzcd} 1^{[0]} \arrow[r] & 2^{[0]} \arrow[r] & 3^{[0]} \end{tikzcd}$$

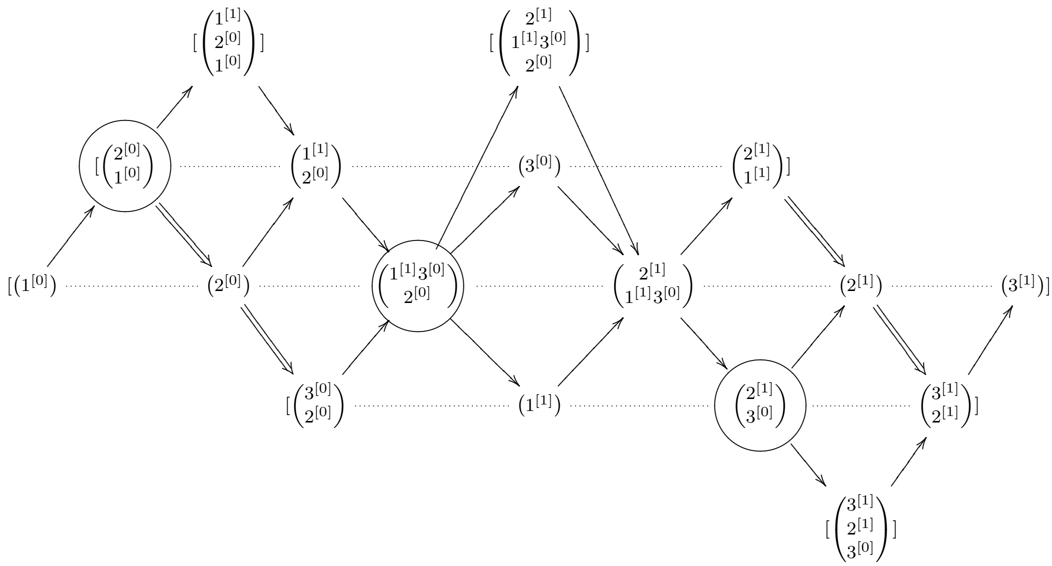
Then Γ_A is given as follows (double arrows present a section).

$$\begin{tikzcd}[column sep=small]
 & {[\begin{pmatrix} 2^{[0]} \\ 1^{[0]} \end{pmatrix}]} \arrow[dr, Rightarrow] & & & \\
{[(1^{[0]})} \arrow[ur] \arrow[rr, dotted, no head] & & (2^{[0]}) \arrow[dr, Rightarrow] \arrow[rr, dotted, no head] & & (3^{[0]})] \\
 & & & {[\begin{pmatrix} 3^{[0]} \\ 2^{[0]} \end{pmatrix}]} \arrow[ur] &
\end{tikzcd}$$

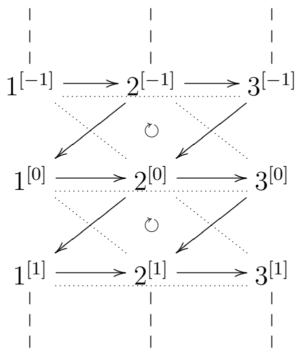
Therefore A is a tilted algebra of type A_3. Moreover $B = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} = \begin{bmatrix} A^{[0]} & 0 \\ (DA)^{[0]} & A^{[1]} \end{bmatrix}$ is an algebra given by following quiver with relations.

$$\begin{tikzcd}
1^{[0]} \arrow[r] & 2^{[0]} \arrow[r] \arrow[dl] \arrow[d, phantom, "\circlearrowright"] & 3^{[0]} \arrow[dl] \\
1^{[1]} \arrow[r] & 2^{[1]} \arrow[r] & 3^{[1]}
\end{tikzcd}$$

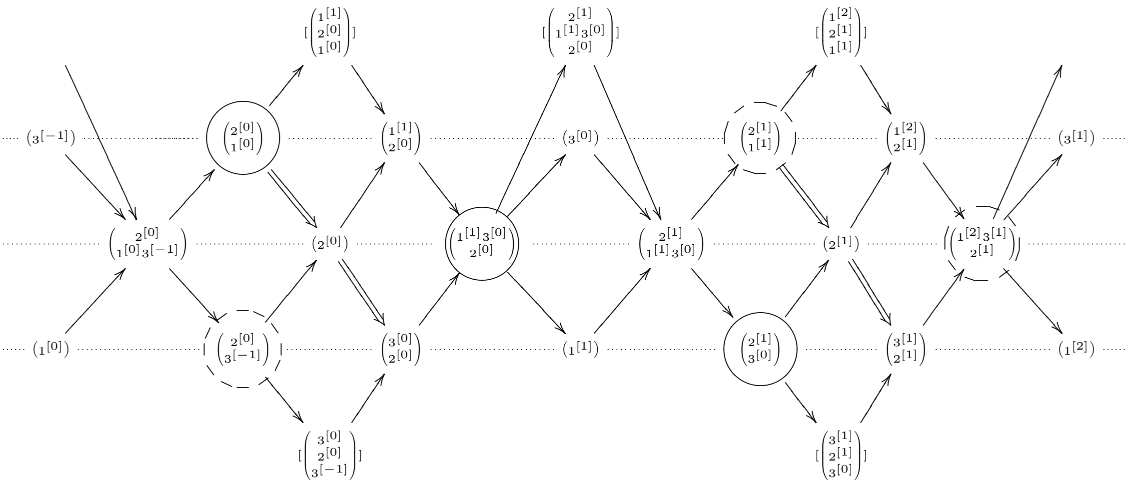
Then Γ_B is given as follows (elements of $\mathcal{C}_B$ are encircled).



In the above, $\mathcal{H}_B$ is given by the full subquiver consisting of vertices between the left section and the right section. $\hat{A}$ is given by the following quiver with relations.



Then $\Gamma_{\hat{A}}$ is follows (each element of $\mathcal{C}_{\hat{A}}$ is encircled by a broken or solid line, in particular solid circles present elements of $\mathcal{C}_B$). In this case we have $m_\Delta = 3$.



The following is immediate from Proposition 3.13.

Corollary 3.15. *We have* $\mathcal{C}_{\hat{A}} = \tau^{\mathbb{Z}m_\Delta}\sigma(\mathcal{C}_B)$.

By this corollary, Problem 2 is reduced to the following.

Problem 3. Let A be a tilted algebra of Dynkin type Δ, and B as above. Then give the configuration $\mathcal{C}_B$ from A.

The purpose of this section is to solve Problem 3.

Definition 3.16. (1) We define an ideal $\mathcal{PI}$ of $\operatorname{mod} B$ as follows and set $\widetilde{\operatorname{mod}} B := (\operatorname{mod} B)/\mathcal{PI}$. For each $X, Y \in (\operatorname{mod} B)_0$

$\mathcal{PI}(X,Y) := \{f \in \operatorname{Hom}_B(X,Y) \,|\, f$ factors through a projective-injective B-module$\}$

Let $\tilde{(?)}\colon \operatorname{mod} B \to \widetilde{\operatorname{mod}} B$ be the canonical functor and set

$$\widetilde{\operatorname{Hom}}_B(\tilde{X}, \tilde{Y}) := (\widetilde{\operatorname{mod}} B)(\tilde{X}, \tilde{Y})$$

for all $X, Y \in \operatorname{mod} B$. Thus $\tilde{X} = X$ for all $X \in (\operatorname{mod} B)_0$ and $\tilde{f} = f + \mathcal{PI}(X,Y)$ for all $f \in \operatorname{Hom}_B(X,Y)$.

(2) We denote by $\operatorname{mod}_{\mathcal{PI}} B$ the full subcategory of $\operatorname{mod} B$ consisting of B-modules without projective-injective direct summands.

(3) Let X and $Y \in \operatorname{mod}_{\mathcal{PI}} B$. Then it is well known that $\mathcal{PI}(X,Y) \subseteq \operatorname{rad}_B(X,Y)$. We set $\widetilde{\operatorname{rad}}_B(X,Y) := \operatorname{rad}_B(X,Y)/\mathcal{PI}(X,Y)$.

Definition 3.17. For AR-quiver Γ_B of B, we define the full translation subquiver $\tilde{\Gamma}_B$ as follows.

$$(\tilde{\Gamma}_B)_0 := \{X \in (\Gamma_B)_0 \mid X \text{ is } \textit{not} \text{ projective-injective.} \}$$

Moreover we set

$$\operatorname{supp}(s_X) := \{Y \in (\tilde{\Gamma}_B)_0 \mid s_X(Y) \neq 0\},$$

where the map $s_X : (\tilde{\Gamma}_B)_0 \to \mathbb{Z}_{\geq 0}$ is defined by $s_X(Y) := \dim \widetilde{\mathrm{Hom}}_B(\tilde{X}, \tilde{Y})$ $(Y \in (\tilde{\Gamma}_B)_0)$ for all $X \in (\tilde{\Gamma}_B)_0$.

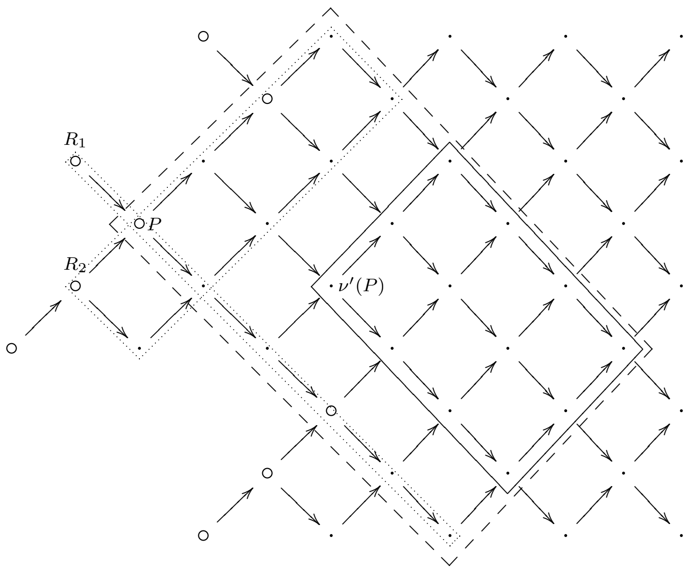
Definition 3.18. Let P be a projective indecomposable A-module, and $\operatorname{rad} P = \bigoplus_{i=1}^r R_i$ with R_i indecomposable for all i. Then we define a full subquiver $\mathcal{R}_P$ of $\tilde{\Gamma}_B$ by

$$(\mathcal{R}_P)_0 := \operatorname{supp}(s_P) \setminus \left(\bigcup_{i=1}^r \operatorname{supp}(s_{R_i}) \right).$$

Definition 3.19. We regard the subquiver $\mathcal{R}_P$ as a poset by Remark 3.11. For a projective indecomposable A-module P, we set

$$\nu'(P) := \min \mathcal{R}_P.$$

Example 3.20. In the following figure, the vertices inside broken lines form $\operatorname{supp}(s_P)$ and those inside doted lines form $(\bigcup_{i=1}^r \operatorname{supp}(s_{R_i}))$. Therefore the subquiver $\mathcal{R}_P$ consists of the vertices inside solid lines, and $\nu'(P)$ is the minimum element of $\mathcal{R}_P$. Projective vertices are presented by white circles $\circ$.



We have the following the proof of which is omitted.

Proposition 3.21. *Let P be a projective indecomposable A-module. then $\nu'(P) \cong \operatorname{top} P$.*

We will give an alternative definition of the map ν' below, which is easier to compute than the first one.

Definition 3.22. Let $P \in \operatorname{mod} B$ be projective.

(1) Let $\mathcal{P}_P$ be the full subcategory of $\operatorname{mod} B$ consisting of projective modules Q such that P is not a direct summand of Q.

(2) We define an ideal $\mathcal{I}_P$ of $\operatorname{mod} B$ and the factor category $\underline{\operatorname{mod}}^P B := \operatorname{mod} B/\mathcal{I}_P$ of $\operatorname{mod} B$ by setting

$$\mathcal{I}_P(X,Y) := \{f \in \operatorname{Hom}_B(X,Y) \,|\, f \text{ factors through an object in } \mathcal{P}_P\},$$

and set

$$\underline{\operatorname{Hom}}_B^P(X,Y) := \operatorname{Hom}_B(X,Y)/\mathcal{I}_P(X,Y)$$

for all $X, Y \in \operatorname{mod} B$. Let $\underline{(?)}\colon \operatorname{mod} B \to \underline{\operatorname{mod}}^P B$ be the canonical functor. Thus $\underline{X} = X$ for all $X \in (\operatorname{mod} B)_0$ and $\underline{f} = f + \mathcal{I}_P(X,Y)$ for all $f \in \operatorname{Hom}_B(X,Y)$.

$$\operatorname{supp}(s'_P) := \{X \in (\tilde{\Gamma}_B)_0 \mid s'_P(X) \neq 0\} \subseteq (\tilde{\Gamma}_B)_0$$

where the map $s'_P : (\tilde{\Gamma}_B)_0 \to \mathbb{Z}_{\geq 0}$ is defined by $s_P(X) := \dim \underline{\operatorname{Hom}}_B^P(P,X)$ $(X \in (\tilde{\Gamma}_B)_0)$ for all $P \in (\tilde{\Gamma}_B)_0$.

The easier way to compute ν' is given by the following three statements, which we state without proofs.

Lemma 3.23. *Let Q and X be in $\operatorname{mod} B$. If Q is projective and there is an epimorphism $Q \to X$, then the projective cover of X is a direct summand of Q.*

Lemma 3.24. *If $f : X \to \operatorname{top} P$ is nonzero in $\operatorname{mod} B$, then $\underline{f} \neq 0$.*

Proposition 3.25. *Let P be a projective indecomposable A-module. Then we have*

$$\max \operatorname{supp}(s'_P) \cong \operatorname{top} P.$$

Thus $\nu'(P) = \max \operatorname{supp}(s'_P)$.

Next we define a map sending a simple A-module to an element of the configurations.

Lemma 3.26. *Let S be a simple A-module, and Q the injective hull of S in $\operatorname{mod} B$. Then the left $(\widetilde{\operatorname{mod}}\, B)$-module $\widetilde{\operatorname{Hom}}_B(S,\text{-})$ has a simple socle, and*

$$\operatorname{soc} \widetilde{\operatorname{Hom}}_B(S,-) \cong \widetilde{\operatorname{Hom}}_B(\operatorname{rad} Q,-)/\widetilde{\operatorname{rad}}(\operatorname{rad} Q,-).$$

It follows by the lemma above that the poset $\operatorname{supp}(s_S)$ has the maximum element for each simple A-module S. We then set $\nu_B(S)$ to be the maximum element. The following is immediate.

Proposition 3.27. *Let S be a simple A-module, and Q the injective hull of S in $\operatorname{mod} B$. Then we have $\nu_B(S) \cong \operatorname{rad} Q$.*

We finally obtain the following by Propositions 3.25 and 3.27.

Theorem 3.28. *Let $\mathcal{P}$ be a complete set of representatives of isoclass of indecomposable projective A-modules. Then we have*

$$\mathcal{C}_B = \nu_B(\nu'(\mathcal{P})).$$

Hence as is stated before, $\mathcal{C}_\Lambda$ is obtained as follows.

Theorem 3.29.

$$\mathcal{C}_\Lambda = \mathcal{C}_{\hat{A}}/\langle\phi\rangle = (\tau^{\mathbb{Z}m_\Delta}\sigma(\mathcal{C}_B))/\langle\phi\rangle = (\tau^{\mathbb{Z}m_\Delta}\sigma\nu_B\nu'(\mathcal{P}))/\langle\phi\rangle.$$

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ON THE DECOMPOSITION OF THE HOCHSCHILD COHOMOLOGY GROUP OF A MONOMIAL ALGEBRA SATISFYING A SEPARABILITY CONDITION

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Abstract. This paper is based on [14]. In this paper, we consider the finite connected quiver Q having two subquivers $Q^{(1)}$ and $Q^{(2)}$ with $Q = Q^{(1)} \cup Q^{(2)} = (Q_0^{(1)} \cup Q_0^{(2)}, Q_1^{(1)} \cup Q_1^{(2)})$. Suppose that $Q^{(i)}$ is not a subquiver of $Q^{(j)}$ where $\{i, j\} = \{1, 2\}$. For a monomial algebra $\Lambda = kQ/I$ obtained by the quiver Q, when the set $AP(n)$ ($n \geq 0$) of overlaps constructed inductively by linking generators of I satisfies a certain separability condition, we propose the method so that we easily construct a minimal projective resolution of Λ as a right Λ^{e}-module and calculate the Hochschild cohomology group of Λ.

Key Words: Monomial algebra, associated sequence of path, Hochschild cohomology, path algebra.

2010 *Mathematics Subject Classification*: 16E40, 16G20.

1. Introduction

For a finite-dimensional algebra A over a field k, the Hochschild cohomology groups $\mathrm{HH}^n(A)$ of A is defined by

$$\mathrm{HH}^n(A) := \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, A)\ (n \geq 0),$$

where $A^{\mathrm{e}} := A^{\mathrm{op}} \otimes_k A$ is the enveloping algebra of A. Note that there is a natural one to one correspondence between the family of A-A-bimodules and that of right A^{e}-modules. Moreover, the Hochschild cohomology rings $\mathrm{HH}^*(A)$ of A is the graded algebra defined by

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^{\mathrm{e}}}(A, A) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A)$$

with the Yoneda product.

The low-dimensional Hochschild cohomology groups are described as follows:

- $\mathrm{HH}^0(A) = Z(A)$ is the center of A.
- $\mathrm{HH}^1(A)$ is the space of derivations modulo the inner derivations. A derivation is a k-linear map $f : A \to A$ such that $f(ab) = af(b) + f(a)b$ for all $a, b \in A$. A derivation $f : A \to A$ is an inner derivation if there is some $x \in A$ such that $f(a) = ax - xa$ for all $a \in A$.

One important property of Hochschild cohomology is its invariance under Morita equivalence, stable equivalence of Morita type and derived equivalence.

The detailed version of this paper has been submitted for publication elsewhere.

Let k be an algebraically closed field and Q a finite connected quiver. Then kQ denotes the path algebra of Q over k in this paper. Let I be an admissible ideal of kQ. If I is generated by a finite number of paths in Q, then I is called a monomial ideal and $\Lambda := kQ/I$ a monomial algebra. For a finite-dimensional monomial algebra $\Lambda = kQ/I$, using a certain set $AP(n)$ of overlaps constructed inductively by linking generators of I, Bardzell gave a minimal projective Λ^{e}-resolution $(P_\bullet, \phi_\bullet)$ of Λ in [3] (so called Bardzell's resolution). By using Bardzell's resolution, the Hochschild cohomology of monomial algebras are studied in the following papers [11], [12], [9], etc.

In general, it is not easy to calculate the Hochschild cohomology of a finite-dimensional algebra. In order to calculate the Hochschild cohomology groups of a quiver algebra, can we use calculations of the Hochschild cohomology groups of quiver algebras obtained by subquivers of the original quiver?

In this paper, for a finite-dimensional monomial algebra Λ, we propose a method so that we easily calculate the Hochschild cohomology groups of Λ under some conditions. Let Q be a finite connected quiver and $Q^{(i)}$ $(i = 1, 2)$ a subquiver of Q such that $Q = Q^{(1)} \cup Q^{(2)} = (Q_0^{(1)} \cup Q_0^{(2)}, Q_1^{(1)} \cup Q_1^{(2)})$. Let $I^{(1)} = \langle X \rangle$ (resp. $I^{(2)} = \langle Y \rangle$) be a monomial ideal of $kQ^{(1)}$ (resp. $kQ^{(2)}$) for X (resp. Y) a set of paths of $kQ^{(1)}$ (resp. $kQ^{(2)}$) and $I = \langle X, Y \rangle$ a monomial ideal of kQ. We assume that I and $I^{(i)}$ $(i = 1, 2)$ are admissible ideals. Then we define $\Lambda = kQ/I$, $\Lambda_{(1)} = kQ^{(1)}/I^{(1)}$ and $\Lambda_{(2)} = kQ^{(2)}/I^{(2)}$. Hence Λ and $\Lambda_{(i)}$ are finite-dimensional monomial algebras for $i = 1, 2$. For the monomial algebra Λ, under a *separability condition* (i.e. $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$), we investigate the minimal projective Λ^{e}-module resolution of Λ given by Bardzell ([3]). Moreover, under an additional condition, we show that, for $n \geq 2$, the Hochschild cohomology group $\mathrm{HH}^n(\Lambda)$ of Λ is isomorphic to the direct sum of the Hochschild cohomology groups $\mathrm{HH}^n(\Lambda_{(1)})$ and $\mathrm{HH}^n(\Lambda_{(2)})$.

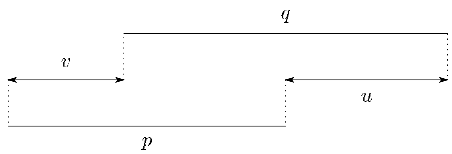
Throughout this paper, for all arrows a of Q, we denote the origin of a by $o(a)$ and the terminus of a by $t(a)$. Also, for simplicity, we denote $\otimes_k$ by $\otimes$.

2. The set $AP(n)$ of overlaps and Bardzell's resolution

2.1. The set $AP(n)$ of overlaps.

In this section, following [3] and [11], we will summarize the definition of the set $AP(n)$ $(n \geq 0)$ of overlaps.

Definition 1. A path $q \in kQ$ overlaps a path $p \in kQ$ with overlap pu if there exist u, v such that $pu = vq$ and $1 \leq l(u) \leq l(q)$, where $l(x)$ denotes the length of a path $x \in kQ$. Note that we allow $l(x) = 0$ here.



A path q properly overlaps a path p with overlap pu if q overlaps p and $l(v) \geq 1$.

Let $\Lambda = kQ/I$ be a finite-dimensional monomial algebra where $I = \langle \rho \rangle$ has a minimal set of generators ρ of paths of length at least 2.

Definition 2. For $n = 0, 1, 2$, we set

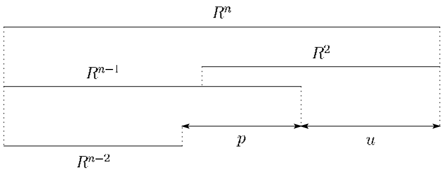
- $AP(0) := Q_0 =$(the set of all vertices of Q);

- $AP(1) := Q_1$ =(the set of all arrows of Q);
- $AP(2) := \rho$.

For $n \geq 3$, we define the set $AP(n)$ of all overlaps R^n formed in the following way: We say that $R^2 \in AP(2)$ maximally overlaps $R^{n-1} \in AP(n-1)$ with overlap $R^n = R^{n-1}u$ if

(1) $R^{n-1} = R^{n-2}p$ for some path p and $R^{n-2} \in AP(n-2)$;
(2) R^2 overlap p with overlap pu;
(3) there is no element of $AP(2)$ which overlaps p with overlap being a proper prefix of pu.

The construction of the paths in $AP(n)$ may be illustrated with the following picture of R^n:



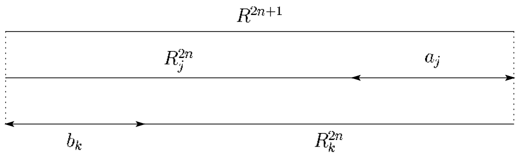
In short, overlaps are constructed by linking generators of an admissible monomial ideal I. A sequence of those generators of I is called the associated sequence of paths ([10]).

2.2. **Bardzell's resolution.** For a monomial algebra $\Lambda = kQ/I$, by using the set $AP(n)$, Bardzell determined a minimal projective Λ^{e}-resolution $(P_\bullet, \phi_\bullet)$ of Λ in [3].

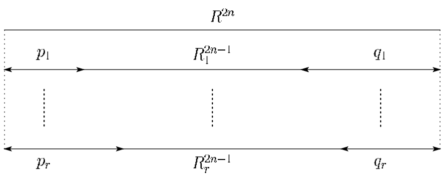
Definition 3. Let $(P_\bullet, \phi_\bullet)$ be the minimal projective Λ^{e}- resolution of Λ in [3]. Then, for $n \geq 0$, we set

$$P_n = \coprod_{R^n \in AP(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda.$$

From [3], if $R^{2n+1} \in AP(2n+1)$, then there uniquely exist R_j^{2n}, $R_k^{2n} \in AP(2n)$ and some paths a_j, b_k such that $R^{2n+1} = R_j^{2n} a_j = b_k R_k^{2n}$.



For even degree elements $R^{2n} \in AP(2n)$, there exist $r \geq 1$, $R_l^{2n-1} \in AP(2n-1)$ and paths p_l, q_l for $l = 1, 2, \ldots, r$ such that $R^{2n} = p_1 R_1^{2n-1} q_1 = \cdots = p_r R_r^{2n-1} q_r$.



Remark 4. Note that $o(R_j^{2n}) \otimes a_j \in \Lambda o(R_j^{2n}) \otimes t(R_j^{2n})\Lambda$ and $b_k \otimes t(R_k^{2n}) \in \Lambda o(R_k^{2n}) \otimes t(R_k^{2n})\Lambda$. Also, note that $p_l \otimes q_l \in \Lambda o(R_l^{2n-1}) \otimes t(R_l^{2n-1})\Lambda$.

Definition 5. The map $\phi_{2n+1} : P_{2n+1} \longrightarrow P_{2n}$ is given as follows. If $R^{2n+1} = R_j^{2n} a_j = b_k R_k^{2n} \in AP(2n+1)$, then

$$o(R^{2n+1}) \otimes t(R^{2n+1}) \longmapsto o(R_j^{2n}) \otimes a_j - b_k \otimes t(R_k^{2n}).$$

The map $\phi_{2n} : P_{2n} \longrightarrow P_{2n-1}$ is given as follows. If $R^{2n} = p_1 R_1^{2n-1} q_1 = \cdots = p_r R_r^{2n-1} q_r$, then

$$o(R^{2n}) \otimes t(R^{2n}) \longmapsto \sum_{l=1}^{r} p_l \otimes q_l.$$

The following result is the main theorem in [3].

Bardzell's Theorem ([3, Theorem 4.1]) Let Q be a finite quiver, and suppose that $\Lambda = kQ/I$ is a monomial algebra with an admissible ideal I. Then the sequence

$$\cdots \to P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \to 0$$

is a minimal projective resolution of Λ as a right Λ^{e}-module, where π is the multiplication map.

3. The decomposition of Hochschild cohomology groups

We recall our setting.

- $Q = Q^{(1)} \cup Q^{(2)}$,
- $I^{(1)} = \langle X \rangle$ be a monomial ideal generated by X a set of paths of $kQ^{(1)}$,
- $I^{(2)} = \langle Y \rangle$ a monomial ideal generated by Y a set of paths of $kQ^{(2)}$,
- $I = \langle X, Y \rangle$ a monomial ideal of kQ,
- $\Lambda = kQ/I$, $\Lambda_{(1)} = kQ^{(1)}/I^{(1)}$, $\Lambda_{(2)} = kQ^{(2)}/I^{(2)}$: finite-dimensional algebras,
- $AP(2) := X \cup Y$, $AP^{(1)}(2) := X$, $AP^{(2)}(2) := Y$.

Then, as in the definition of $AP(n)$ of overlaps, we define $AP^{(1)}(n)$, $AP^{(2)}(n)$. Moreover, we define projective Λ^{e}-modules as follows:

$$P_n^{(1)} = \coprod_{R^n \in AP^{(1)}(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda,$$
$$P_n^{(2)} = \coprod_{R^n \in AP^{(2)}(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda,$$
$$P_n = \coprod_{R^n \in AP(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda.$$

To prove our main result, we need the following lemma. As mentioned in Introduction, we consider the *separability condition* $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$.

Lemma 6. *Let $i \in \{1, 2\}$. If we assume $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$, then we have the following*:

(a) *For all $n \geq 1$, $AP(n) = AP^{(1)}(n) \cup AP^{(2)}(n)$.*

(b) *For all $n \geq 1$, $AP^{(1)}(n) \cap AP^{(2)}(n) = \varnothing$.*

(c) *Let $n \geq 1$ and $p^n \in AP(n)$. Then R^n is a path of $kQ^{(i)}$ if and only if $R^n \in AP^{(i)}(n)$.*

By Bardzell's Theorem and Lemma 6, we have the following proposition.

Proposition 7. ([14, Proposition 3.2]) *If the condition $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$ holds, then, in the following minimal projective resolution of Λ:*

$$\cdots \to P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} P_{n-1} \to \cdots \xrightarrow{\phi_3} P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0,$$

for any $n \geq 1$, P_n is isomorphic to $P_n^{(1)} \oplus P_n^{(2)}$ as right Λ^e-modules and $\phi_{n+1} = \phi_{n+1}^{(1)} \oplus \phi_{n+1}^{(2)}$, where $\phi_{n+1}^{(i)} : P_{n+1}^{(i)} \to P_n^{(i)}$ $(i = 1, 2)$ is the restriction of ϕ_{n+1}.

Remark 8. For $i = 1, 2$, $b_k \in \Lambda_{(i)} o(R_k^{2n})$, $a_j \in t(R_j^{2n})\Lambda_{(i)}$, $p_l \in \Lambda_{(i)} o(R_j^{2n+1})$ and $q_l \in t(R_l^{2n+1})\Lambda_{(i)}$ actually hold. So, for $n \geq 1$, $\phi_{n+1}^{(i)}$ sends $\coprod_{R^{n+1} \in AP^{(i)}(n+1)} \Lambda_{(i)} o(R^{n+1}) \otimes t(R^{n+1})\Lambda_{(i)}$ to $\coprod_{R^n \in AP^{(i)}(n)} \Lambda_{(i)} o(R^n) \otimes t(R^n)\Lambda_{(i)}$ (not just to $\coprod_{R^n \in AP(n)} \Lambda o(R^n) \otimes t(R^n)\Lambda$). Therefore, $(\coprod_{R^n \in AP^{(i)}(n)} \Lambda_{(i)} o(R^n) \otimes t(R^n)\Lambda_{(i)}; \phi_{n+1}^{(i)})_{n \geq 1}$ is exactly a part of degree $n \geq 1$ for the minimal projective resolution of $\Lambda_{(i)}$ $(i = 1, 2)$.

The following theorem is our main result.

Theorem 9. ([14, Theorem 3.3]) *If the condition $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$ holds and, for each $i = 1, 2$, $o(R^n)\Lambda t(R^n) = o(R^n)\Lambda_{(i)} t(R^n)$ holds for any $n \geq 1$ and any $R^n \in AP^{(i)}(n)$, then we have the direct sum decomposition of Hochschild cohomology groups*

$$\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$$

for any $n \geq 2$.

Remark 10. For $n = 0, 1$, the above equation fails in general (see Example 14 for the case $n = 1$).

If $Q_0^{(1)} \cap Q_0^{(2)} = \{v_0\}$ and $v_0 \Lambda v_0 = k v_0$, then we have $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$. Also, by Lemma 6 and Theorem 9, we have the following corollary.

Corollary 11. ([14, Corollary 3.4]) *In the case $Q_0^{(1)} \cap Q_0^{(2)} = \{v_0\}$ and $v_0 \Lambda v_0 = k v_0$, we have the direct sum decomposition of the Hochschild cohomology groups*

$$\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$$

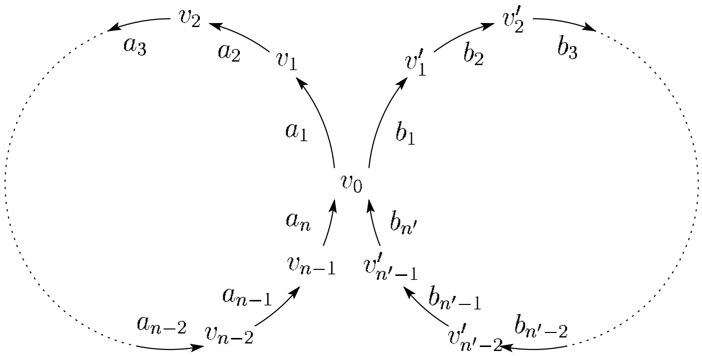
for any $n \geq 2$.

Remark 12. Hence, for a finite dimensional monomial algebra obtained by linking some quivers bound by monomial relations successively, we can also decompose the Hochschild cohomology groups as in Corollary 11.

4. Examples

In this section, we give two examples of monomial algebras satisfying the condition $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$.

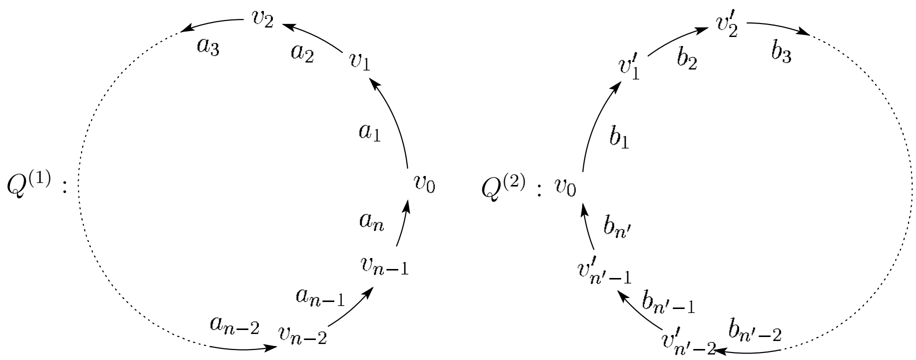
Example 13. Let Q be a quiver



bound by

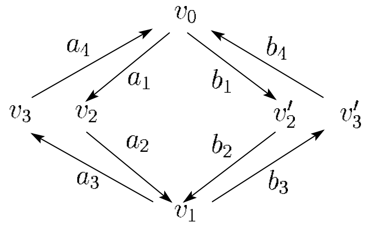
$$\begin{aligned} I =& \langle a_1a_2\cdots a_m, a_2a_3\cdots a_{m+1}, \dots, a_na_1\cdots a_{-n+m+1}, \\ & b_1b_2\cdots b_{m'}, b_2b_3\cdots b_{m'+1}, \dots, b_{n'}b_1\cdots b_{-n'+m'+1}\rangle \end{aligned}$$

for any integers m, $m' \geq 2$ with $m \leq n$ and $m' \leq n'$. We set the algebra $\Lambda = kQ/I$. Let $Q^{(1)}$ be the subquiver of Q bound by $I^{(1)} = \langle a_1a_2\cdots\ a_m, a_2a_3\cdots a_{m+1}, \dots, a_na_1\cdots a_{-n+m+1}\rangle$ and $Q^{(2)}$ be the subquiver of Q bound by $I^{(2)} = \langle b_1b_2\cdots\ b_{m'}, b_2b_3\cdots b_{m'+1}, \dots,\ b_{n'}b_1 \cdots b_{-n'+m'+1}\rangle$, where $Q_0^{(1)} \cap Q_0^{(1)} = \{v_0\}$ and $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$. We set $\Lambda_{(i)} = kQ^{(i)}/I^{(i)}$



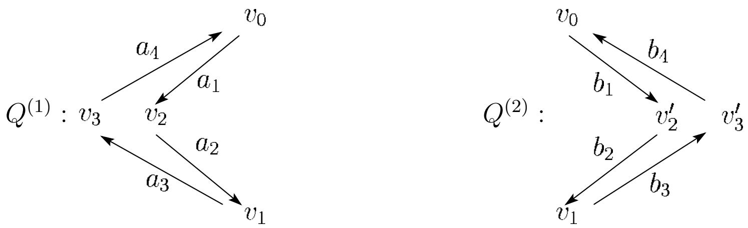
for $i = 1, 2$. Then the condition of Corollary 11 is satisfied. Applying Corollary 11, we obtain the direct sum decomposition of the Hochschild cohomology groups $\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$ for any $n \geq 2$. Also, since $\Lambda_{(i)}$ $(i = 1, 2)$ is a self-injective Nakayama algebra, we know the dimension of $\mathrm{HH}^n(\Lambda_{(i)})$ from [5, Propositions 4.4, 5.3] for $i = 1, 2$, and so we have the dimension of $\mathrm{HH}^n(\Lambda)$ by the decomposition above.

Example 14. Let Q be a quiver



bound by $I = \langle a_1a_2, a_2a_3, a_3a_4, a_4a_1, b_1b_2, b_2b_3, b_3b_4, b_4b_1\rangle$. We set the algebra $\Lambda = kQ/I$. Let $Q^{(1)}$ be the subquiver of Q bound by $I^{(1)} = \langle a_1a_2, a_2a_3,\ a_3a_4, a_4a_1\rangle$ and $Q^{(2)}$ be the subquiver of Q bound by $I^{(2)} = \langle b_1b_2, b_2b_3,\ b_3b_4, b_4b_1\rangle$, where $Q_0^{(1)} \cap Q_0^{(1)} = \{v_0, v_1\}$ and $Q_1^{(1)} \cap Q_1^{(2)} = \varnothing$.

We set $\Lambda_{(i)} = kQ^{(i)}/I^{(i)}$ for $i = 1, 2$. Then $AP^{(1)}(1) \cap AP^{(2)}(1) = \varnothing$ holds and for each $i = 1, 2$, $o(R^n)\Lambda t(R^n) = o(R^n)\Lambda_{(i)}t(R^n)$ holds for any $n \geq 1$ and any $R^n \in AP^{(i)}(n)$. Applying Theorem 9, we obtain the direct sum decomposition of the Hochschild cohomology groups $\mathrm{HH}^n(\Lambda) \cong \mathrm{HH}^n(\Lambda_{(1)}) \oplus \mathrm{HH}^n(\Lambda_{(2)})$ for any $n \geq 2$.



On the other hand, by direct computations, we have $\dim_k \mathrm{HH}^1(\Lambda) = 3$ and $\dim_k \mathrm{HH}^1(\Lambda_{(i)}) = 1$ $(i = 1, 2)$. Hence the above decomposition does not hold for $n = 1$.

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DUALITIES IN STABLE CATEGORIES

MITSUO HOSHINO, NORITSUGU KAMEYAMA AND HIROTAKA KOGA

Abstract. We provide a sufficient condition for a left and right noetherian ring A to have finite selfinjective dimension on one side and, as a corollary to it, we also provide a necessary and sufficient condition for A to have finite selfinjective dimension on both sides.

Let A be a left and right coherent ring. We denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely presented right A-modules. We consider left A-modules as right A^{op}-modules, where A^{op} denotes the opposite ring of A. For each $n > 0$ we denote by $\mathcal{G}_A^n$ the full subcategory of mod-A consisting of $X \in$ mod-A with $\mathrm{Ext}_A^i(X, A) = 0$ for $1 \le i \le n$ and, for convenience's sake, we set $\mathcal{G}_A^0 =$ mod-A. We set $C_A = \oplus\, E_A(S)$, where S runs over the non-isomorphic simple modules in Mod-A. Such a module C_A is unique up to isomorphism and called a minimal cogenerator for Mod-A. Extending [9, Lemma A] to coherent rings, we showed in [5] that if flat dim $C_{A^{\mathrm{op}}} < \infty$ and flat dim $C_A < \infty$ then flat dim $C_{A^{\mathrm{op}}} =$ flat dim C_A.

In this note, we first show that for any $n \ge 0$ we have flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \le n$ if and only if for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with Y Gorenstein projective and proj dim $Z \le n - 1$.

Next, we provide a condition which implies flat dim $C_{A^{\mathrm{op}}} \le n$. It is obvious that $C_{A^{\mathrm{op}}}$ is flat if and only if $\mathcal{G}_A^0 = \mathcal{G}_A^1$. Since for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^1$ and Z projective, it follows that flat dim $C_{A^{\mathrm{op}}} \le 1$ if and only if $\mathcal{G}_A^1 = \mathcal{G}_A^2$. So, in the following, we assume $n \ge 2$.

We denote by $D(-)$ both $\mathbf{R}\mathrm{Hom}_A^\bullet(-, A)$ and $\mathbf{R}\mathrm{Hom}_{A^{\mathrm{op}}}^\bullet(-, A)$. Our main theorem states that flat dim $C_{A^{\mathrm{op}}} \le n$ if the following three conditions are satisfied: (a) $\mathcal{G}_A^n = \mathcal{G}_A^{n+1}$; (b) $\mathrm{H}^i(D\sigma'_{\ge n-1}(\sigma_{\le n}(DX))) = 0$ for all $X \in \mathcal{G}_A^{n-2}$ and $i \le -2$; (c) for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^{n-2}$ and proj dim $Z \le n - 1$. In the above, the condition (c) is always satisfied for $n = 2$ and 3. Also, as a corollary to this theorem, we show that flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \le n$ if and only if the following three conditions are satisfied: (a) $\mathcal{G}_A^n$ consists only of Gorenstein projectives; (b) $\mathrm{H}^i(D\sigma'_{\ge n-1}(\sigma_{\le n}(DX))) = 0$ for all $X \in \mathcal{G}_A^{n-2}$ and $i \le -2$; (c) for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^{n-2}$ and proj dim $Z \le n - 1$.

The detailed version of this paper will be submitted for publication elsewhere.

1. Stable module theory

For a ring A, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. In particular, we denote by $\mathrm{Hom}_A(-,-)$ (resp., $\mathrm{Hom}_{A^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-A (resp., Mod-A^{op}).

In this note, complexes are cochain complexes and modules are considered as complexes concentrated in degree zero. We denote by $\mathcal{K}(\text{Mod-}A)$ the homotopy category of complexes over Mod-A, by $\mathcal{K}^-(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}(\text{Mod-}A)$ consisting of bounded above complexes over $\mathcal{P}_A$ and by $\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}^-(\mathcal{P}_A)$ consisting of complexes with bounded cohomology. We denote by $\mathcal{D}(\text{Mod-}A)$ the derived category of complexes over Mod-A. Also, we denote by $\mathrm{Hom}^\bullet_A(-,-)$ the associated single complex of the double hom complex and by $\mathbf{R}\mathrm{Hom}^\bullet_A(-,A)$ the right derived functor of $\mathrm{Hom}^\bullet_A(-,A)$. We refer to [2], [4] and [8] for basic results in the theory of derived categories.

Definition 1. For a complex $X^\bullet$ and an integer $n \in \mathbb{Z}$, we denote by $\mathrm{Z}^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the nth cycle, the nth cocycle and the nth cohomology, respectively, and define the following truncations:

$$\sigma_{\le n}(X^\bullet) : \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Z}^n(X^\bullet) \to 0 \to \cdots ,$$
$$\sigma'_{\ge n}(X^\bullet) : \cdots \to 0 \to \mathrm{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots .$$

Note that for each $n \in \mathbb{Z}$ we have additive functors

$$\sigma_{\le n}(-), \sigma'_{\ge n}(-) : \mathcal{D}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$$

which are not exact.

Definition 2 ([3])**.** A module $X \in \text{Mod-}A$ is said to be coherent if it is finitely generated and every finitely generated submodule of it is finitely presented. A ring A is said to be left (resp., right) coherent if it is coherent as a left (resp., right) A-module.

Throughout the rest of this note, A is assumed to be a left and right coherent ring. Note that mod-A consists of coherent modules and is a thick abelian subcategory of Mod-A in the sense of [4].

We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ the full triangulated subcategory of $\mathcal{D}(\text{Mod-}A)$ consisting of complexes over mod-A with bounded cohomology. Note that the canonical functor $\mathcal{K}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$ gives rise to an equivalence of triangulated categories $\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$.

We denote by $D(-)$ both $\mathbf{R}\mathrm{Hom}^\bullet_A(-,A)$ and $\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-,A)$. There exists a bifunctorial isomorphism

$$\theta_{M^\bullet,X^\bullet} : \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A^{\mathrm{op}})}(M^\bullet, DX^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, DM^\bullet)$$

for $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ and $M^\bullet \in \mathcal{D}(\text{Mod-}A^{\mathrm{op}})$. For each $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ we set

$$\eta_{X^\bullet} = \theta_{DX^\bullet,X^\bullet}(\mathrm{id}_{DX^\bullet}) : X^\bullet \to D^2X^\bullet = D(DX^\bullet).$$

Definition 3 ([1] and [7]). A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have finite Gorenstein dimension if $DX^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})$ and if $\eta_{X^\bullet}$ is an isomorphism. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes of finite Gorenstein dimension.

For a module $X \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$, we set

$$\text{G-dim } X = \sup\{\, i \geq 0 \mid \mathrm{Ext}^i_A(X, A) \neq 0\}$$

if $X \neq 0$, and G-dim $X = 0$ if $X = 0$. Also, we set G-dim $X = \infty$ for a module $X \in \text{mod-}A$ with $X \notin \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$. Then G-dim X is called the Gorenstein dimension of $X \in \text{mod-}A$. A module $X \in \text{mod-}A$ is said to be Gorenstein projective if it has Gorenstein dimension zero.

Note that a module $X \in \text{mod-}A$ is Gorenstein projective if and only if it is reflexive, i.e., the canonical homomorphism

$$X \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A), x \mapsto (f \mapsto f(x))$$

is an isomorphism and $\mathrm{Ext}^i_A(X, A) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A) = 0$ for $i \neq 0$.

Definition 4. For each $X \in \text{mod-}A$, taking a projective resolution $P^\bullet \to X$ in mod-A, we set $\Omega^n X = \mathrm{Z}'^{-n}(P^\bullet)$ for $n \geq 0$ and $\mathrm{Tr}X = \mathrm{Z}'^1(\mathrm{Hom}^\bullet_A(P^\bullet, A))$.

We denote by $\underline{\text{mod}}$-A the residue category mod-$A/\mathcal{P}_A$ and by $\underline{\mathrm{Hom}}_A(-,-)$ the morphism set in $\underline{\text{mod}}$-A. Then we have additive functors

$$\mathrm{Tr} : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A^{\mathrm{op}} \quad \text{and} \quad \Omega^n : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A$$

for $n \geq 0$. We set $\Omega = \Omega^1$. Then Ω^n is the nth power of Ω for $n \geq 0$.

Proposition 5. *For any $n \geq 0$ there exists a bifunctorial isomorphism*

$$\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{Tr}(\Omega^n X), M) \xrightarrow{\sim} \underline{\mathrm{Hom}}_A(\mathrm{Tr}(\Omega^n M), X)$$

for $X \in \text{mod-}A$ and $M \in \text{mod-}A^{\mathrm{op}}$.

For each $n > 0$ we denote by $\mathcal{G}^n_A$ the full subcategory of mod-A consisting of $X \in \text{mod-}A$ with $\mathrm{Ext}^i_A(X, A) = 0$ for $1 \leq i \leq n$ and, for convenience's sake, we set $\mathcal{G}^0_A = \text{mod-}A$.

Corollary 6 (cf. [6, Proposition 1.1.1]). *For any $n \geq 0$ the pair of functors*

$$\mathrm{Tr} \circ \Omega^n : \mathcal{G}^n_A/\mathcal{P}_A \to \mathcal{G}^n_{A^{\mathrm{op}}}/\mathcal{P}_{A^{\mathrm{op}}} \quad \textit{and} \quad \mathrm{Tr} \circ \Omega^n : \mathcal{G}^n_{A^{\mathrm{op}}}/\mathcal{P}_{A^{\mathrm{op}}} \to \mathcal{G}^n_A/\mathcal{P}_A$$

defines a duality.

Lemma 7. *For any $n \geq 0$ the following are equivalent.*

(1) $\mathcal{G}^n_A = \mathcal{G}^{n+1}_A$.
(2) $\mathcal{G}^n_{A^{\mathrm{op}}}$ *consists only of torsionless modules.*

Lemma 8. *For any $n \geq 1$ and $X \in \text{mod-}A$ the following are equivalent.*

(1) G-dim $X \leq n$.
(2) *There exists an exact sequence $0 \to Z \to Y \to X \to 0$ in* mod-A *with Y Gorenstein projective and* proj dim $Z \leq n - 1$.

Lemma 9. *For any $X \in \text{mod-}A$ there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in* mod-A *with $Y \in \mathcal{G}^1_A$ and $Z \in \mathcal{P}_A$.*

2. Applications

In the following, we denote by $E_A(-)$ an injective envelope of a module in Mod-A and set $C_A = \oplus\, E_A(S)$, where S runs over the non-isomorphic simple modules in Mod-A. Such a module C_A is unique up to isomorphism and called a minimal cogenerator for Mod-A. We have seen in [5] that if flat dim $C_{A^{\mathrm{op}}} < \infty$ and flat dim $C_A < \infty$ then flat dim $C_{A^{\mathrm{op}}} =$ flat dim C_A.

According to Lemma 8, [5, Theorem 3.6] implies the following.

Proposition 10. *For any $n \geq 0$ the following are equivalent.*

(1) flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \leq n$.

(2) *For any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with Y Gorenstein projective and* proj dim $Z \leq n-1$.

Remark 11. For any $n \geq 0$, flat dim $C_{A^{\mathrm{op}}} \leq n$ if and only if $\mathrm{Ext}_A^{n+1}(-, A)$ vanishes on mod-A. In particular, $C_{A^{\mathrm{op}}}$ is flat if and only if $\mathcal{G}_A^0 = \mathcal{G}_A^1$. Also, Lemma 9 implies that flat dim $C_{A^{\mathrm{op}}} \leq 1$ if and only if $\mathcal{G}_A^1 = \mathcal{G}_A^2$.

Throughout the rest of this note, we fix an integer $n \geq 2$.

Theorem 12. *We have* flat dim $C_{A^{\mathrm{op}}} \leq n$ *if the following three conditions are satisfied:*

(a) $\mathcal{G}_A^n = \mathcal{G}_A^{n+1}$;

(b) $\mathrm{H}^i(D\sigma'_{\geq n-1}(\sigma_{\leq n}(DX))) = 0$ *for all $X \in \mathcal{G}_A^{n-2}$ and $i \leq -2$;*

(c) *for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^{n-2}$ and* pd $Z \leq n-1$.

In the above, the condition (c) is trivially satisfied if $n = 2$. Also, it follows by Lemma 9 that the condition (c) is satisfied for $n = 3$.

Corollary 13. *We have* flat dim $C_{A^{\mathrm{op}}} =$ flat dim $C_A \leq n$ *if and only if the following three conditions are satisfied:*

(a) $\mathcal{G}_A^n$ *consists only of Gorenstein projectives;*

(b) $\mathrm{H}^i(D\sigma'_{\geq n-1}(\sigma_{\leq n}(DX))) = 0$ *for all $X \in \mathcal{G}_A^{n-2}$ and $i \leq -2$;*

(c) *for any $X \in$ mod-A there exists an exact sequence $0 \to Z \to Y \to X \to 0$ in mod-A with $Y \in \mathcal{G}_A^{n-2}$ and* pd $Z \leq n-1$.

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BRAUER INDECOMPOSABILITY OF SCOTT MODULES

HIROKI ISHIOKA

Abstract. Let p be a prime number and k an algebraically closed field of characteristic p. Let G be a finite group and P a p-subgroup of G. In this article, we study the relationship between the fusion system $\mathcal{F}_P(G)$ and the Brauer indecomposability of the Scott kG-module in the case that P is not necessarily abelian. We give an equivalent condition for the Scott kG-module with vertex P to be Brauer indecomposable.

1. Introduction

Let p be a prime number, G a finite group, and k an algebraically closed field of prime characteristic p. For a kG-module M and a p-subgroup Q of G, we denote by $M(Q)$ the Brauer quotient of M with respect to Q. The Brauer quotient $M(Q)$ has naturally the structure of a $kN_G(Q)$-module.

Definition 1. A kG-module M is said to be Brauer indecomposable if $M(Q)$ is indecomposable or zero as a $kQC_G(Q)$-module for any p-subgroup Q of G.

Brauer indecomposability of p-permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2]).

Let P be a p-subgroup of G. We denote by $\mathcal{F}_P(G)$ the fusion system of G over P. In [1], a relationship between fusion system $\mathcal{F}_P(G)$ and Brauer indecomposability of p-permutation modules with vertex P was given. One of the main result in [1] is the following.

Theorem 2 ([1, Theorem 1.1]). *Let P be a p-subgroup of G and M an indecomposable p-permutation kG-module with vertex P. If M is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.*

In the special case that P is abelian and M is the Scott kG-module $S(G, P)$, the converse of the above theorem holds.

Theorem 3 ([1, Theorem 1.2]). *Let P be an abelian p-subgroup of G. If $\mathcal{F}_P(G)$ is saturated, then $S(G, P)$ is Brauer indecomposable.*

In general, the above theorem does not hold for non-abelian P. However, there are some cases in which the Scott kG-module $S(G, P)$ is Brauer indecomposable, even if P is not necessarily abelian.

We study the condition that $S(G, P)$ to be Brauer indecomposable where P is not necessarily abelian. The following result gives an equivalent condition for Scott kG-module with vertex P to be Brauer indecomposable.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 4. *Let G be a finite group and P a p-subgroup of G. Suppose that $M = S(G, P)$ and that $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent.*

(i) *M is Brauer indecomposable.*

(ii) *For each fully normalized subgroup Q of P, the module $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable.*

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

The following theorem shows that $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable if Q satisfies some conditions.

Theorem 5. *Let G be a finite group, P a p-subgroup of G and Q a fully normalized subgroup of P. Suppose that $\mathcal{F}_P(G)$ is saturated. Moreover, we assume that there is a subgroup H_Q of $N_G(Q)$ satisfying following two conditions:*

(i) $N_P(Q) \in Syl_p(H_Q)$

(ii) $|N_G(Q) : H_Q| = p^a$ $(a \geq 0)$

Then $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable.

The following is a consequence of above two theorems.

Corollary 6. *Let G be a finite group and P a p-subgroup of G. Suppose that $\mathcal{F}_P(G)$ is saturated. If for every fully normalized subgroup Q of P there is a subgroup H_Q of $N_G(Q)$ satisfies the conditions of 5, then $S(G, P)$ is Brauer indecomposable.*

Throughout this article, we denote by $L \cap_G H$ the set $\{{}^gL \cap H \mid g \in G\}$ for subgroups L and K of G.

2. Preliminaries

2.1. **Scott modules.** First, We recall the definition of Scott modules and some of its properties:

Definition 7. For a subgroup H of G, the Scott kG-module $S(G, H)$ with respect to H is the unique indecomposable summand of $\mathrm{Ind}^G_H k_H$ that contains the trivial kG-module.

If P is a Sylow p-subgroup of H, then $S(G, H)$ is isomorphic to $S(G, P)$. By definition, the Scott kG-module $S(G, P)$ is a p-permutation kG-module.

By Green's indecomposability criterion, the following result holds.

Lemma 8. *Let H be a subgroup of G such that $|G : H| = p^a$(for some $a \geq 0$). Then $\mathrm{Ind}^G_H k_H$ is indecomposable. In particular, we have that*

$$S(G, H) \cong \mathrm{Ind}^G_H.$$

Hence, for p-subgroup P of G, if there is a subgroup H of G such that P is a Sylow p-subgroup of H and $|G : H| = p^a$, then we have that

$$S(G, P) \cong \mathrm{Ind}^G_H k_H.$$

The following theorem gives us information of restrictions of Scott modules.

Theorem 9 ([3, Theorem 1.7]). *Let H be a subgroup of G and P a p-subgroup of G. If Q is a maximal element of $P \cap_G H$, then $S(H, Q)$ is a direct summand of $\mathrm{Res}^G_H S(G, P)$.*

2.2. **Brauer quotients.** Let M be a kG-module and H a subgroup of G. Let M^H be the set of H-fixed elements in M. For subgroups L of H, we denote by Tr^G_H the trace map $\mathrm{Tr}^H_L : M^L \longrightarrow M^H$. Brauer quotients are defined as follows.

Definition 10. Let M be a kG-module. For a p-subgroup Q of G, the Brauer quotient of M with respect to Q is the k-vector space

$$M(Q) := M^Q/(\sum_{R<Q} \mathrm{Tr}^Q_R(M^R)).$$

This k-vector space has a natural structure of $kN_G(Q)$-module.

Proposition 11. *Let P be a p-subgroup of G and $M = S(G, P)$. Then $M(P) \cong S(N_G(P), P)$.*

Proposition 12. *Let M be an indecomposable p-permutation kG-module with vertex P. Let Q be a p-subgroup of G. Then $Q \leq_G P$ if and only if $M(Q) \neq 0$.*

2.3. **Fusion systems.** For a p-subgroup P of G, the fusion system $\mathcal{F}_P(G)$ of G over P is the category whose objects are the subgroups of P, and whose morphisms are the group homomorphisms induced by conjugation in G.

Definition 13. Let P be a p-subgroup of G

(i) A subgroup Q of P is said to be fully normalized in $\mathcal{F}_P(G)$ if $|N_P({}^xQ)| \leq |N_P(Q)|$ for all $x \in G$ such that ${}^xQ \leq P$.

(ii) A subgroup Q of P is said to be fully automized in $\mathcal{F}_P(G)$ if $p \nmid |N_G(Q) : N_P(Q)C_G(Q)|$.

(iii) A subgroup Q of P is said to be receptive in $\mathcal{F}_P(G)$ if it has the following property: for each $R \leq P$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}_P(G)}(R, Q)$, if we set

$$N_\varphi := \{g \in N_P(Q) \mid \exists h \in N_P(R), c_g \circ \varphi = \varphi \circ c_h\},$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\overline{\varphi}\mid_R = \varphi$.

Saturated fusion systems are defined as follows.

Definition 14. Let P be a p-subgroup of G. The fusion system $\mathcal{F}_P(G)$ is saturated if the following two conditions are satisfied:

(i) P is fully normalized in $\mathcal{F}_P(G)$.

(ii) For each subgroup Q of P, if Q is fully normalized in $\mathcal{F}_P(G)$, then Q is receptive in $\mathcal{F}_P(G)$.

For example, if P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G)$ is saturated.

3. Sketch of Proof

In this section, let P be a p-subgroup of G and M the Scott module $S(G, P)$.

Lemma 15. *If $Q \leq P$ is fully normalized in $\mathcal{F}_P(G)$, then $N_P(Q)$ is a maximal element of $P \cap_G N_G(Q)$.*

By above lemma, we can show that $S(N_G(Q), N_P(Q))$ is a direct summand of $M(Q)$ for each fully normalized subgroup Q of P. Therefore, we have that Theorem 4 (i) implies 4 (ii).

Assume that Theorem 4 (ii) holds. We prove that $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)}(M(Q))$ is indecomposable for each $Q \leq P$ by induction on $|P : Q|$. Without loss of generality, we can assume that Q is fully normalized. If $M(Q)$ is decomposable, then by the following lemma, we can show that there is a subgroup R such that $Q < R \leq P$ and $\mathrm{Res}^{N_G(R)}_{RC_G(R)}$ is decomposable, this contradicts the induction hypothesis.

Lemma 16. *Suppose that a subgroup Q of P is fully automized and receptive. Then for any $g \in G$ such that $Q \leq {}^gP$, we have that $N_{{}^gP}(Q) \leq_{N_G(Q)} N_P(Q)$.*

Hence, $M(Q)$ is indecomposable, and isomorphic to $S(N_G(Q), N_P(Q))$. Consequently, Theorem 4 (ii) implies 4 (i).

Theorem 5 is proved by using properties of Scott modules and the following lemma.

Lemma 17. *If Q is fully automized subgroup of P, and there is a subgroup $H_Q \leq N_G(Q)$ containing $N_P(Q)$ such that $|N_G(Q) : H_Q| = p^a$, then $C_G(Q)H_Q = N_G(Q)$.*

4. Example

We set $p = 2$ and

$$\begin{aligned} G &:= \langle a, x, y \mid a^4 = x^2 = e, a^2 = y^2, \\ &\qquad\qquad xax = a^{-1}, ay = ya, xy = yx\rangle, \\ P &:= \langle a, xy\rangle. \end{aligned}$$

Then G is a finite group of order 16, and P is isomorphic to the quaternion group of order 8. Hence, P is a non-abelian p-subgroup of G. One can easily show that G and P satisfy the hypothesis of the Corollary 6. Therefore, $S(G, P)$ is Brauer indecomposable.

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NOTES ON THE HOCHSCHILD HOMOLOGY DIMENSION AND TRUNCATED CYCLES

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Abstract. In this paper, we show that if an algebra KQ/I with an ideal I of KQ contained in R_Q^m for an integer $m \geq 2$ has an m-truncated cycle, then this algebra has infinitely many nonzero Hochschild homology groups, where R_Q denotes the arrow ideal. Consequently, such an algebra of finite global dimension has no m-truncated cycles and satisfies an m-truncated cycles version of the no loops conjecture.

1. Introduction

In [8], Happel remarks that if all the higher Hochschild cohomology groups vanish for a finite dimensional algebra, then does the algebra have finite global dimension? This is called "Happel's question". It is shown in [3] that this does not hold in general.

On the other hand, in [7], Han conjectures the homology version of Happel's question, that is, if all the higher Hochschild homology groups of a finite dimensional algebra vanish, then is the algebra of finite global dimension? Moreover, he shows that the counter example of Happel's question in [3] satisfies Han's conjecture in [7].

In [4], Han's conjecture is approached with focusing on the combinatorics of quivers of algebras. Specifically, it is shown that all algebras having a 2-truncated cycle in which the product of two consecutive arrows is always zero, have infinitely many nonzero Hochschild homology groups. Consequently, 2-truncated cycles version of the well-known "no loops conjecture" holds: algebras of finite global dimension have no 2-truncated cycles. In addition, for arbitrary integer $m \geq 2$, an m-truncated cycles version of the "no loops conjecture" is conjectured. In particular, it is shown that monomial algebras satisfy an m-truncated cycles version of the "no loops conjecture". For finite dimensional elementary algebras, in [9], it is shown that the no loops conjecture can be derived from an earlier result of Lenzing in [12] (cf. [10]).

In this paper, we show the following assertion: Let K be a field, Q a finite quiver, R_Q the arrow ideal of KQ and $m \geq 2$ a positive integer. If an algebra KQ/I with an ideal $I \subset KQ$ contained in R_Q^m has an m-truncated cycle, then KQ/I has infinitely many nonzero Hochschild homology groups (Theorem 6). Consequently, in the case I is an admissible ideal of KQ which is contained in R_Q^m, then KQ/I satisfies an m-truncated cycles version of the "no loops conjecture". That is, if KQ/I has finite global dimension, then it contains no m-truncated cycles (Corollary 7). This result generalizes the result [4, Corollary 3.3].

The detailed version of this paper has been published in Archiv der Mathematik.

2. Preliminaries

Let K be a commutative ring and A a unital K-algebra. Thus, there exists a nonzero ring homomorphism $K \to A$, whose image is contained in the center of A. We assume that A is finitely generated as a K-module. Throughout the paper, $\otimes$ denotes $\otimes_K$ for the sake of simplicity.

For each $n \geq 1$, we denote the n-fold tensor product $A \otimes \cdots \otimes A$ of A over K by $A^{\otimes n}$ and the enveloping algebra of A by A^e.

Definition 1 ([13])**.** The Hochschild complex is the following complex:

$$\cdots \to M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M,$$

where M is a left A^e-module, the module $M \otimes A^{\otimes n}$ is in degree n, and the map $b : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ is given by the formula

$$\begin{aligned} &b(x \otimes a_1 \otimes \cdots \otimes a_n) := xa_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i (x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n a_n x \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

The n-th Hochschild homology group $HH_n(A, M)$ of A with coefficients in the left A^e-module M is defined by the n-th homology group of the Hochschild complex above. In particular, $HH_n(A, A)$ is simply called the n-th Hochschild homology group of A, which is denoted by $HH_n(A)$.

It is well known that if the unital K-algebra A is a projective K-module, then the n-th Hochschild homology group $HH_n(A)$ is given by $\mathrm{Tor}_n^{A^e}(A, A)$. Now we recall the definition of the bar resolution of A.

Definition 2 ([13])**.** Let A be a unital K-algebra. The following resolution of the left A^e-module A denoted by C^{bar} is called the *bar resolution*:

$$C^{\mathrm{bar}} : \longrightarrow A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \longrightarrow \cdots \longrightarrow A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0,$$

where μ is multiplication and b' is defined by $b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$.

Let A and B be two K-algebras and suppose that $f : A \to B$ is a K-algebra homomorphism. Then f is a homomorphism of rings, the composition map of f and the map $K \to A$ giving the K-algebra structure of A is equal to the map $K \to B$ giving the K-algebra structure of B. This implies that $bf^{\otimes(n+1)} = f^{\otimes n} b$, therefore $\{f^{\otimes n}\}_{n \in \mathbb{N}}$ is a chain map between the Hochschild complex of A and the one of B. For each $n \geq 0$, this map of Hochschild complexes induces a map $f^{\otimes(n+1)} : HH_n(A) \to HH_n(B)$ of Hochschild homology groups. The following fact is the key of the main theorem in [4]: if we can show that the image of $HH_n(A) \to HH_n(B)$ is nonzero, then this forces $HH_n(A)$ to be nonzero. This fact is also important for our main theorem.

Finally, in [4], the Hochschild homology dimension of the algebra A is defined by

$$\mathrm{HHdim}\, A = \sup\{n \in \mathbb{Z} \mid HH_n(A) \neq 0\},$$

which is treated in the main theorem.

3. The Hochschild homology of truncated quiver algebras

In this section, for a truncated quiver algebra we give elements in the complex, induced by Sköldberg's projective resolution $\boldsymbol{P}$, which correspond to nonzero homology classes.

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. For an arrow $\alpha \in Q_1$, its source and target are denoted by $s(\alpha)$ and $t(\alpha)$, respectively. A path in Q is a sequence of arrows $\alpha_1\alpha_2\cdots\alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$. The set of all paths of length n is denoted by Q_n.

For a path γ of Q, $|\gamma|$ denotes the length of γ. A path γ is said to be a cycle if $|\gamma| \geq 1$ and its source and target coincide. The period of a cycle γ is defined by the smallest integer i such that $\gamma = \delta^j$ $(j \geq 1)$ for a cycle δ of length i, which is denoted by $\operatorname{per}\gamma$. A cycle is said to be a basic cycle if the length of the cycle coincides with its period. It is also called a proper cycle [7]. Denote by Q_n^{c} (respectively Q_n^{b}) the set of cycles (respectively basic cycles) of length n. Let $G_n = \langle g \rangle$ be the cyclic group of order n and the path $\alpha_1 \cdots \alpha_{n-1}\alpha_n$ a cycle where α_i is an arrow in Q. Then we define the action of G_n on Q_n^{c} by $g \cdot (\alpha_1 \cdots \alpha_{n-1}\alpha_n) := \alpha_n\alpha_1 \cdots \alpha_{n-1}$, and Q_n^{c}/G_n denotes the set of all G_n-orbits on Q_n^{c}. Similarly, G_n acts on Q_n^{b}, and Q_n^{b}/G_n denotes the set of all G_n-orbits on Q_n^{b}. For $\bar{\gamma} \in Q_n^{\mathrm{c}}/G_n$, we denote by $\operatorname{per}\bar{\gamma}$ the period of γ, that is $\operatorname{per}\bar{\gamma} := \operatorname{per}\gamma$. For convenience we use the notation Q_0^{c}/G_0 for the set of vertices Q_0.

Sköldberg gives an projective resolution $\boldsymbol{P}$ of a truncated quiver algebra A. Moreover, by means of the complex $\bigoplus_i \bigoplus_{\bar{\gamma} \in Q_i^{\mathrm{c}}/G_i} K_{\bar{\gamma},n}$ given by the following isomorphism:

$$A \otimes_{A^e} P_n \xrightarrow{\varphi} A \otimes_{KQ_0^e} K\Gamma^{(n)} \xrightarrow{\sim} \bigoplus_i \bigoplus_{\bar{\gamma} \in Q_i^{\mathrm{c}}/G_i} K_{\bar{\gamma},n},$$

he gives the module structure of $HH_n(A)$, where the set $\Gamma^{(*)}$ is given by

$$\Gamma^{(i)} = \begin{cases} Q_{cm} & \text{if } i = 2c\ (c \geq 0), \\ Q_{cm+1} & \text{if } i = 2c+1\ (c \geq 0). \end{cases}$$

In order to prove our main theorem, we investigate elements in $A \otimes_{KQ_0^e} \Gamma^{(*)}$ which correspond to nonzero homology classes.

Lemma 3. *Let K be a field and $A = KQ/R_Q^m$ a truncated quiver algebra. For an element $\bar{\gamma} \in Q_{cm}^{\mathrm{c}}/G_{cm}$ with $\gamma = \alpha_1 \cdots \alpha_{cm}(\alpha_1, \ldots, \alpha_{cm} \in Q_1)$, the following elements correspond to non-zero homology classes:*

$$\alpha_{(c-1)m+i+1} \cdots \alpha_{cm}\alpha_1 \cdots \alpha_{i-1} \otimes \alpha_i \cdots \alpha_{(c-1)m+i} \in A \otimes_{KQ_0^e} \Gamma^{((c-1)m+1)},$$

where $d = \gcd(m, \operatorname{per}\bar{\gamma})$ and $i = 1, 2, \ldots, d-1$.

Lemma 4. *Let K be a field and $A = KQ/R_Q^m$ a truncated quiver algebra. For an element $\bar{\gamma} \in Q_{cm+e}^{\mathrm{c}}/G_{cm+e}(1 \leq e \leq m-1)$ with $\gamma = \alpha_1 \cdots \alpha_{cm+e}(\alpha_1, \ldots, \alpha_{cm+e} \in Q_1)$, the following element corresponds to a non-zero homology class:*

$$\alpha_{cm+1} \cdots \alpha_{cm+e} \otimes \alpha_1 \cdots \alpha_{cm} \in A \otimes_{KQ_0^e} \Gamma^{(cm)}.$$

We note that there is the following chain map in [6], which we denote by θ. This chain map θ induces a quasi-isomorphism $\mathrm{id}_A \otimes \theta : A \otimes_{A^e} C^{\mathrm{bar}} \to A \otimes_{A^e} \boldsymbol{Q}$, which we denote by θ for the sake of simplicity.

A chain map π from Cibils' projective resolution $\boldsymbol{Q}$ to $\boldsymbol{P}$ given in [1] induces a quasi-isomorphism $\bar{\pi} = \mathrm{id}_{\mathrm{A}} \otimes \pi : A \otimes_{A^e} \boldsymbol{Q} \longrightarrow A \otimes_{A^e} \boldsymbol{P}$. We use the following composition map of chain maps from the Hochschild complex to Sköldberg's complex by Φ;

$$\begin{array}{l} A \otimes_{A^e} Q_n \xleftarrow{\theta} A \otimes_{A^e} (C^{\mathrm{bar}})_n = A \otimes_{A^e} A^{\otimes(n+2)} \xleftarrow{\psi} A^{\otimes(n+1)} \\ \quad \Big\downarrow \bar{\pi} \\ A \otimes_{A^e} P_n \xrightarrow{\varphi} A \otimes_{KQ_0^e} K\Gamma^{(n)} \xrightarrow{\sim} \bigoplus_i \bigoplus_{\bar{\gamma} \in Q_i^c/G_i} K_{\bar{\gamma},n}, \end{array}$$

where ψ is given by $\psi(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes_{A^e} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$.

4. The m-truncated cycles version of the "no loops conjecture"

Let K be a field, Q a finite quiver, R_Q the arrow ideal of KQ and $m \geq 2$ a positive integer. In this section, we show that if an algebra KQ/I with $I \subset R_Q^m$ has an m-truncated cycle (see Definition 5), then the algebra has infinite Hochschild homology dimension. Moreover, we show that the algebra satisfies an m-truncated cycles version of the "no loops conjecture".

If $I \subset R_Q^2$ is an ideal in the path algebra KQ, then a finite sequence $\alpha_1, \dots, \alpha_u$ of arrows which satisfies the equations $t(\alpha_i) = s(\alpha_{i+1})$ $(i = 1, \dots, u-1)$ and $t(\alpha_u) = s(\alpha_1)$ is called *a cycle* in KQ/I in [4].

Definition 5 ([4]). A cycle $\alpha_1, \dots, \alpha_u$ in KQ/I is *m-truncated* for an integer $m \geq 2$ if

$$\alpha_i \cdots \alpha_{i+m-1} = 0 \quad \text{and} \quad \alpha_i \cdots \alpha_{i+m-2} \neq 0 \quad \text{in } KQ/I$$

for all i, where the indices are modulo u.

By means of composition map Φ, we have the following our main theorem by the Lemma 3 and 4.

Theorem 6. *Let K be a field, Q a finite quiver and $I \subset KQ$ an ideal contained in R_Q^m. Suppose that KQ/I contains an m-truncated cycle $\alpha_1, \dots, \alpha_u$. Then the following holds:*

(i) *Assume that* $\gcd(m, \mathrm{per}(\alpha_1 \cdots \alpha_u)) \neq 1$. *For every $n \geq 1$ with $un \equiv 0 \pmod{m}$, the element*

$$\begin{aligned} &\alpha_{(c-1)m+2} \cdots \alpha_{cm} \otimes \alpha_1 \otimes \alpha_2 \cdots \alpha_m \otimes \alpha_{m+1} \\ &\quad \otimes \alpha_{m+2} \cdots \alpha_{2m} \otimes \alpha_{2m+1} \otimes \cdots \otimes \alpha_{(c-2)m+2} \cdots \alpha_{(c-1)m} \otimes \alpha_{(c-1)m+1}, \end{aligned}$$

where $c = un/m$, represents a nonzero element in $HH_{2c-1}(KQ/I)$.

(ii) *Let e be an integer with $1 \leq e \leq m-1$. For every $n \geq 1$ with $un \equiv e \pmod{m}$, the element*

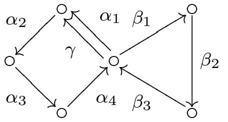
$$\begin{aligned} \sum_{0 \leq j_1, \dots, j_c \leq m-2} &\alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{un} \\ &\otimes \alpha_1 \cdots \alpha_{1+j_1} \otimes \alpha_{2+j_1} \otimes \alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2} \otimes \alpha_{4+j_1+j_2} \otimes \cdots \\ &\quad \otimes \alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c} \otimes \alpha_{2c+j_1+\cdots+j_c}, \end{aligned}$$

where $c = (un-e)/m$, represents a nonzero element in $HH_{2c}(KQ/I)$.

In particular, the Hochschild homology dimension $\mathrm{HHdim}\,(KQ/I) = \infty$.

Corollary 7. *Let K be a field, Q a finite quiver and I an admissible ideal in KQ with $I \subset R_Q^m$. If the algebra KQ/I has finite global dimension, then it contains no m-truncated cycles.*

Example 8. Let B be an algebra given by the quiver with relations:



$$\alpha_i\alpha_{i+1}\alpha_{i+2} = \beta_1\beta_2\beta_3 = \beta_3\gamma\alpha_2 = 0,$$
$$\beta_2\beta_3\alpha_1 = \beta_2\beta_3\gamma,$$

where the indices of α_i are modulo 4 ($1 \leq i \leq 4$). Then B has the 3-truncated cycle $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. By the Theorem 6, we have $\mathrm{HHdim}\, B = \infty$. Therefore, the global dimension of B is infinite.

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ON A GENERALIZATION OF COMPLEXES AND THEIR DERIVED CATEGORIES.

OSAMU IYAMA AND HIROYUKI MINAMOTO

Abstract. When we want to understand the reason why the equation $d^2 = 0$ has the beautiful consequences, one way is to consider generalizations of it and research how its properties vary. One natural candidate of a generalization is the notion of N-complex, that is, gradeds object equipped with a morphism d of degree 1 such that $d^N = 0$. This was introduced by Kapranov [5] and Sarkaria [7] independently. Nowadays there is a vast collection of literatures on the subject.

For an N-complex X, there are several cohomology functors. More precisely, for $1 \leq r \leq N-1$, we define a cohomorogy functor to be

$$\mathrm{H}^i_{(r)}(X) := \frac{\mathrm{Ker}[d^r : X^i \to X^{i+r}]}{\mathrm{Im}[d^{N-r} : X^{i-N+r} \to X^i]}.$$

As a new feature, it is observed that there are several relations between these cohomology functors [5, 1].

On the other hands, Iyama-Kato-Miyachi [4] construct and study the homotopy category $\mathsf{K}_N(R)$, the derived category $\mathsf{D}_N(R)$ of N-complexes. They showed that the derived category $\mathsf{D}_N(R)$ is equivalent as triangulated categories to the derived category (in the ordinary sense) $\mathsf{D}(R \otimes_{\mathbf{k}} \mathbf{k}\overrightarrow{A}_{N-1})$. Inspired by their results, we introduce the notion of A-complexes for a graded self-injective algebra A. We construct and study the homotopy category, the derived category of and the cohomology functors. As a consequence, we see that the relations between various cohomology functors of N-complexes comes from representation theory of the graded algebra $\mathbf{k}[\delta]/(\delta^N)$ with $deg\mathbf{k} = 0, \deg\delta = 1$.

1. N-complexes (Kapranov, Sarkaria, G. Kato, Dubois-Violette, Hiramatsu-G. Kato, Iyama-K. Kato-Miyachi ...)

1.1. N-complexes.
Our setup is the followings:

- $N \geq 2$ is an integer greater than 1.
- R is an algebra over a field $\mathbf{k}$.

For simplicity, in this note N-(A-)complexes are that of R-modules.

Definition 1. An *N-complex* X (of R-modules) is a graded R-module $\bigoplus_{i \in \mathbb{Z}} X^i$ equipped with an endomorphism d_X of degree 1 (the differential of X) such that $d_X^N = 0$.

$$d_X^N = d_X \circ d_X \circ \cdots d_X \qquad (N \text{ times }).$$

$$\cdots \to X^{i-1} \xrightarrow{d_X} X^i \xrightarrow{d_X} X^{i+1} \to \cdots$$

The detailed version of this paper will be submitted for publication elsewhere.

A morphism $f : X \to Y$ of N-complexes is a morphism of graded R-modules which is compatible with the differentials d_X and d_Y.

$$
\begin{tikzcd}
\arrow[r, "d_X"] & X^{i-1} \arrow[r, "d_X"] \arrow[d, "f^{i-1}"'] & X^i \arrow[r, "d_X"] \arrow[d, "f^i"'] & X^{i+1} \arrow[r, "d_X"] \arrow[d, "f^{i+1}"'] & {} \\
\arrow[r, "d_Y"] & Y^{i-1} \arrow[r, "d_Y"] & Y^i \arrow[r, "d_Y"] & Y^{i+1} \arrow[r, "d_Y"] & {}
\end{tikzcd}
$$

The category $\mathsf{C}_N(R)$ of N-complexes is abelian.

The notion of N-complexes is so natural that it have been studied by many researchers from various point of views.

1.2. Cohomology group $\mathrm{H}^i_{(n)}(X)$ of N-complexe X.

Definition 2. For $i \in \mathbb{Z}$ and $0 < n < N$, we define the cohomology group $\mathrm{H}^i_{(n)}(X)$ of N-complexe X which has i-th degree and n-th position to be

$$\mathrm{H}^i_{(n)}(X) := \frac{\mathrm{Ker}[d_X^n : X^i \to X^{i+n}]}{\mathrm{Im}[d_X^{N-n} : X^{i-N+n} \to X^i]}.$$

For N-complexes we have cohomology long exact sequences.

Theorem 3 (Dubois-Violette)**.** *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of N-complexes. Then we have the following exact sequence:*

$$\cdots \to \mathrm{H}^i_{(n)}(X) \to \mathrm{H}^i_{(n)}(Y) \to \mathrm{H}^i_{(n)}(Z) \to$$

$$\to \mathrm{H}^{i+n}_{(N-n)}(X) \to \mathrm{H}^{i+n}_{(N-n)}(Y) \to \mathrm{H}^{i+n}_{(N-n)}(Z) \to$$

$$\to \mathrm{H}^{i+N}_{(n)}(X) \to \mathrm{H}^{i+N}_{(n)}(Y) \to \mathrm{H}^{i+N}_{(n)}(Z) \to$$

$$\to \mathrm{H}^{i+n+N}_{(N-n)}(X) \to \mathrm{H}^{i+n+N}_{(N-n)}(Y) \to \mathrm{H}^{i+n+N}_{(N-n)}(Z) \to \cdots$$

Note that this sequence is 6-periodic up to degree shift.

There is another long exact sequence for cohomology groups of N-complexes.

Theorem 4 (Second long exact sequence (Dubois-Violett))**.** *Let $n, m > 0$ be natural numbers such that $n + m < N$. Then, for an N-complex X, we have*

$$\cdots \to \mathrm{H}^i_{(n)}(X) \to \mathrm{H}^i_{(n+m)}(X) \to \mathrm{H}^{i+n}_{(m)}(X) \to$$

$$\to \mathrm{H}^{i+n}_{(N-n)}(X) \to \mathrm{H}^{i+n+m}_{(N-n-m)}(X) \to \mathrm{H}^{i+n+m}_{(N-m)}(X) \to$$

$$\to \mathrm{H}^{i+N}_{(n)}(X) \to \mathrm{H}^{i+N}_{(n+m)}(X) \to \mathrm{H}^{i+n+N}_{(m)}(X) \to$$

$$\to \mathrm{H}^{i+n+N}_{(N-n)}(X) \to \mathrm{H}^{i+n+m+N}_{(N-n-m)}(X) to \mathrm{H}^{i+n+m+N}_{(N-m)}(X) \to \cdots$$

We remark that for the ordinary complexes (i.e., the case where $N = 2$) the condition for n and m is empty. We note that this sequence is also 6-periodic up to degree shift.

1.3. **Results of Iyama-Kato-Miyachi.** Iyama-Kato-Miyachi showed that $\mathsf{C}_N(R)$ has a Frobenious structure. Then they defined the homotopy category$\mathsf{K}_N(R)$ to be the stable category $\mathsf{K}_N(R) := \underline{\mathsf{C}_N(R)}$ and of $\mathsf{C}_N(R)$ with respect to this Frobenious structure, and the derived category $\mathsf{D}_N(R)$ to be the Verdier quotient of $\mathsf{K}_N(R)$ by the thick subcategories consisting of acyclic N-complexes $\mathsf{D}_N(R) := \dfrac{\mathsf{K}_N(R)}{(\text{Acyclic } N\text{-complexes})}$.

I heard that one of their mtivation to define a derived category of N-complexes is to get a triangulated category of new kind. But they showed that derived category of N-complexes is no new. It turns out to be equivalent to an ordinary derived category. More precisely we have the following equivalence of triangulated categories:

Theorem 5 (Iyama-Kato-Miyachi)**.**

$$\mathsf{D}_N(R) \simeq \mathsf{D}(\mathbf{k}\overrightarrow{\mathrm{A}}_{N-1} \otimes R)$$

The right hand side is the ordinary derived category of the algebra $\mathbf{k}\overrightarrow{\mathrm{A}}_{N-1} \otimes R$ where $\mathbf{k}\overrightarrow{\mathrm{A}}_{N-1}$ is the path algebra of A_{N-1}-quiver.

Since there are interesting results on N-complexes, now we would like to ask why $d^N = 0$? For this purpose, we try to find a further generalization of N-complexes.

2. A-complexes

2.1. **An observation on N-complexes.** We observe that the notion of N-complexes and related things can be reformulated in terms of a graded algebra and its modules.

We define a graded algebra B_N to be $B_N := \mathbf{k}[\delta]/\delta^N$ with $\deg \delta = 1$. A point is that an N-complex X is nothing but a graded module over the graded algebra $B_N \otimes R$ and

$$\mathsf{C}_N(R) = (B_N \otimes R)\,\mathrm{GRMod}$$

where we consider $\deg R = 0$.

2.2. **A-complexes and their cohomologies.** We define a notion of A-complex by replacing the graded algebra B_N with a graded algebra A satisfying some conditions, which allow us to develop general theory.

Let $A := \bigoplus_{i \in \mathbb{Z}} A^i$ be a finite dimensional graded Frobenius algebra having Gorenstein parameter $\ell \in \mathbb{Z}$, i.e., $\mathrm{Hom}_k(A, \mathbf{k}) \cong A(\ell)$ for some $\ell \in \mathbb{Z}$.

Definition 6. An A-complex is a graded $A \otimes R$-module. We set the category $\mathsf{C}_A(R)$ of A-complexes to be the category of graded $A \otimes R$-modules.

$$\mathsf{C}_A(R) := (A \otimes R)\,\mathrm{GRMod}\,.$$

Remark 7. The above definition and the following results can be generalized to the case where A is a self-injective $\mathbf{k}$-linear category with a Serre functor satisfying some conditions.

For A-complex X we have a notion of cohomology groups $\mathrm{H}^t(X)$. The indexes t are not integers any more.

Definition 8. Let t be a graded A-module. We define t-th cohomology group of an A-complexes X to be

$$\mathrm{H}^t(X) := \mathrm{Ext}^1_{A\,\mathrm{GRMod}}(t, X)$$

The cohomology group $\mathrm{H}^t(X)$ is functorial in X and hence gives a functor

$$\mathrm{H}^t(-) : \mathsf{C}_A(R) \to RMod, X \mapsto \mathrm{H}^t(X).$$

Theorem 9 (Cohomology long exact sequences for A-complexes)**.**
Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of A-complexes. Then we have the following exact sequence

$$\begin{gathered}\to \mathrm{H}^{\Omega^{-1}(t)}(X) \to \mathrm{H}^{\Omega^{-1}(t)}(Y) \to \mathrm{H}^{\Omega^{-1}(t)}(Z) \to \cdots \\ \to \mathrm{H}^t(X) \to \mathrm{H}^t(Y) \to \mathrm{H}^t(Z) \to \\ \to \mathrm{H}^{\Omega(t)}(X) \to \mathrm{H}^{\Omega(t)}(Y) \to \mathrm{H}^{\Omega(t)}(Z) \to \cdots\end{gathered}$$

where Ω and Ω^{-1} denote the syzygy functor and co-syzygy functor.

Theorem 10 (Cohomology long exact sequence for indexes)**.**
Let $0 \to s \to t \to u \to 0$ be an exact sequence of graded A-modules. Then, for an A-complex X, we have the following long exact sequence

$$\begin{gathered}\to \mathrm{H}^{\Omega^{-1}(u)}(X) \to \mathrm{H}^{\Omega^{-1}(t)}(X) \to \mathrm{H}^{\Omega^{-1}(s)}(X) \to \\ \to \mathrm{H}^u(X) \to \mathrm{H}^t(X) \to \mathrm{H}^s(X) \to \\ \to \mathrm{H}^{\Omega(u)}(X) \to \mathrm{H}^{\Omega(t)}(X) \to \mathrm{H}^{\Omega(s)}(X) \to\end{gathered}$$

Now we discuss a Frobenius Structure in $\mathsf{C}_A(R)$.

Lemma 11. *Let $\mathcal{E}$ be the class of exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathsf{C}_A(R)$ which become a split exact sequence when they are considered as graded R-modules. Then $\mathcal{E}$ gives a Frobenius structure in $\mathsf{C}_A(R)$.*

Definition 12. We define the homotopy category $\mathsf{K}_A(R)$ of A-complexes to be the stable category of $\mathsf{C}_A(R)$ with respect to the above Frobenious structure.

$$\mathsf{K}_A(R) := \underline{\mathsf{C}_A(R)}$$

Remark 13. There exists a notion of *homotopy equivalence* for a morphism $f : X \to Y$ of A-complexes. It can be proved that the homotopy category $\mathsf{K}_A(R)$ is isomorphic to the residue category of $\mathsf{C}_A(R)$ modulo homotopy equivalences.

The cohomology functor $\mathrm{H}^t(X)$ descend to

$$\mathrm{H}^t(-) : \mathsf{K}_A(R) \to RMod, X \mapsto \mathrm{H}^t(X).$$

An A-complex X is said to be *acyclic* if $\mathrm{H}^t(X) = 0$ for all A-module t.

Definition 14. We define the derived category $\mathsf{D}_A(R)$ of A-complexes to be the Verdier quotient of $\mathsf{K}_A(R)$ by the acyclic A-complexes.

$$\mathsf{D}_A(R) := \frac{\mathsf{K}_A(R)}{(\text{Acyclic } A\text{-complexes})}$$

An A-complex X is said to be K-*projective* if we have $\mathrm{Hom}_{\mathsf{K}_A(R)}(X, Y) = 0$ for any acyclic A-complex Y. We denote by $\mathsf{K}_A{-}\mathsf{Proj}$ the full subcategory of $\mathsf{K}_A(R)$ consisting of K-projective A-complexes.

Proposition 15. *(1) There is a semi-orthogonal decomposition*

$$\mathsf{K}_A(R) = \langle \mathsf{K}_A\text{−}\mathsf{Proj}, (\text{Acyclic } A\text{-complexes})\rangle$$

(2) The canonical functor induces an equivalence

$$\mathsf{K}_A\text{−}\mathsf{Proj} \to \mathsf{K}_A(R) \to \mathsf{D}_A(R).$$

3. Back to N-complexes

Let $B_N = \mathbf{k}[\delta]/\delta^N$ with $\deg \delta = 1$. Recall that $\mathsf{C}_N(R) = \mathsf{C}_{B_N}(R)$.

Definition 16. For $i \in \mathbb{Z}, 0 < n < N$, we define a graded B_N-module $t(i,n)$ to be

$$t(i,n) := (\mathbf{k}[\delta]/\delta^{N-n})(N-n-i)$$

Then we have

$$\mathrm{H}^{t(i,n)}(X) = \mathrm{H}^i_{(n)}(X)$$

where in the left hand side X is considered as a B_N-complex and in the right hand side as an N-complex Moreover,

$$\Omega(t(i,n)) = t(i+n, N-n).$$

Now it can be easily seen that the cohomology long exact sequence of N-complexes (Theorem 3) is nothing but that of B_N-complexes (Theorem 9). More precisely, the sequence

$$\to \mathrm{H}^i_{(n)}(X) \to \mathrm{H}^i_{(n)}(Y) \to \mathrm{H}^i_{(n)}(Z) \to \mathrm{H}^{i+n}_{(N-n)}(X) \to \mathrm{H}^{i+n}_{(N-n)}(Y) \to \mathrm{H}^{i+n}_{(N-n)}(Z) \to$$

is equal to the sequence

$$\to \mathrm{H}^{t(i,n)}(X) \to \mathrm{H}^{t(i,n)}(Y) \to \mathrm{H}^{t(i,n)}(Z) \to \mathrm{H}^{\Omega(t(i,n))}(X) \to \mathrm{H}^{\Omega(t(i,n))}(Y) \to \mathrm{H}^{\Omega(t(i,n))}(Z) \to$$

Now we see that the periodicity of the cohomology long exact sequence of N-complexes is a consequence of the well-known fact that the syzygy functor Ω_{B_N} is 2-periodic up to degree $-N$-shift: $\Omega^2_{B_N} \cong (-N)$.

In the same way, we can see that the second cohomology long exact sequence for N-complexes (Theorem 4) is nothing bu the cohomology long exact sequence for indexes (Theorem 10), by using the following exact sequence of graded B_N-modules:

$$0 \to \frac{\mathbf{k}[\delta]}{\delta^m}(-n) \to \frac{\mathbf{k}[\delta]}{\delta^{n+m}} \to \frac{\mathbf{k}[\delta]}{\delta^n} \to 0.$$

4. Iyama-Kato-Miyachi equivalence for A-complexes (Ogawa)

The Iyama-Kato-Miyachi equivalence (Theorem 5) is generalized for A-complexes by Y. Ogawa.

Theorem 17 (Ogawa)**.** *We assume that* $\mathbf{k}$ *is an algebraically closed field. Let* Λ *be a finite dimensional algebra and* $A := \Lambda \oplus \Lambda^*$ *the trivial extension algebra equipped with the grading that* $\deg \Lambda = 0, \deg \Lambda^* = 1$*. Then there is an equivalence of triangulated categories:*

$$\mathsf{D}_A(R) \simeq \mathsf{D}(\Lambda \otimes R).$$

In a nutshell, this is a relative version of Happel's equivalence ([2]):

$$\underline{\mathrm{grmod}}\, A \simeq \mathsf{D}(\Lambda).$$

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CLASSIFICATION OF CATEGORICAL SUBSPACES OF LOCALLY NOETHERIAN SCHEMES

RYO KANDA

Abstract. This paper is an announcement of our results in [2]. We classify the prelocalizing subcategories of the category of quasi-coherent sheaves on a locally noetherian scheme. In order to give the classification, we introduce the notion of a local filter of subobjects of the structure sheaf. We also classify the localizing subcategories and the closed subcategories in terms of filters.

Key Words: Locally noetherian scheme, Prelocalizing subcategory, Localizing subcategory, Closed subcategory, Local filter.

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1. Gabriel's results

Let $\mathcal{A}$ be a Grothendieck category. For example, the category $\operatorname{Mod}\Lambda$ of right modules over a ring Λ and the category $\operatorname{QCoh}X$ of quasi-coherent sheaves on a scheme X are Grothendieck categories. In this paper, we deal with the following classes of subcategories.

Definition 1. Let $\mathcal{Y}$ be a full subcategory of $\mathcal{A}$.

(1) $\mathcal{Y}$ is called a *prelocalizing subcategory* (or a *weakly closed subcategory*) if $\mathcal{Y}$ is closed under subobjects, quotient objects, and arbitrary direct sums.
(2) $\mathcal{Y}$ is called a *closed subcategory* if $\mathcal{Y}$ is a prelocalizing subcategory closed under arbitrary direct products.
(3) $\mathcal{Y}$ is called a *localizing subcategory* if $\mathcal{Y}$ is a prelocalizing subcategory closed under extensions.

For a ring Λ, Gabriel [1] classified the prelocalizing subcategories and the localizing subcategories of $\operatorname{Mod}\Lambda$ by using the notion of filters. We define filters for objects in Grothendieck categories.

Definition 2. Let M be an object in $\mathcal{A}$. A *filter* (of subobjects) of M in $\mathcal{A}$ is a set $\mathcal{F}$ of subobjects of M satisfying the following conditions.

(1) $M \in \mathcal{F}$.
(2) If $L \subset L'$ are subobjects of M with $L \in \mathcal{F}$, then $L' \in \mathcal{F}$.
(3) If $L_1, L_2 \in \mathcal{F}$, then $L_1 \cap L_2 \in \mathcal{F}$.

For each subobject L of M, denote by $\mathcal{F}(L)$ the filter consisting of all subobjects L' of M with $L \subset L'$. A filter of the form $\mathcal{F}(L)$ is called a *principal filter*.

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Remark 3. The principal filter $\mathcal{F}(L)$ is closed under arbitrary intersection. Conversely, if a filter $\mathcal{F}$ of M is closed under arbitrary intersection, then $\mathcal{F} = \mathcal{F}(L)$, where L is the smallest element of $\mathcal{F}$.

Definition 4. For a ring Λ, we say that a filter $\mathcal{F}$ (of right ideals) of Λ in $\operatorname{Mod}\Lambda$ is *prelocalizing* if for each $L \in \mathcal{F}$ and $a \in \Lambda$, the right ideal

$$a^{-1}L = \{\, b \in \Lambda \mid ab \in L \,\}$$

of Λ belongs to $\mathcal{F}$.

Note that every filter $\mathcal{F}$ of a commutative ring R is prelocalizing.

The following theorem is the motivating result of our study.

Theorem 5 ([1, Lemma V.2.1]). *Let Λ be a ring. Then the map*

$$\{\, \textit{prelocalizing subcategories of } \operatorname{Mod}\Lambda \,\} \to \{\, \textit{prelocalizing filters of } \Lambda \textit{ in } \operatorname{Mod}\Lambda \,\}$$

given by

$$\mathcal{Y} \mapsto \left\{ L \subset \Lambda \textit{ in } \operatorname{Mod}\Lambda \;\middle|\; \frac{\Lambda}{L} \in \mathcal{Y} \right\}$$

is bijective. The inverse map is given by

$$\begin{aligned} \mathcal{F} &\mapsto \{\, M \in \operatorname{Mod}\Lambda \mid \operatorname{Ann}_\Lambda(x) \in \mathcal{F} \textit{ for every } x \in M \,\} \\ &= \left\langle \frac{\Lambda}{L} \in \operatorname{Mod}\Lambda \;\middle|\; L \in \mathcal{F} \right\rangle_{\mathrm{preloc}}, \end{aligned}$$

where $\langle \mathcal{S} \rangle_{\mathrm{preloc}}$ is the smallest prelocalizing subcategory containing the set $\mathcal{S}$ of objects.

By considering the principal filters, we can recover the classification of the closed subcategories of $\operatorname{Mod}\Lambda$ due to Rosenberg [3].

Theorem 6 (Gabriel [1, Lemma V.2.1] and Rosenberg [3, Proposition III.6.4.1]). *Let Λ be a ring. Then there exist bijections between the following sets.*

(1) *The set of closed subcategories of* $\operatorname{Mod}\Lambda$.
(2) *The set of principal prelocalizing filters of right ideals of Λ.*
(3) *The set of two-sided ideals of Λ.*

The bijection between (1) and (2) is induced by the bijection in Theorem 5.

The bijection between (1) and (3) is given by

$$(1) \to (3) : \; \mathcal{Y} \mapsto \bigcap_{M \in \mathcal{Y}} \operatorname{Ann}_\Lambda(M),$$

$$(3) \to (1) : \; I \mapsto \{\, M \in \operatorname{Mod}\Lambda \mid MI = 0 \,\} = \left\langle \frac{\Lambda}{I} \right\rangle_{\mathrm{preloc}}.$$

Gabriel [1] also classified the localizing subcategories of $\operatorname{Mod}\Lambda$. For more details, see [2, section 10].

2. Classification for QCoh X

In this section, let X be a locally noetherian scheme. Its structure sheaf is denoted by $\mathcal{O}_X$. We give classifications of the three classes of subcategories of QCoh X. In order to do that, we need to refine the notion of filters.

Definition 7. Let X be a locally noetherian scheme. We say that a filter $\mathcal{F}$ of subobjects of $\mathcal{O}_X$ in QCoh X is a *local filter* of $\mathcal{O}_X$ if it satisfies the following condition: let I be a subobject of $\mathcal{O}_X$, and assume that for each $x \in X$, there exist an open neighborhood U of x in X and $I' \in \mathcal{F}$ such that $I'|_U \subset I|_U$ as a subobject of $\mathcal{O}_U$. Then we have $I \in \mathcal{F}$.

We can show that every principal filter of $\mathcal{O}_X$ is a local filter. In the case where X is noetherian, every filter of $\mathcal{O}_X$ is a local filter.

The following theorem is our main result.

Theorem 8. *Let X be a locally noetherian scheme.*

(1) *The map*

$$\{\textit{ prelocalizing subcategories of } \operatorname{QCoh} X\,\} \to \{\textit{ local filters of } \mathcal{O}_X \textit{ in } \operatorname{QCoh} X\,\}$$

given by

$$\mathcal{Y} \mapsto \left\{ I \subset \mathcal{O}_X \textit{ in } \operatorname{QCoh} X \;\middle|\; \frac{\mathcal{O}_X}{I} \in \mathcal{Y} \right\}$$

is bijective. The inverse map is given by

$$\mathcal{F} \mapsto \left\langle \frac{\mathcal{O}_X}{I} \in \operatorname{QCoh} X \;\middle|\; I \in \mathcal{F} \right\rangle_{\mathrm{preloc}}.$$

(2) *The bijection in (1) induces bijections*

$$\{\textit{ closed subcategories of } \operatorname{QCoh} X\,\} \to \{\textit{ principal filters of } \mathcal{O}_X\,\}$$

and

$$\{\textit{ localizing subcategories of } \operatorname{QCoh} X\,\} \to \{\textit{ local filters of } \mathcal{O}_X \textit{ closed under products}\}.$$

Corollary 9. *There exist bijections between the following sets.*

(1) *The set of closed subcategories of* QCoh X.
(2) *The set of subobjects of $\mathcal{O}_X$ in* QCoh X.
(3) *The set of closed subschemes of X.*

The key of the proof is the fact that every prelocalizing subcategory $\mathcal{Y}$ of QCoh X has the description

$$\mathcal{Y} = \{\, M \in \operatorname{QCoh} X \mid M_x \in \mathcal{Y}_x \text{ for each } x \in X \,\}.$$

Example 10. Let k be an algebraically closed field, and consider the projective line $X = \mathbb{P}^1_k$. Denote by Φ the set of closed points in X. For each $r \in \prod_{x\in\Phi}(\mathbb{Z}_{\geq 0} \cup \{\infty\})$, we define the prelocalizing subcategory $\mathcal{Y}_r$ of QCoh X by

$$\mathcal{Y}_r = \{\, M \in \operatorname{QCoh} X \mid M_x \mathfrak{m}_x^{r(x)} = 0 \text{ for each } x \in \Phi \text{ with } r(x) \neq \infty \,\}.$$

The set of prelocalizing subcategories of $\operatorname{QCoh} X$ is

$$\left\{ \mathcal{Y}_r \;\middle|\; r \in \prod_{x \in \Phi} (\mathbb{Z}_{\geq 0} \cup \{\infty\}) \right\} \cup \{\operatorname{QCoh} X\},$$

the set of localizing subcategories of $\operatorname{QCoh} X$ is

$$\left\{ \mathcal{Y}_r \;\middle|\; r \in \prod_{x \in \Phi} \{0, \infty\} \right\} \cup \{\operatorname{QCoh} X\},$$

and the set of closed subcategories of $\operatorname{QCoh} X$ is

$$\left\{ \mathcal{Y}_r \;\middle|\; r \in \bigoplus_{x \in \Phi} \mathbb{Z}_{\geq 0} \right\} \cup \{\operatorname{QCoh} X\}.$$

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TAKING TILTING MODULES FROM THE POSET OF SUPPORT TILTING MODULES

RYOICHI KASE

Abstract. C. Ingalls and H. Thomas defined support tilting modules for path algebras. From τ-tilting theory introduced by T. Adachi, O. Iyama and I. Reiten, a partial order on the set of basic tilting modules defined by D. Happel and L. Unger is extended as a partial order on the set of support tilting modules. In this report, we study a combinatorial relationship between the poset of basic tilting modules and basic support tilting modules. We will show that the subposet of tilting modules is uniquely determined by the poset structure of the set of support tilting modules.

1. Introduction

Tilting theory first appeared in an article by S. Brenner and M.C.R. Butler [2]. In that article the notion of a tilting module for finite dimensional algebras was introduced. Let T be a tilting module for a finite dimensional algebra Λ and let $B = \mathrm{End}_A(T)$. Then D. Happel showed that the two bounded derived categories $\mathcal{D}^{\mathrm{b}}(A)$ and $\mathcal{D}^{\mathrm{b}}(B)$ are equivalent as triangulated category [3]. Therefore, classifying tilting modules is an important problem.

Tilting mutation introduced by C. Riedtmann and A. Schofield [7] is an approach to this problem. It is an operation which gives a new tilting module from given one by replacing an indecomposable direct summand. They also introduced a tilting quiver whose vertices are (isomorphism classes of) basic tilting modules and arrows correspond to mutations. D. Happel and L. Unger showed that there is a partial order on the set of (isomorphism classes of) basic tilting modules suct that its Hasse quiver coincides to tilting quiver [4, 5]. However, tilting mutation is often impossible. Support τ-tilting modules introduced by T. Adachi,O. Iyama and I. Reiten [1] are generalization of tilting modules. They showed that a mutation (resp. a partial order) on the set of (isomorphism classes of) basic tilting modules is extended as an operation (resp. a partial order) on the set of (isomorphism classes of) support τ-tilting modules and improved behavior of tilting mutation.

In path algebras case, it is known that a support τ-tilting module is a support tilting module introduced by C. Ingalls and H. Thomas [6]. Then the main result of this report is the following.

Theorem 1. *Let Λ be a finite dimensional path algebra. Then the set of basic tilting modules of Λ is determined by poset structure of the set of basic support tilting modules.*

The detailed version of this paper has been submitted for publication elsewhere.

2. Path algebras

Let k be an algebraically closed field and let Q be a finite quiver (=oriented graph). We denote by Q_0 (resp. Q_1) the set of vertices (resp. edges) of Q. For an edge $\alpha : a \to b$, we set $s(\alpha) := a,\ t(\alpha) := b$.

Definition 2. A sequence $w = (\alpha_1|\alpha_2|\cdots|\alpha_l)$ of Q_1 is a path on Q if $t(\alpha_i) = s(\alpha_{i+1})$ holds for any i. Then we call l the length of w and put $s(w) := s(\alpha_1)\ t(w) := \alpha_l$. We regard a vertex $a \in Q_0$ as a path of length 0 with $s(e_a) = a = t(e_a)$ and denote it by e_a.

Then a path algebra $\Lambda = kQ$ is defined as follows:

(1) $\Lambda = \bigoplus_{w:\text{path}} k \cdot w$.
(2) For two paths $w = (\alpha_1|\alpha_2|\cdots|\alpha_l),\ w' = (\beta_1|\beta_2|\cdots|\beta_{l'})$, we define

$$w \cdot w' = \begin{cases} (\alpha_1|\alpha_2|\cdots|\alpha_l \mid \beta_1|\beta_2|\cdots|\beta_{l'}) & \text{if } t(w) = s(w') \\ 0 & \text{if } t(w) \neq s(w'). \end{cases}$$

From now on, we assume that $\Lambda = kQ$ and Q has no oriented cycles ($\Leftrightarrow \dim \Lambda < \infty$).

3. Tilting modules and support tilting modules

In this section, we recall definitions of poset of tilting modules and poset of support tilting modules. For a module $M \in \operatorname{mod} \Lambda$ with indecomposable decomposition

$$M \simeq \oplus_{i=1}^m M_i^{r_i}\ (i \neq j \Rightarrow M_i \not\simeq M_j),$$

we put $|M| := m$. M is said to be basic if $r_i = 1$ $(\forall i)$.

Definition 3. $T \in \operatorname{mod} \Lambda$ is a tilting module if T satisfies following properties.

(1) $\operatorname{Ext}^1_\Lambda(T, T) = 0$.
(2) $|T| = \#Q_0$.

We denote by $\operatorname{tilt} \Lambda$ the set of (isomorphism classes of) basic tilting modules.

Proposition 4. [4, 5] *The following relation induces a partial order on* $\operatorname{tilt} \Lambda$.

$$T \geq T' \Leftrightarrow \operatorname{Ext}^1_\Lambda(T, T') = 0.$$

For a module $M \in \operatorname{mod} \Lambda$, we put $\mathrm{supp(M)} := \{\mathrm{a \in Q_0 \mid \dim Me_a > 0}\}$ and denote by $Q(M)$ the full subquiver of Q with $Q(M)_0 = \mathrm{supp(M)}$.

Remark 5. We can regard M as $kQ(M)$-module.

Definition 6. $T \in \operatorname{mod} \Lambda$ is a support tilting module if T satisfies following properties.

(1) $\operatorname{Ext}^1_\Lambda(T, T) = 0$.
(2) $|T| = \#\mathrm{supp(T)}$.

We denote by $\operatorname{stilt}\Lambda$ the set of (isomorphism classes of) support tilting module.

We note that T is support tilting if and only if $\Lambda(T)$ is tilting as $kQ(T)$-module.

Proposition 7. [1, 6] *The following relation induces a partial order on* $\operatorname{stilt} \Lambda$.

$$T \geq T' \Leftrightarrow \operatorname{Ext}^1_\Lambda(T, T') = 0 \ \&\ \mathrm{supp(T') \subset supp(T)}.$$

Example 8. Let $Q = 1 \to 2$. Then stiltΛ is given by the following.

$$\begin{tikzcd}
& P(1) \oplus P(2) \arrow[dl] \arrow[dr] & \\
P(1) \oplus I(1) \arrow[d] & & P(2) \arrow[d] \\
I(1) \arrow[rr] & & 0
\end{tikzcd}$$

4. Outline of a proof

By definition of support tilting modules, we have

$$T \in \mathrm{stilt}\Lambda \text{ is a tilting module} \Leftrightarrow T \geq I_\Lambda = \oplus_{a \in Q_0} I(a).$$

For a non negative integer i, we define a subset V_i of Q_0 as follows.

- $V_0 = \emptyset$.
- $V_i = V_{i-1} \cup \{a \in Q_0 \mid a \text{ is a source of } Q \setminus V_{i-1}\}$.

We set $I_i := \oplus_{a \in V_i} I(a)$ $(I_0 = 0)$. Then we note that $I_i \in \mathrm{stilt}\Lambda$.

Lemma 9. *Let $i \geq 0$. Then I_{i+1} is a minimum element of*

$$\bigcap_{X \in \mathrm{idp(I_i)}} \{T \in \mathrm{stilt}\Lambda \mid T \geq X\},$$

where idp(I$_\mathrm{i}$) I_i *is the set of injective direct predecessors of* I_i.

Lemma 1 shows that it is sufficient to determine idp(I$_\mathrm{i}$) by poset structure of stiltΛ.

4.1. **Deleting non injective direct predecessors of I_i.** Non injective direct predecessor T satisfies one of the following.

(1) $\#\mathrm{supp(T)} = \#\mathrm{supp(I_i)} + 1$.
(2) $\#\mathrm{supp(T)} = \#\mathrm{supp(I_i)}$.

We denote by $\mathcal{N}_i(p)$ $(p = 1, 2)$ the set of non injective direct predecessors of I_i which satisfies (p).

Lemma 10. *Let a, $b \in Q_0$. Then There is an edge $a \to b$ in Q if and only if there are $X \in \mathrm{dp(S(a))}$, $Y \in \mathrm{dp(S(b))}$ such that $X < Y$.*

Since $S(a)$ is injective if and only if $a \in Q_0$ is a source, we can determine idp(I_0) by poset structure of stilt Λ.

Lemma 11. *Let $T \in \mathcal{N}_i(1)$. Then there are $T' \in \mathrm{dp(I_i)}$, $X \in \mathrm{dp(T)}$, $Y \in \mathrm{dp(T')}$ such that $X > Y$.*

Lemma 12. *Let $T \in \mathrm{idp(I_i)}$. Then for any $T' \in \mathrm{dp(I_i)}$, $X \in \mathrm{dp(T)}$, $Y \in \mathrm{dp(T')}$, we have $X \not> Y$.*

Lemma 3 and Lemma 4 implies that we can delete $\mathcal{N}_i(1)$. For $T \in \mathrm{dp}(\mathrm{I_i})$ and $r \in \mathbb{Z}_{\geq 1}$, we set

$$\mathcal{F}(i, T, r) := \{((X_k)_{k \in \{0, \cdots, r\}}, (T_k)_{k \in \{0 \cdots, r-1\}}, (Y_k)_{k \in \{1, \cdots, r-1\}}) \mid (\star)\}$$

where the condition $(\star)$ is as follows: $(\star) := \begin{cases} \bullet \ X_0 = I_i, \ T_0 = T \\ \bullet \ X_1 \in \mathrm{ds(I_i)}, \ \mathrm{X_{k+1} \in ds(X_k)} \\ \bullet \ T_k \in \mathrm{dp(X_k)} \setminus \{\mathrm{X_{k-1}}\} \\ \bullet \ Y_k \in \mathrm{dp(T_k)} \\ \bullet \ Y_1 \geq T, \ Y_{k+1} \geq T_k \end{cases}$

Lemma 13. *Let $T \in \mathcal{N}_i(2)$. Then there are $r \in \mathbb{Z}_{\geq 1}$ and $((X_k), (T_k), (Y_k)) \in \mathcal{F}(i, T, r)$ such that for any $T_r \in \mathrm{dp(X_r)} \setminus \{\mathrm{X_{r-1}}\}$ and $Y_r \in \mathrm{dp(T_r)}$, we have $Y_r \not\geq T_{r-1}$.*

Lemma 14. *Let $T \in \mathrm{idp(I_i)}$. Then for any $r \in \mathbb{Z}_{\geq 1}$ and $((X_k), (T_k), (Y_k)) \in \mathcal{F}(i, T, r)$, there are $T_r \in \mathrm{dp(X_r)} \setminus \{\mathrm{X_{r-1}}\}$ and $Y_r \in \mathrm{dp(T_r)}$ such that $Y_r \geq T_{r-1}$.*

Thus we can also delete $\mathcal{N}_i(2)$.

Corollary 15. *Let Λ and Γ be two path algebras, ρ be a poset isomorphism*

$$\rho : \mathrm{stilt}\Lambda \simeq \mathrm{stilt}\Gamma.$$

Then the restriction of ρ to $\mathrm{tilt}\Lambda$ *induces a poset isomorphism*

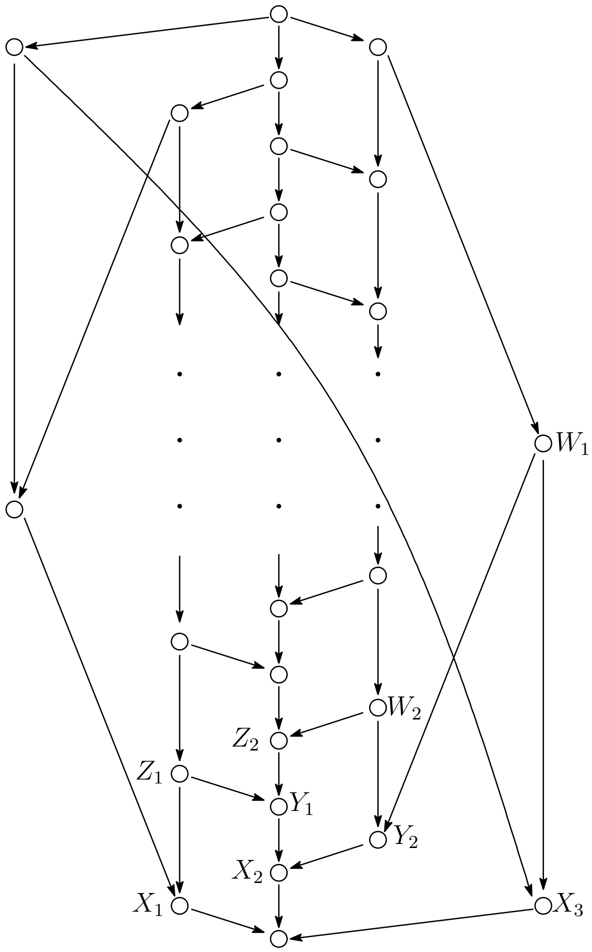
$$\rho|_{\mathrm{tilt}\Lambda} : \mathrm{tilt}\Lambda \simeq \mathrm{tilt}\Gamma.$$

5. Example

We consider the following quiver Q.

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$

Then stilt Λ is given by the following.

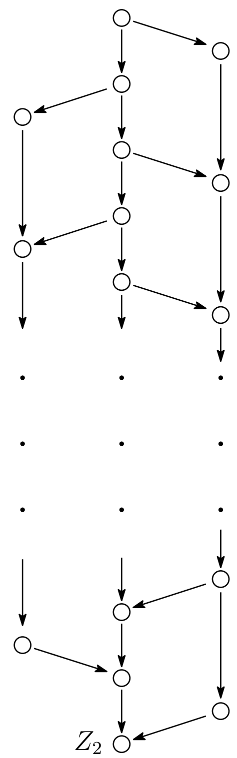


step 1 By applying Lemma 3 and Lemma 4 to $\{0,\ X_1,\ X_2,\ Y_1,\ Z_1\}$, we can see that X_1 is not injective. Similarly we have X_3 is not injective. Therefore X_2 is injective.

step 2 By applying Lemma 5 and Lemma 6 to $\{X_2,\ Y_1,\ Y_2,\ Z_2,\ W_2\}$, we have Y_2 is not injective. Hence Y_1 is injective.

step 3 We consider $\mathcal{F}(1, Z_1, Y_1) \ni ((Y_1, X_2), (Z_1), \emptyset)$. Then Y_2 is a unique direct predecessor of X_2 and $\{W_1, W_2\}$ is the set of direct predecessors of Y_2. Since $W_p \not\geq Z_1$ ($p = 1, 2$), Lemma 5 implies that Z_1 is not injective. Therefore we have $I_\Lambda = Z_2$.

In particular, tiltΛ is given by the following.



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ON ISOMORPHISMS OF GENERALIZED MULTIFOLD EXTENSIONS OF ALGEBRAS WITHOUT NONZERO ORIENTED CYCLES

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Abstract. We show that an algebra of the form $\hat{A}/\langle\phi\rangle$ where A is an algebra and ϕ is an automorphism of $\hat{A}$ such that $\phi(A^{[0]}) = A^{[n]}$ for some integer n is isomorphic to an algebra of the form $\hat{A}/\langle\hat{\phi_0}\nu_{\hat{A}}^n\rangle$ where $\hat{\phi_0}$ is an automorphism of $\hat{A}$ induced by ϕ and $\nu_{\hat{A}}$ is the Nakayama automorphism of $\hat{A}$ if A has no nonzero oriented cycles. Throughout this paper we do not assume that the action of groups (or automorphisms of $\hat{A}$) are free. Therefore this result give us applying a derived equivalence classification in [1] and [3] to $n = 0$.

1. Introduction

Throughout this paper $\Bbbk$ is an algebraically closed field, algebras are basic finite-dimensional $\Bbbk$-algebras and categories are $\Bbbk$-categories.

We say that an algebra is a generalize multifold extension of algebra A if it has the form $\hat{A}/\langle\phi\rangle$ where $\hat{A}$ is the repetitive category of $\hat{A}$ and ϕ is an automorphism of $\hat{A}$ with jump n for some integer n (see Definition 1 and Proposition 2). In [3], we gave a derived equivalence classification of generalized multifold extensions of algebras which are piecewise hereditary of tree type (i.e., algebras are derived equivalent to some hereditary algebra whose ordinary quiver is oriented tree) if automorphisms act on algebras have positive jump. To give a classification, we showed that for a positive integer $n \in \mathbb{Z}$, a generalized n-fold extension $\hat{A}/\langle\phi\rangle$ is derived equivalent to $T^n_{\phi_0}(A) := \hat{A}/\langle\hat{\phi_0}\nu_{\hat{A}}^n\rangle$ where $\hat{\phi_0}$ is the automorphism of $\hat{A}$ naturally induced from automorphism $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu_{\hat{A}}^{-n}\phi\mathbb{1}^{[0]}$ of A and $\nu_{\hat{A}}$ is the Nakayama automorphism of $\hat{A}$. Also, we posed a following question

Problem. If A is piecewise hereditary of tree type, when are the algebras $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ isomorphic?

In this paper we will give the answer to this question.

Acknowledgments. I would like to express my gratitude to Junichi Miyachi for suggesting me the problem above. I would also like to express my gratitude to Hideto Asasiba for his generous supports. The idea, which was essential to solve this problem, to combine an idea in Saorín's paper [7] and Lemma 12 taken from [2] was suggested by Asashiba. I would also like to express my thanks to Manuel Saorín for sending me the paper [7] (which is the original version of [6]) through Asashiba and his results are cited in section 3. Finally I would also like to thank Steffen Koenig for informing me the proof of Lemma 18 through Asashiba.

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

For a category R we denote by R_0 and R_1 the class of objects and morphisms of R, respectively. A category R is said to be *locally bounded* if it satisfies the following:

- Distinct objects of R are not isomorphic;
- $R(x,x)$ is a local algebra for all $x \in R_0$;
- $R(x,y)$ is finite-dimensional for all $x, y \in R_0$; and
- The set $\{y \in R_0 \mid R(x,y) \neq 0 \text{ or } R(y,x) \neq 0\}$ is finite for all $x \in R_0$.

A category is called *finite* if it has only a finite number of objects.

A pair (A, E) of an algebra A and a complete set $E := \{e_1, \dots, e_n\}$ of orthogonal primitive idempotents of A can be identified with a locally bounded and finite category R by the following correspondences. Such a pair (A, E) defines a category $R_{(A,E)} := R$ as follows: $R_0 := E$, $R(x,y) := yAx$ for all $x, y \in E$, and the composition of R is defined by the multiplication of A. Then the category R is locally bounded and finite. Conversely, a locally bounded and finite category R defines such a pair (A_R, E_R) as follows: $A_R := \bigoplus_{x,y \in R_0} R(x,y)$ with the usual matrix multiplication (regard each element of A as a matrix indexed by R_0), and $E_R := \{(\mathbb{1}_x \delta_{(i,j),(x,x)})_{i,j \in R_0} \mid x \in R_0\}$. We always regard an algebra A as a locally bounded and finite category by fixing a complete set A_0 of orthogonal primitive idempotents of A.

Definition 1. Let A be a locally bounded category.

(1) The *repetitive category* $\hat{A}$ of A is a $\mathbb{k}$-category defined as follows ($\hat{A}$ turns out to be locally bounded again):

- $\hat{A}_0 := A_0 \times \mathbb{Z} = \{x^{[i]} := (x, i) \mid x \in A_0, i \in \mathbb{Z}\}$.
- $\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} \mid f \in A(x,y)\} & \text{if } j = i, \\ \{\phi^{[i]} \mid \phi \in DA(y,x)\} & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$ for all $x^{[i]}, y^{[j]} \in \hat{A}_0$.
- For each $x^{[i]}, y^{[j]}, z^{[k]} \in \hat{A}_0$ the composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is given as follows.
  - (i) If $i = j, j = k$, then this is the composition of A $A(y,z) \times A(x,y) \to A(x,z)$.
  - (ii) If $i = j, j+1 = k$, then this is given by the right A-module structure of DA: $DA(z,y) \times A(x,y) \to DA(z,x)$.
  - (iii) If $i+1 = j, j = k$, then this is given by the left A-module structure of DA: $A(y,z) \times DA(y,x) \to DA(z,x)$.
  - (iv) Otherwise, the composition is zero.

(2) We define an automorphism ν_A of $\hat{A}$, called the *Nakayama automorphism* of $\hat{A}$, by $\nu_A(x^{[i]}) := x^{[i+1]}$, $\nu_A(f^{[i]}) := f^{[i+1]}$, $\nu_A(\phi^{[i]}) := \phi^{[i+1]}$ for all $i \in \mathbb{Z}, x \in A_0, f \in A_1, \phi \in \bigcup_{x,y \in A_0} DA(y,x)$.

(3) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $\hat{A}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow \hat{A}, x \mapsto x^{[n]}$, the embedding functor.

We cite the following [3, Proposition 1.6.].

Proposition 2. *Let A be an algebra, n an integer, and ϕ an automorphism of $\hat{A}$. Then the following are equivalent:*

(1) *ϕ is an automorphism with jump n;*
(2) *$\phi(A^{[i]}) = A^{[i+n]}$ for some integer i;*
(3) *$\phi(A^{[j]}) = A^{[j+n]}$ for all integers j; and*
(4) *$\phi = \phi_L \nu_A^n$ for some automorphism ϕ_L of $\hat{A}$ with jump 0.*
(5) *$\phi = \nu_A^n \phi_R$ for some automorphism ϕ_R of $\hat{A}$ with jump 0.*

We cite the following from [1, Lemma 2.3].

Lemma 3. *Let $\psi\colon A \to B$ be an isomorphism of locally bounded categories. Denote by $\psi_x^y\colon A(y,x) \to B(\psi y, \psi x)$ the isomorphism defined by ψ for all $x, y \in A$. Define $\hat{\psi}\colon \hat{A} \to \hat{B}$ as follows.*

- *For each $x^{[i]} \in \hat{A}, \hat{\psi}(x^{[i]}) := (\psi x)^{[i]}$;*
- *For each $f^{[i]} \in \hat{A}(x^{[i]}, y^{[i]}), \hat{\psi}(f^{[i]}) := (\psi f)^{[i]}$; and*
- *For each $\phi^{[i]} \in \hat{A}(x^{[i]}, y^{[i+1]}), \hat{\psi}(\phi^{[i]}) := (D((\psi_x^y)^{-1})(\phi))^{[i]} = (\phi \circ (\psi_x^y)^{-1})^{[i]}$.*

Then

(1) *$\hat{\psi}$ is an isomorphism.*
(2) *Given an isomorphism $\rho\colon \hat{A} \to \hat{B}$, the following are equivalent.*
  (a) *$\rho = \hat{\psi}$;*
  (b) *ρ satisfies the following.*
    (i) *$\rho\nu_A = \nu_B\rho$;*
    (ii) *$\rho(A^{[0]}) = A^{[0]}$;*
    (iii) *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \mathbf{1}^{[0]}}\downarrow & & \downarrow{\scriptstyle \mathbf{1}^{[0]}} \\ A^{[0]} & \xrightarrow[\rho]{} & B^{[0]} \end{array}$$

is commutative; and
    (iv) *$\rho(\phi^{[0]}) = (\phi \circ (\psi_x^y)^{-1})^{[0]}$ for all $x, y \in A$ and all $\phi \in DA(y, x)$.*

3. Automorphisms of repetitive category with jump 0

Throughout this section A is an algebra. We set $\mathrm{Aut}^0(\hat{A})$ to be the group of all automorphisms of $\hat{A}$ with jump 0.

Lemma 4. *Let $\phi \in \mathrm{Aut}^0(\hat{A})$. Then ϕ gives a family of $\Bbbk$-linear maps $(\phi_i, f_i)_{i\in\mathbb{Z}}$, where ϕ_i is an automorphism of A and $f_i : A \to A$ is a bijective ϕ_i-ϕ_{i+1}-semilinear map for all $i \in \mathbb{Z}$.*

Proof. Let $i \in \mathbb{Z}$. Then by definition, we have $\phi(A^{[i]}) = A^{[i]}$. We set $\phi_i := (\mathbb{1}_A^{[i]})^{-1}\phi\mathbb{1}_A^{[i]} : A \to A$, then ϕ_i is an automorphism of A. On the other hand, also by definition, we have $\phi(DA^{[i]}) = DA^{[i]}$. Hence we get a bijective $\Bbbk$-linear map $D(f_i^{-1}) := D(\mathbb{1}_{A^{[i]}})\phi(D(\mathbb{1}_{A^{[i]}}))^{-1} : A \to A$. For morphisms $a, b \in A_1$ and $\mu^* \in DA_1$, $b^{[i+1]}\mu^{*[i]}a^{[i]} = (a\mu^*b)^{[i]} \in DA^{[i]}$ and

$$\phi(b^{[i+1]}\mu^{*[i]}a^{[i]}) = \phi(b^{[i+1]})\phi(\mu^{*[i]})\phi(a^{[i]}).$$

Since

$$LHS = (D(f_i^{-1})(a\mu^* b))^{[i]} = ((a\mu^* b)f_i^{-1})^{[i]}$$

and

$$RHS = \phi_{i+1}(b)^{[i+1]}(D(f_i^{-1})(\mu^*))^{[i]}\phi_i(a)^{[i]} = \phi_{i+1}(b)^{[i+1]}(\mu^* f_i^{-1})^{[i]}\phi_i(a)^{[i]},$$

we have $f_i(a\alpha b) = \phi_i(a)f_i(\alpha)\phi_{i+1}(b)$ for each $\alpha \in A_1$, which shows that f_i is ϕ_i-ϕ_{i+1}-semilinear. □

We identify ϕ with $(\phi_i, f_i)_{i\in\mathbb{Z}}$ and write $\phi = (\phi_i, f_i)_{i\in\mathbb{Z}}$.

For $\psi \in \mathrm{Aut}(\hat{A})$ with jump $n \in \mathbb{Z}$, we also get a family of $\Bbbk$-linear maps by following way. By Proposition 2, there exists an automorphism $\psi_R = (\psi_{Ri}, f_i)_{i\in\mathbb{Z}}$ of $\hat{A}$ with jump 0 such that $\psi = \nu_A^n \psi_R$. We can define $(\psi_i, g_i)_{i\in\mathbb{Z}}$ by $\psi_i := \psi_{Ri}$, $g_i := f_i$ for all $i \in \mathbb{Z}$.

Remark 5. We can define a group homomorphism $\Psi : \mathrm{Aut}^0(\hat{A}) \to \mathrm{Aut}(A)$ by $\Psi(\phi) := \phi_0$ for all $\phi \in \mathrm{Aut}^0(\hat{A})$. Then we have $\hat{\sigma} \in \mathrm{Aut}^0(\hat{A})$ and $\Psi(\hat{\sigma}) = \sigma$ for all $\sigma \in \mathrm{Aut}(A)$ by lemma 3. Thus Ψ is an epimorphism, in particular split epimorhism.

Clearly an automorphism ϕ in the kernel of Ψ is whose ϕ_0 is the identity of A. Therefore to see the kernel of Ψ more particularly, we are interested to construct an automorphism of $\hat{A}$ from the identity of A.

Definition 6. We define a map $\xi : (\Bbbk^\times)^{A_0} \to \mathrm{Aut}(A)$ by

$$\xi(\lambda)(e) := e$$

and

$$\xi(\lambda)(a) := \lambda(t(a))^{-1}\lambda(s(a))a$$

for all $\lambda = (\lambda(x))_{x\in A_0} \in (\Bbbk^\times)^{A_0}$, all objects e and morphisms a in A.

Then ξ is a group homomorphism.

Lemma 7. *Let $\lambda = (\lambda_i)_{i\in\mathbb{Z}} \in (\Bbbk^\times)^{\hat{A}_0}$ (We regard $(\Bbbk^\times)^{\hat{A}_0} = ((\Bbbk^\times)^{A_0})^{\mathbb{Z}}$ by the canonical isomorphism $(\Bbbk^\times)^{\hat{A}_0} = (\Bbbk^\times)^{A_0\times\mathbb{Z}} \cong ((\Bbbk^\times)^{A_0})^{\mathbb{Z}}$). Then a family $(\phi_i, f_i)_{i\in\mathbb{Z}}$ of maps where*

$$\phi_i := \begin{cases} \xi(\lambda_i\lambda_{i+1}\cdots\lambda_{-1}) & \text{if } i < 0 \\ \mathbb{1}_A & \text{if } i = 0 \\ \xi(\lambda_0\lambda_1\cdots\lambda_{i-1}) & \text{if } i > 0 \end{cases}$$

and $f_i : A \to A$ is defined by $f_i(a) := \lambda_i(s(a))\phi_i(a)(= \lambda_i(t(a))\phi_{i+1}(a))$ for $a \in A_1$, gives an automorphism of $\hat{A}$ with jump 0.

We assume the following property which is necessary for our purpose.

Definition 8. If $eAe \cong \Bbbk$ for all primitive idempotents of A, then A is said to have no nonzero oriented cycles.

Let $A := \Bbbk Q/I$ where Q is a quiver and I is an admissible ideal of $\Bbbk Q$. The definition 8 means that I contains all cycles in Q.

Proposition 9. *Assume that A has no nonzero oriented cycles. Then there is an exact sequence of groups*

$$1 \to (\Bbbk^\times)^{\hat{A}_0} \xrightarrow{\Phi} \mathrm{Aut}^0(\hat{A}) \xrightarrow{\Psi} \mathrm{Aut}(A) \to 1.$$

Proof. For $\lambda \in (\Bbbk^\times)^{\hat{A}_0}$, we define $\Phi(\lambda)$ the automorphism constructed by Lemma 7. Since ξ is group homomorphism, clearly Φ is a group homomorphism. First, we show that Φ is injective. If $\Phi(\lambda) = \mathbb{1}_{\hat{A}}$, then $\phi_i = \mathbb{1}_A$ and $D(f_i^{-1}) = \mathbb{1}_{DA}$ for all $i \in \mathbb{Z}$. By induction, the former implies that $\xi(\lambda_i) = \mathbb{1}_A$ for all $i \in \mathbb{Z}$. Hence for each $i \in \mathbb{Z}$, we get an element $k_i \in \Bbbk^\times$ such that $f_i = k_i \mathbb{1}_A$ for all $x \in A_0$. Therefore $D(f_i^{-1}) = \lambda_i^{-1}\mathbb{1}_{DA} = \mathbb{1}_{DA}$, so that $k_i = 1$ for all $i \in \mathbb{Z}$. This shows that Φ is injective.

Next we show that $\operatorname{Im}\Phi = \operatorname{Ker}\Psi$. We easily have $\Psi\Phi = 1$ by definition, hence it is enough to show $\operatorname{Im}\Phi \supseteq \operatorname{Ker}\Psi$. Let $\psi = (\psi_i, g_i)_{i\in\mathbb{Z}} \in \operatorname{Ker}\Psi$. Since g_i is a ψ_i-ψ_{i+1}-semilinear bijection, the equality $\psi_0 = \mathbb{1}_A$ imply that $g_i(x) = g_i(x^3) = \psi_i(x)g_i(x)\psi_{i+1}(x) = xg_i(x)x$ for all $x \in A_0$. Hence $g_i(x) \in xAx = \Bbbk x$, because A have no nonzero oriented cycles. Therefore we get $\lambda_i(x) \in \Bbbk^\times$ such that $g_i(x) = \lambda_i(x)x$ for each $i \in \mathbb{Z}$ and $x \in A_0$. We claim that $\psi = \Phi(\lambda)$. Set $\Phi(\lambda) = (\phi_i, f_i)_{i\in\mathbb{Z}}$. To see that, we take an arbitrary morphism $a \in A(x,y)$. If $\phi_i = \psi_i$ for all $i \in \mathbb{Z}$, then

$$\begin{aligned}
f_i(a) &= \lambda_i(x)\phi_i(a) \\
&= \lambda_i(x)\psi_i(a) \\
&= \lambda_i(x)\psi_i(ax) \\
&= \psi_i(a)\lambda_i(x)\psi_i(x) \\
&= \psi_i(a)\lambda_i(x)x \\
&= \psi_i(a)g_i(x) \\
&= g_i(a).
\end{aligned}$$

Hence we check that $\psi_i = \xi(\lambda_0\lambda_1\cdots\lambda_{i-1})$ for all $0 \le i \in \mathbb{Z}$ and $\psi_i = \xi(\lambda_i\lambda_{i+1}\cdots\lambda_{-1})^{-1}$ for all $0 > i \in \mathbb{Z}$. We prove by induction on $0 \le i \in \mathbb{Z}$ the first equality, the other one following in a similar way. Since $\psi_0 = \mathbb{1}_A = \phi_0$, it is enough to show it in the case that $1 \le i$. For any morphism $a \in A(x,y)$,

$$\begin{aligned}
\phi_i(a) &= \xi(\lambda_0\lambda_1\cdots\lambda_{i-1})(a) \\
&= \lambda_0\cdots\lambda_{i-1}(x)(\lambda_0\cdots\lambda_{i-1}(y))^{-1}a \\
&= \lambda_{i-1}(x)(\lambda_{i-1}(y))^{-1}\psi_{i-1}(a) \\
&= \lambda_{i-1}(x)(\lambda_{i-1}(y))^{-1}\psi_{i-1}(a)x \\
&= (\lambda_{i-1}(y))^{-1}\psi_{i-1}(a)g_{i-1}(x) \\
&= (\lambda_{i-1}(y))^{-1}g_{i-1}(a) \\
&= (\lambda_{i-1}(y))^{-1}g_{i-1}(ya) \\
&= (\lambda_{i-1}(y))^{-1}g_{i-1}(y)\psi_i(a) \\
&= y\psi_i(a) \\
&= \psi_i(a)
\end{aligned}$$

as desired. □

Remark 10.

(1) By Remark 5, the exact sequence in Proposition 9 splits. Therefore an automorphism of $\hat{A}$ with jump is characterized by an automorphism of A and a map

from $\hat{A}_0$ to $\Bbbk^\times$. Let $\phi = (\phi_i, f_i)_{i\in\mathbb{Z}}$ be an automorphism of $\hat{A}$ with jump. For all morphism $a \in A(x,y)$,

$$f_i(a) = \phi_i(a)f_i(x) = f_i(y)\phi_{i+1}(a)$$

and

$$f_i(x) = f_i(x^3) = \phi_i(x)f_i(x)\phi_{i+1}(x).$$

By Proposition 2, $\phi_i(x) = \phi_{i+1}(x)$ therefore $f_i(x) \in \phi_i(x)A\phi_i(x) = \Bbbk\phi_i(x)$. Hence we get $\lambda_i(x) \in \Bbbk$ such that $f_i(x) = \lambda_i(x)\phi_i(x)$ and

$$\phi_{i+1}(a) = \lambda_i(x)(\lambda_i(y))^{-1}\phi_i(a).$$

(2) In [5, section 3] automorphisms of repetitive category with jump 0 is characterized in general case i.e., it does not assume that algebras have no nonzero oriented cycles, and automorphisms are "algebra automorphisms". In their results, the left term of exact sequence is given by $\mathrm{U}(A)^{\mathbb{Z}}$ where $\mathrm{U}(A)$ is the set of all units in A.

4. Orbit categories

Throughout this section G is a group. A pair $(\mathcal{C}, A)$ of a category and a group homomorphism $A : G \to \mathrm{Aut}(\mathcal{C})$ (we write $A_\alpha := A(\alpha)$) is called a category with G-action.

We cite the following definition and lemma from [2, Section 4].

Definition 11. Let $(\mathcal{C}, A)$, $(\mathcal{C}', A')$ be categories with G-actions and $F : \mathcal{C} \to \mathcal{C}'$ a functor. Then an *equivariance adjuster* of F is a family $\eta = (\eta_\alpha)_{\alpha\in G}$ of natural isomorphisms $\eta_\alpha : A'_\alpha F \Rightarrow FA_\alpha$ $(\alpha \in G)$ such that the following diagram commutes for each $\alpha, \beta \in G$

$$\begin{tikzcd}
A'_{\beta\alpha}F = A'_\beta A'_\alpha F \arrow[r, "A'_\beta\eta_\alpha"] \arrow[dr, "\eta_{\beta\alpha}"'] & A'_\beta F A_\alpha \arrow[d, "\eta_\beta A_\alpha"] \\
 & FA_{\beta\alpha} = FA_\beta A_\alpha
\end{tikzcd}$$

and a pair (F, η) is called a *G-equivariant* functor.

Lemma 12. *Let $(\mathcal{C}, A)$, $(\mathcal{C}', A')$ be categories with G-actions, and $(F, \eta) : \mathcal{C} \to \mathcal{C}'$ a G-equivariant equivalence. Then $\mathcal{C}/G$ and $\mathcal{C}'/G$ are equivalent.*

Proposition 13. *Let R be a locally bounded category, and g, h automorphisms of R. If there exists a map $\rho\colon R_0 \to \Bbbk^\times$ such that $\rho(y)g(f) = h(f)\rho(x)$ for all morphisms $f \in R(x,y)$, then $R/\langle g\rangle \cong R/\langle h\rangle$.*

Remark 14. Proposition 13 does not assume free actions. Therefore we extend a derived equivalence classification in [3] to "0-fold" case.

5. Main results

Throughout this section we assume that A is an algebra without nonzero oriented cycles unless we note.

Lemma 15. *Let ϕ and ψ be automorphisms of $\hat{A}$ with jump $n \in \mathbb{Z}$. If there exists a map $\rho_0\colon A_0 \to \Bbbk^\times$ such that $\rho_0(y)\phi_0(a) = \psi_0(a)\rho_0(x)$ for all morphisms $a \in A(x,y)$, then $\hat{A}/\langle\phi\rangle$ and $\hat{A}/\langle\psi\rangle$ are isomorphic.*

Theorem 16. *Let ϕ and ψ be automorphisms of $\hat{A}$ with jump $n \in \mathbb{Z}$. If there exist $i, j \in \mathbb{Z}$ and $\rho : A_0 \to \Bbbk^\times$ such that $\rho(y)\phi_i(a) = \psi_j(a)\rho(x)$ for all morphisms $a \in A(x, y)$, then $\hat{A}/\langle\phi\rangle$ and $\hat{A}/\langle\psi\rangle$ are isomorphic.*

Proof. By Remark 10(1), we get each of the elements $((\lambda_k(x))_{x\in A_0})_{k\in\mathbb{Z}}, ((\mu_k(x))_{x\in A_0})_{k\in\mathbb{Z}} \in ((\Bbbk^\times)^{A_0})^{\mathbb{Z}}$ from ϕ and ψ. Define $\rho_0 : A_0 \to \Bbbk^\times$ by

$$\rho_0(x) := \begin{cases} (\lambda_j \cdots \lambda_{-1}(x))^{-1}\mu_i \cdots \mu_{-1}(x)\rho(x) & \text{if } i, j < 0 \\ \lambda_0 \cdots \lambda_j(x)\mu_i \cdots \mu_{-1}(x)\rho(x) & \text{if } i < 0, j > 0 \\ (\lambda_j \cdots \lambda_{-1}(x))^{-1}(\mu_0 \cdots \mu_i(x))^{-1}\rho(x) & \text{if } i > 0, j < 0 \\ \lambda_0 \cdots \lambda_j(x)(\mu_0 \cdots \mu_i(x))^{-1}\rho(x) & \text{if } i, j > 0 \end{cases}$$

for all $x \in A_0$. Then for a morphism $a \in A(x, y)$,

$$\begin{aligned} \rho(y)\phi_i(a) &= \rho(y)((\lambda_{i-1}(x))^{-1}\lambda_{i-1}(y)\phi_{i-1}(a)) \\ &= \rho(y)(\lambda_{i-1}(x))^{-1}\lambda_{i-1}(y)((\lambda_{i-2}(x))^{-1}\lambda_{i-2}(y)\phi_{i-2}(a)) \\ &\vdots \\ &= \rho(y)(\lambda_0 \cdots \lambda_{i-2}\lambda_{i-1}(x))^{-1}\lambda_0 \cdots \lambda_{i-2}\lambda_{i-1}(y)\phi_0(a) \end{aligned}$$

and similarly

$$\psi_j(a)\rho(x) = (\mu_0 \cdots \mu_{j-2}\mu_{j-1}(x))^{-1}\mu_0 \cdots \mu_{j-2}\mu_{j-1}(y)\psi_0(a)\rho(x).$$

Hence we get $\rho_0(y)\phi_0(a) = \psi_0(a)\rho_0(x)$. By Lemma 15, $\hat{A}/\langle\phi\rangle$ and $\hat{A}/\langle\psi\rangle$ are isomorphic. □

Corollary 17. *Let ϕ be an automorphism of $\hat{A}$ with jump $n \in \mathbb{Z}$. Then $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are isomorphic.*

What we want to know is when $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are isomorphic if A is piecewise hereditary algebra of tree type. The following lemma gives us the answer.

Lemma 18. *A piecewise hereditary algebra has no nonzero oriented cycles.*

Proof. If A is a piecewise hereditary algebra, then there is a tilting complex T on a hereditary algebra H such that $A \cong \mathrm{End}(T)$. For all idempotents e in A, eAe is isomorphic to $\mathrm{End}(T_e)$ where T_e is a direct summand of T. By [4, Corollary 5.5], eAe is a piecewise hereditary algebra because T_e is a partial tilting complex. Since piecewise hereditary algebras have finite global dimension and eAe is local, eAe is isomorphic to $\Bbbk$. Hence A have no nonzero oriented cycles if A is a piecewise hereditary algebra. □

Corollary 19. *Let A be a piecewise hereditary algebra and ϕ be an automorphism of $\hat{A}$ with jump $n \in \mathbb{Z}$. Then $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are isomorphic.*

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TILTING OBJECTS IN STABLE CATEGORIES OF PREPROJECTIVE ALGEBRAS

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Abstract. In this paper, we construct a tilting object in stable categories of factor algebras of preprojective algebras. In [4], for a finite acyclic quiver Q and its preprojective algebra Π, Buan-Iyama-Reiten-Scott introduced and studied the factor algebra Π_w associated with an element w in the Coxeter group of Q. The algebra Π_w has a natural $\mathbb{Z}$-grading, and we prove that $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ has a tilting object if w is a c-sortable element.

1. Introduction

The preprojective algebra Π of a finite acyclic quiver Q has an important role in representation theory of algebras. One of them is categorifications of cluster algebras introduced by Fomin-Zelevinsky [6]. In the study of categorifications of cluster algebras, 2-Calabi-Yau triangulated categories (2-CY for short) and their cluster tilting objects are important.

If Q is a Dynkin quiver, then the preprojective algebra Π of Q is a finite dimensional selfinjective algebra and Geiss-Leclerc-Schröer showed that the stable category $\underline{\mathsf{mod}}\,\Pi$ is a 2-CY category and $\underline{\mathsf{mod}}\,\Pi$ has cluster tilting objects [7]. If Q is finite acyclic non-Dynkin quiver, Buan-Iyama-Reiten-Scott introduced and studied the factor algebra Π_w associated with an element w in the Coxeter group of Q [4]. They showed that the stable category of $\mathsf{Sub}\,\Pi_w$ is a 2-CY category and has cluster tilting objects, where $\mathsf{Sub}\,\Pi_w$ is the full subcategory of $\mathsf{mod}\,\Pi_w$ of submodules of finitely generated free Π_w-modules.

There are other classes of 2-CY triangulated categories. For a finite dimensional algebra A of finite global dimension, the cluster category $\mathcal{C}_A$ were introduced [1, 5]. The category $\mathcal{C}_A$ is a 2-CY category and has cluster tilting objects. Amiot-Reiten-Todorov [3] showed that there are close connections between 2-CY categories $\underline{\mathsf{Sub}}\,\Pi_w$ and $\mathcal{C}_A$. That is, for any finite acyclic quiver Q and any element w of the Coxeter group, there is a triangle equivalence

$$\underline{\mathsf{Sub}}\,\Pi_w \simeq \mathcal{C}_{A_w}$$

for some finite dimensional algebra A_w of global dimension at most two.

The aim of this paper is to construct a derived category version of this equivalence. More precisely, we regard Π_w as a $\mathbb{Z}$-graded algebra and consider the stable category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ of graded Π_w-submodules of graded free Π_w-modules. Then we construct a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$.

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

Through out this paper, let k be an algebraically closed field. By a module, we mean a left module unless stated otherwise. In this section, we give definitions used in the next section.

Definition 1. Let Q be a finite acyclic quiver.

(1) The *double quiver* $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, s, t)$ of Q is defined by $\overline{Q}_0 = Q_0$, $\overline{Q}_1 = Q_1 \sqcup \{\bar{\alpha} \mid \alpha \in Q_1\}$, where $s(\bar{\alpha}) = t(\alpha)$, $t(\bar{\alpha}) = s(\alpha)$ for all $\alpha \in Q_1$.

(2) Then we have the *preprojective algebra* Π of Q by

$$\Pi := k\overline{Q}/\langle \sum_{\beta \in Q_1} \beta\bar{\beta} - \bar{\beta}\beta \rangle.$$

In this paper, we assume Q is non-Dynkin quiver, that is, the underlying graph of Q is not a simply laced Dynkin diagram. Note that, if Q is non-Dynkin quiver, then the preprojective algebra of Q is not a finite dimensional algebra. Next we define the Coxeter group of Q.

Definition 2. The *Coxeter group* W of a quiver Q is the group generated by the set $\{s_i \mid i \in Q_0\}$ with relations

- $s_i^2 = 1$,
- $s_j s_i = s_i s_j$ if there are no arrows between i and j,
- $s_i s_j s_i = s_j s_i s_j$ if there is exactly one arrow between i and j.

An expression $w = s_{i_1} s_{i_2} \dots s_{i_l}$ is *reduced* if for any other expression $w = s_{i_1} s_{i_2} \cdots s_{i_m}$, we have $l \leq m$.

Let i be a vertex of Q. We define the two-sided ideal I_i of Π by

$$I_i := \Pi(1 - e_i)\Pi,$$

where e_i is the idempotent associated to i. Let $w = s_{i_1} s_{i_2} \dots s_{i_l}$ be a reduced expression of w. We define a two-sided ideal I_w of Π by

$$I_w := I_{i_1} I_{i_2} \cdots I_{i_l}.$$

Note that, an ideal I_w is independent of the choice of a reduced expression of w by [4, Theorem Ⅲ. 1.9]. In [4], the authors studied the algebra Π/I_w.

Let $\mathsf{mod}\,\Pi_w$ be the category of finitely generated Π_w-modules. We denote by $\mathsf{Sub}\,\Pi_w$ the full subcategory of $\mathsf{mod}\Pi_w$ of submodules of finitely generated free Π_w-modules.

Proposition 3. [4] *Let Q be a finite acyclic non-Dynkin quiver. For an element w of the Coxeter group of Q, we have the following results.*

(a) *The algebra Π_w is finite dimensional and* $\mathsf{inj.dim}({}_{\Pi_w}\Pi_w) \leq 1$.

(b) *The category $\mathsf{Sub}\,\Pi_w$ is a Frobenius category.*

(c) *The stable category $\underline{\mathsf{Sub}}\,\Pi_w$ is 2-Calabi-Yau triangulated category, that is, for any objects $X, Y \in \underline{\mathsf{Sub}}\,\Pi_w$, there is a functorial isomorphism $\underline{\mathrm{Hom}}_{\Pi_w}(X, Y) \simeq \mathrm{D}\,\underline{\mathrm{Hom}}_{\Pi_w}(Y, X[2])$, where $\mathrm{D} = \mathrm{Hom}_k(\ , k)$.*

(d) *For any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$, the object $T = \bigoplus_{j=1}^{l} \Pi_{s_{i_1} s_{i_2} \cdots s_{i_j}}$ is a cluster tilting object of $\underline{\mathsf{Sub}}\,\Pi_w$.*

Next we consider the grading of a preprojective algebra. We regard the path algebra $k\overline{Q}$ as a $\mathbb{Z}$-graded k-algebra by the following grading:

$$\deg \beta = \begin{cases} 1 & \beta = \bar{\alpha}, \alpha \in Q_1 \\ 0 & \beta = \alpha, \alpha \in Q_1. \end{cases}$$

Since the element $\sum_{\beta \in Q_1} (\beta\bar{\beta} - \bar{\beta}\beta)$ in $k\overline{Q}$ is homogeneous of degree 1, the grading of $k\overline{Q}$ naturally gives a grading on the preprojective algebra $\Pi = \bigoplus_{i \geq 0} \Pi_i$.

Remark 4. (a) We have $\Pi_0 = kQ$, since Π_0 is spanned by all paths of degree 0.

(b) For any $w \in W$ the ideal I_w of Π is a graded ideal of Π since so is each I_i.

(c) In particular, the quotient algebra Π_w is a graded algebra.

For a graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ and an integer j, we define a new graded module $M(j)$ by $(M(j))_i = M_{i+j}$. For any integer j, we define a graded submodule $M_{\geq j}$ of M by

$$(M_{\geq j})_i = \begin{cases} M_i & i \geq j \\ 0 & \text{else} \end{cases}$$

and a graded factor module of M by $M_{\leq j} = M/M_{\geq j+1}$.

Let $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ be the category of finitely generated $\mathbb{Z}$-graded Π_w-modules with degree zero morphisms. We denote by $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ the full subcategory of $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ of submodules of graded free Π_w-modules, that is,

$$\mathsf{Sub}^{\mathbb{Z}}\Pi_w = \left\{ X \in \mathsf{mod}^{\mathbb{Z}}\Pi_w \mid X \subset \bigoplus_{j=1}^{m} \Pi_w(i_j),\ i_j \in \mathbb{Z} \right\}.$$

By Proposition 3 (a), $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ is a Frobenius category. Then we have a triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. In this paper, we get a tilting object in this category.

3. c-sortable words and grading

In this section, we define a c-sortable words of the Coxeter group of Q and calculate the graded structure of Π_w.

Definition 5. Let Q be a finite acyclic quiver with vertices $Q_0 = \{1, 2, \ldots, n\}$ and W be the Coxeter group of Q.

(1) An element c in W is called a *Coxeter element* if c has an expression $c = s_{i_1}s_{i_2}\ldots s_{i_n}$, where $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$.

(2) A Coxeter element $c = s_{i_1}s_{i_2}\ldots s_{i_n}$ in W is said to be *admissible with respect to the orientation of* Q if c satisfies $e_{i_j}(kQ)e_{i_k} = 0$ for $k < j$.

Since Q is acyclic, W has a Coxeter element c admissible with respect to the orientation of Q. There are some expression of $c = s_{i_1}s_{i_2}\ldots s_{i_n}$ satisfying $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$ and $e_{i_j}(kQ)e_{i_k} = 0$ for $k < j$. However, it is shown that c is uniquely determined as an element of W. From now on, we call a Coxeter element admissible with respect to the orientation of Q simply a Coxeter element.

Then we define a c-sortable words.

Definition 6. Let c be a Coxeter element of W. An element $w \in W$ is said to be *c-sortable* if there is a reduced expression $w = c^{(0)}c^{(1)}\cdots c^{(l)}$, where each $c^{(i)}$ is subsequence of c and

$$\mathsf{Supp}(c^{(l)}) \subset \mathsf{Supp}(c^{(l-1)}) \subset \cdots \subset \mathsf{Supp}(c^{(0)}) \subset Q_0,$$

where $\mathsf{Supp}(c^{(i)})$ is the set of i_j such that s_{i_j} appears in $c^{(i)}$.

Example 7. Let $Q = \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. A Coxeter element is $c = s_3s_2s_1$. Then an element $w = s_3s_2s_1s_3s_2s_3$ is a c-sortable element. Actually, $c^{(0)} = s_3s_2s_1$, $c^{(1)} = s_3s_2$, and $c^{(2)} = s_3$.

If $w = c^{(0)}c^{(1)}\cdots c^{(l)}$ is a c-sortable element, then the grading of Π_w is calculated as follows.

Proposition 8. *Let $w = c^{(0)}c^{(1)}\cdots c^{(l)} \in W$ be a c-sortable element. If $i \leq l$, then we have $(\Pi_w)_{\leq i} = (\Pi_{c^{(0)}c^{(1)}\cdots c^{(i)}})_{\leq i} = \Pi_{c^{(0)}c^{(1)}\cdots c^{(i)}}$. If $i > l$, then we have $(\Pi_w)_{\geq i} = 0$.*

4. Main theorem

In this section, we state the main theorem of this paper. Let $\mathcal{T}$ be a triangulated category. Recall that, an object M in $\mathcal{T}$ is called a *tilting object* if following holds.

- $\mathsf{Hom}_{\mathcal{T}}(M, M[j]) = 0$ for any $j \neq 0$,
- $\mathsf{thick}M = \mathcal{T}$, where $\mathsf{thick}M$ is the smallest triangulated full subcategory of $\mathcal{T}$ containing M and closed under direct summands.

Let $\mathcal{T}$ be the stable category of a Frobenius category, and assume that $\mathcal{T}$ is Krull-Schmidt. If there is a tilting object M in $\mathcal{T}$, then it follows from [8, (4.3)] that we have a triangle equivalence

$$\mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\underline{\mathsf{End}}_{\mathcal{T}}(M)),$$

where $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathsf{End}_{\mathcal{T}}(M))$ is the homotopy category of bounded complexes of projective $\mathsf{End}_{\mathcal{T}}(M)$-modules.

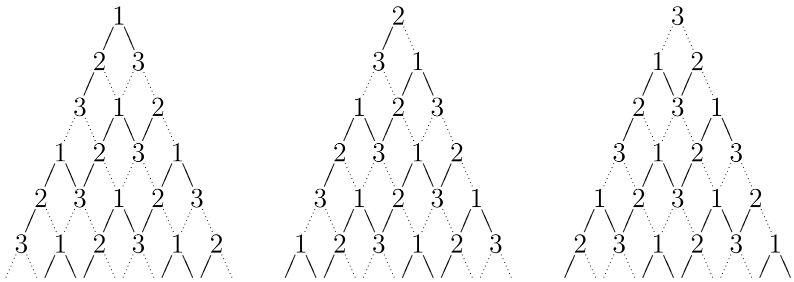
Theorem 9. *Let $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ be a c-sortable element. For an integer $1 \leq j \leq l$, let m_j be the number of integers $1 \leq k \leq j-1$ satisfying $i_j = i_k$. Then $M = \bigoplus_{j=1}^{l} \Pi/I_{s_{i_1}\cdots s_{i_j}} e_{i_j}(m_j)$ is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$.*

Actually, the module $M = \bigoplus_{j=1}^{l} \Pi/I_{s_{i_1}\cdots s_{i_j}} e_{i_j}(m_j)$ belongs to $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$, since M corresponds to the cluster tilting object of $\underline{\mathsf{Sub}}\,\Pi_w$ of Proposition 3 (d) by forgetting the grading.

The first condition of tilting objects follows from Proposition 8 and calculating a projective resolution of M. The second condition of tilting objects follows from the following Theorem which is shown in [2]. For a c-sortable element $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ and $i \in \mathsf{Supp}(w)$, let t_i be the number of integers $1 \leq k \leq l$ satisfying $i_k = i$.

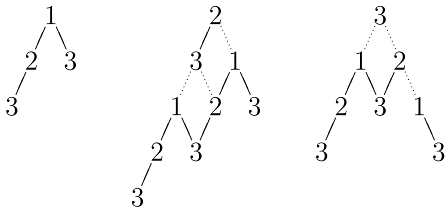
Theorem 10. [2, Theorem 3.11] *Let $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ be a c-sortable element. Then $\bigoplus_{i\in Q'_0}(\Pi_w e_i)_{t_i-1}$ is a tilting kQ'-module, where Q' is the full subquiver of Q such that $Q'_0 = \mathsf{Supp}(w)$.*

Example 11. Let Q be a quiver $\begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Then we have a graded algebra $\Pi = \Pi e_1 \oplus \Pi e_2 \oplus \Pi e_3$, and these are represented by their radical filtrations

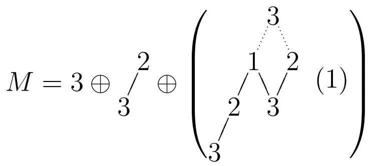


where numbers connected by solid lines are in the same degree, and the tops of the Πe_i are concentrated in degree 0.

Let $w = s_3 s_2 s_1 s_3 s_2 s_3$, then we have a graded algebra $\Pi_w = \Pi_w e_1 \oplus \Pi_w e_2 \oplus \Pi_w e_3$,



and a tilting module



in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$, where graded projective Π_w-modules are removed. The endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$ of M is given by the following quiver with relations

$$\Delta = \bullet \xleftarrow{b} \bullet \xleftarrow{a} \bullet \qquad ba = 0.$$

Since the algebra $k\Delta/\langle ba\rangle$ has global dimension two, we have a triangle equivalence

$$\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, k\Delta/\langle ba\rangle) \simeq \mathsf{D}^{\mathrm{b}}(k\Delta/\langle ba\rangle).$$

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A CHARACTERIZATION OF THE CLASS OF HARADA RINGS

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Abstract. One-sided Harada rings are certain artinian QF-3 rings, which can be regarded as a generalization of QF rings and serial rings (Nakayama rings). It is well-known that every left Harada ring can be represented by a upper staircase factor ring of a block extension of a QF ring. In this paper, we shall give a slightly different construction and characterization of left Harada rings by characterizing the class of left Harada rings.

1. 研究の動機

片側原田環はある種の QF-3 両側アルチン環であり，QF 環や serial 環 (中山環) の一般化と見なすことができる．原田環の性質や構造は，大城を中心として詳しく調べられており，非常に多くの特徴付けが得られている．ここでは次を定義とする．なお，片側原田環であるという性質は森田不変であることが知られているので，基本的 (basic) な場合の定義を与える．

Definition 1. 基本的な両側アルチン環 R が**左原田環**であるとは，次の条件を満たす直交原始べき等元の完全集合 $\{e_{ij} \mid i = 1, \ldots, m,\ j = 1, \ldots, n(i)\}$ をもつことをいう．

(1) 任意の $i = 1, \cdots, m$ に対して，$e_{i1}R$ は入射的右 R 加群である．
(2) 任意の $i = 1, \cdots, m,\ j = 1, \cdots, n(i) - 1$ に対して，$J(e_{ij}R_R) \cong e_{i,j+1}R_R$ が成り立つ．

Remark 2. 上の定義は実際には**右余原田環**と呼ばれる環のそれである．余原田環は，原田環と双対的な性質を満たす環であるが，大城によって**左**原田環と**右**余原田環の概念は一致することが示された．そのため，最近では右余原田環の性質に基づいた記述をする場合でも，左原田環と呼ぶことが多い．本論文でも左原田環と呼ぶが，用いるのはほとんどが右余原田環の性質である．

左原田環について次の構造定理が知られている．

Theorem 3 (大城 [1, Chapter 4])**.** すべての基本的左原田環は，ある QF 環のブロック拡大の上階段型剰余環で表される．

ここでブロック拡大について説明しておこう．一般の場合も同様なので簡単な例で述べると，R が基本的半完全環で，2 個から成る直交原始べき等元の完全集合 $\{e_1, e_2\}$ をもつ場合，R は行列表現 (ピアス分解)

$$R = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$$

The detailed version of this paper will be submitted for publication elsewhere.

をもつ．ただし，$A = e_1Re_1, B = e_2Re_2, X = e_1Re_2, Y = e_2Re_1$ である．ブロック拡大とは，次のような R と森田同値な環 T の部分環 S のことである．

$$S = \begin{pmatrix} A & A & A & X & X \\ J(A) & A & A & X & X \\ J(A) & J(A) & A & X & X \\ Y & Y & Y & B & B \\ Y & Y & Y & J(B) & B \end{pmatrix} \subset \begin{pmatrix} A & A & A & X & X \\ A & A & A & X & X \\ A & A & A & X & X \\ Y & Y & Y & B & B \\ Y & Y & Y & B & B \end{pmatrix} = T.$$

ただし，$J(A), J(B)$ は radical 表す．大城はこの S を R のブロック拡大と呼び，$R(3,2)$ で表した．一般のブロック拡大も同様に定義される．なお，ブロック拡大は blow-up とも呼ばれる ([4, Chapter 6])．

このようにブロック拡大は難しいものではないが，一般に記述が少し大変である．また Theorem 3 の上階段型剰余環については，剰余を取るイデアルの記述自体も容易ではなく，左原田環を得るためにどのようなイデアルで割ることが本質的なのかも分かりにくかった．

本論文の目的は，基本的 QF 環を含み，1 個のべき等元を添加したブロック拡大と，ある種の「単純」イデアルによる剰余環 (組成列の長さが 1 だけ小さくなる) を取るという操作が，基本的左原田環全体のクラスを特徴付けることを示し，このような操作を繰り返すことにより，すべての左原田環が構成されることを述べることである．また，この構成方法について quiver による具体例で詳しく説明する．

以下，本論文において，R は単位元をもつ半完全環とし，右 R 加群 M に対して，$J(M)$ と $S(M)$ でそれぞれ M の radical と socle を表す．$S_i(M)$ $(i = 1, 2, \dots)$ で M の i-th socle を表し，$T(M)$ で M の top を表す．すなわち $T(M) = M/J(M)$ とする．$\mathrm{Pi}(R)$ で，固定した R の 1 つの直交原始べき等元の完全集合を表す．

2. 主結果と例

まず主結果を述べる．そのためには，原田環の研究において重要な役割を果たす，ある拡大環 (単位元は共有しない) を定義しておかなければならない．R を基本的半完全環とするとき，$e \in \mathrm{Pi}(R)$ に対して，環 R_e を次の行列環によって定める．

$$R_e = \begin{pmatrix} R & Re \\ J(eR) & eRe \end{pmatrix}.$$

明らかに R_e も基本的半完全環である．$\mathrm{Pi}(R) = \{e_1, \dots, e_i, \dots, e_n\}$，$e = e_i$ のとき，R_e は大城 [1, Chapter 4] の意味の R のブロック拡大 $R(1, \cdots, 2, \cdots, 1)$ に他ならない．R_e は次の 2 つの原始べき等元をもつ．

$$\tilde{e} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, \ \hat{e} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}.$$

これらについて，右 R_e 加群として $J(\tilde{e}R_e) \cong \hat{e}R_e$ が成り立っている．後で述べる quiver の形からも，R_e は R に e のコピー $\hat{e}$ を添加した拡大環であると見なすことができる．

次が本論文の主結果である．

Theorem 4. 基本的左原田環全体のクラスを $\mathcal{H}$ とするとき，$\mathcal{H}$ は次の性質を満たす．

(I) $\mathcal{H}$ はすべての基本的 QF 環を含んでいる．

(II) $R \in \mathcal{H}$, $e \in \mathrm{Pi}(R)$ ならば $R_e \in \mathcal{H}$ である．

(III) $R \in \mathcal{H}$ で，$e, f \in \mathrm{Pi}(R)$ が
(i) eR_R は入射的で $S(eR_R) \cong T(fR_R)$;
(ii) fR_R は入射的でない
を満たすならば，$R/S(eR_R) \in \mathcal{H}$ である．
(IV) $R \in \mathcal{H}$ で，$e, g \in \mathrm{Pi}(R)$ が
(i) eR_R は入射的である;
(ii) $eR/S(eR_R) \cong J(gR_R)$
を満たすならば，$R/S(eR_R) \in \mathcal{H}$ である．
逆に，$\mathcal{H}$ は性質 (I)–(IV) を満たす最小の環のクラスである．

Remark 5. (1) $\mathcal{H}$ の最小性より，すべての基本的左原田環は
- 基本的 QF 環を取る
- (II), (III), (IV) の操作を繰り返す

ことによって得られる．性質 (I)–(IV) は左原田環の一種の「公理」と見なすことができる．
(2) 記述を簡単にするため (IV) の表現としたが，R が斜体の場合に適用すると，$\mathcal{H}$ は零環を含んでしまう．$R \in \mathcal{H}$ に斜体でないという条件をつけておけば，$\mathcal{H}$ は零環を含まない．

Theorem 4 の状況を例で説明する前に，R が体上道多元環の剰余環の場合の R_e の quiver の形を述べておく．なお，一般のブロック拡大の quiver 表現については山浦 [5] で求められているが，次のように R_e の方が当然記述は簡単になる．

Proposition 6. K を体，$Q = (Q_0, Q_1, s, t)$ を有限 quiver，I を道多元環 KQ の許容イデアルとし，$R = KQ/I$ とおく．$i \in Q_0$ を固定し，対応する R の原始べき等元を e_i とする．このとき，$\tilde{R} = R_{e_i}$ の quiver $\tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1)$ と許容イデアル $\tilde{I}$ は次のようにして与えられる．
【頂点】$\tilde{Q}$ の頂点は，Q の頂点に i のコピー $\hat{i}$ を付け加えたものである: $\tilde{Q}_0 = Q_0 \cup \{\hat{i}\}$.
【矢】
- target が i でないような Q の矢はそのまま $\tilde{Q}$ の矢とする．
- target が i であるような Q の矢 $\alpha : j \to i$ は，target を $\hat{i}$ に変えた $\hat{\alpha} : j \to \hat{i}$ を $\tilde{Q}$ の矢とする．
- $\tilde{Q}$ は $\hat{i}$ から i への特別な矢 $\omega : \hat{i} \to i$ をもつ．(i への矢，$\hat{i}$ からの矢はこれのみである．)

$$\tilde{Q}_1 = \{\alpha \mid \alpha \in Q_1, t(\alpha) \neq i\} \cup \{\hat{\alpha} \mid \alpha \in Q_1, t(\alpha) = i\} \cup \{\omega\}.$$

【関係式 (許容イデアルの生成元)】target を i とする Q の矢 $\alpha : j \to i$ に対して，$\tilde{Q}$ の矢 $\hat{\alpha} : j \to \hat{i}$ と $\omega : \hat{i} \to i$ を合成した $\tilde{Q}$ の道 $\omega\hat{\alpha} : j \to i$ を α と名付ける．Q の関係式 (I の生成元) はそのまま $\tilde{Q}$ の関係式 ($\tilde{I}$ の生成元) であると見なす．ただし，target が i であるような Q の関係式 $\sum_l \alpha_l p_l$ ($\alpha_l : j_l \to i$ は Q の矢，$p_l : k \to j_l$ は Q の道) については，$\sum_l \hat{\alpha}_l p_l$ を $\tilde{Q}$ の関係式とする．

それでは Theorem 4 の状況を詳述しよう．

Example 7. (1) K を体とし，A を quiver と関係式

$$Q_A : \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] & 3 \arrow[l, bend right, "\beta"'] \end{tikzcd}, \quad \{\delta\alpha,\ \gamma\beta,\ \alpha\gamma - \beta\delta\}.$$

で定義される QF 多元環とする．定義より (Theorem 4 (I) より) A は左原田環である．e_i $(i = 1, 2, 3)$ を頂点 i に対応する原始べき等元とする．単純加群 $T(e_i R)$ を "i" で表せば，直既約射影的右 A 加群の Loewy 列は次のようになる．

$$A_A = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{ccccc} & & 2 & & \\ & / & & \backslash & \\ 1 & & & & 3 \\ & \backslash & & / & \\ & & 2 & & \end{array} \oplus \begin{array}{c} 3 \\ 2 \\ 3 \end{array}.$$

(2) A に原始べき等元 e_3 を添加した環 $B = A_{e_3}$ を考える．Theorem 4 (II) は，B は左原田環であることを主張している．Proposition 6 より B の quiver と関係式は次の通りとなる．

$$Q_B : \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\gamma"] \arrow[dr, bend right, "\hat{\delta}"'] & 3 \arrow[l, bend right, "\beta"'] \\ & & \hat{3} \arrow[u, "\omega"'] \end{tikzcd}, \quad \{\hat{\delta}\alpha,\ \gamma\beta,\ \alpha\gamma - \beta\delta\}.$$

ただし δ は道 $\delta = \omega\hat{\delta}$ である．直既約射影的右 B 加群を Loewy 列で表すと次のようになる．($\boxed{\boxed{}}$ と $\boxed{}$ は後の C や D でイデアルとして割る部分．)

$$B_B = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{ccccc} & & 2 & & \\ & / & & \backslash & \\ 1 & & & & 3 \\ & & & & | \\ & \backslash & & & \hat{3} \\ & & 2 & / & \end{array} \oplus \begin{array}{c} 3 \\ \hat{3} \\ 2 \\ 3 \\ \boxed{\boxed{\hat{3}}} \end{array} \oplus \begin{array}{c} \hat{3} \\ 2 \\ 3 \\ \boxed{\hat{3}} \end{array}.$$

$\hat{e_3} = e_{\hat{3}}$ を頂点 $\hat{3}$ に対応する原始べき等元とすると，$e_i B$ $(i = 1, 2, 3)$ は入射的，$J(e_3 B) \cong e_{\hat{3}} B$ で，確かに B は左原田環の定義を満たしている．

(3) 左原田環 B について，$e_3 B$ は入射的，$S(e_3 B) \cong T(e_{\hat{3}} B)$ で，$Be_{\hat{3}}$ は入射的ではないので，Theorem 4 (III) より，剰余環 $C = B/S(e_3 B)$ は再び左原田環 (実際には QF) になる．C の quiver は B の quiver と同じであり，関係式は $\omega\hat{\delta}\beta\omega$ を追加したものである．直既約射影的右 C 加群は，B_B のそれを $\boxed{\boxed{}}$ の部分で割ったものである．

(4) 左原田環 C について，$e_3 C$ は入射的で，$e_{\hat{3}} C / S(e_{\hat{3}} C) \cong J(e_3 C)$ であるから，Theorem 4 (IV) より，剰余環 $D = C/S(e_{\hat{3}} C) = B/(S(e_3 B) \oplus S(e_{\hat{3}} B))$ も左原田環となる．D の quiver も B の quiver と同じであり，関係式は $\omega\hat{\delta}\beta\omega$ と $\hat{\delta}\beta\omega$ を追加したものである．直既約射影的右 D 加群は，B_B のそれを $\boxed{\boxed{}}$ と $\boxed{}$ の部分で割ったものである．

Remark 8. 本論とは直接関係ないが，Example 7 の左原田環 C の大局次元は 6 に等しい．筆者は [3] において，大局次元が 3 以下の左原田環は serial となることを示し (Theorem 2.1)，6 以上の任意の偶数 $2n$ に対して $\mathrm{gl.dim}\, R = 2n$ となる serial でない左原田環の例を構成した (Example 2.2)．大局次元が 6 である serial でない左原田環が C であり，8 以上のものは，Example 7 のように，C を元にべき等元を 1 個付け加えて剰余環を取ることによって帰納的に構成した (Remark 2.1 参照)．

なお，大局次元が有限で serial でないものは，筆者はこの例しか得ていない．大局次元が 4 や 5 であるような serial でない左原田環が存在するかどうかは，まだ分かっていない．

3. 性質 (III), (IV)

それでは，Theorem 4 の前半部分の証明について述べよう．性質 (I) は自明である．性質 (II) は左原田環のブロック拡大の左原田性の特別な場合であり，一般的な形で，例えば大城 [1, Theorem 4.2.2] で述べられている．また，比較的簡単に確かめることができる．したがって，性質 (III), (IV) が問題になる．

R を片側アルチン環，$e \in \mathrm{Pi}(R)$ で eR_R は入射的であるとする．このとき，$S(eR) \cong T(fR)$, $S(Rf) \cong T(Re)$ となるような $f \in \mathrm{Pi}(R)$ が存在し，${}_RRf$ は入射的である．このような (eR, Rf) は ***i*-pair** と呼ばれる．また $S(eR) = S(Rf)$ であるからこれは両側イデアルで，左右いずれの加群としても単純である．

なお，i-pair の概念は原田環の研究においても非常に重要な役割を果たす．以下でも，R が左原田環の場合に剰余環 $\bar{R}$ における直既約射影的右 $\bar{R}$ 加群 $\bar{g}\bar{R}$ の入射性を述べるが，実際には i-pair を用いて証明する．

Theorem 4 の (III), (IV) は，R が左原田環の場合の剰余環 $\bar{R} = R/S(eR)$ の左原田性を扱っている．次の Lemma は eR が入射的であるという条件のみで成り立つ．

Lemma 9. R を基本的左原田環とし，eR_R は入射的 $(e \in \mathrm{Pi}(R))$ であるとする．剰余環 $\bar{R} = R/S(eR)$ について，次が成り立つ．

(1) $gR \cong J(hR)$ $(g, h \in \mathrm{Pi}(R))$ のとき，
  (a) $h \neq e$ であれば，$\bar{g}\bar{R} \cong J(\bar{h}\bar{R})$ である．
  (b) $h = e$ であれば，$\bar{g}\bar{R}_{\bar{R}}$ は入射的である．
(2) gR_R が入射的 $(g \in \mathrm{Pi}(R))$ のとき，$g \neq e$ であれば，$\bar{g}\bar{R}_{\bar{R}}$ も入射的である．

一般に，基本的アルチン R が左原田環であるためには，任意の $g \in \mathrm{Pi}(R)$ に対して，gR が入射的であるか，ある $h \in \mathrm{Pi}(R)$ に対して $gR \cong J(hR)$ であることが必要十分であるから，この Lemma より，eR が入射的のとき，剰余環 $\bar{R} = R/S(eR)$ はかなり左原田環に近いことが分かる．したがって，剰余環 $\bar{R}$ において $\bar{e}\bar{R}$ が入射的または $\bar{e}\bar{R} \cong J(\bar{g}\bar{R})$ $(g \in \mathrm{Pi}(R))$ となるかどうかが問題となる．Theorem 4 (IV) は後者となる場合なので，性質 (IV) も分かった．

Lemma 10. R を基本的左原田環とし，eR_R は入射的で $S(eR) \cong T(fR)$，fR_R は入射的でなく $fR \cong J(gR)$ $(e, f, g \in \mathrm{Pi}(R))$ であるとする．剰余環 $\bar{R} = R/S(eR)$ について，次が成り立つ．

(1) ${}_RRg$ が入射的でないとき，$\bar{e}\bar{R}_{\bar{R}}$ は入射的である．
(2) ${}_RRg$ が入射的のとき，$S(hR) \cong T(gR)$ $(h \in \mathrm{Pi}(R))$ とする．$h_1 = h, h_2, \ldots, h_n \in \mathrm{Pi}(R)$ を，$J(h_iR) \cong h_{i+1}R$ $(i = 1, \ldots, n-1)$，$J(h_{n+1}R)$ は射影的でない，であるようなものとすれば，$\bar{e}\bar{R} \cong J(\bar{h_n}\bar{R})$ が成り立つ．

したがって，Theorem 4 (III) の仮定の下では，$\bar{e}\bar{R}$ は左原田環の条件を満たす射影的右 $\bar{R}$ 加群になるので，(III) も言える．なお，Lemma 9 に比べると Lemma 10 の証明は容易ではなく，いくつかの準備を必要とする．

Theorem 4 (III), (IV) を繰り返し適用すると次が分かる．後で用いるようにこの結果は有用であり，基本的左原田環の剰余環がいつ再び左原田環になるかを示している．

Theorem 11. R を基本的左原田環とし，$e_1, \ldots, e_n, f \in \mathrm{Pi}(R)$ は次を満たすとする．

(1) e_1R は入射的で，$S(e_1R) \cong T(fR)$,
(2) $J(e_iR) \cong e_{i+1}R \quad (i = 1, \ldots, n-1)$.

もし fR が入射的でなければ，R/K_i $(i=1,\cdots,n)$ も左原田環である．ただし，

$$K_i = S(e_1R)\oplus\cdots\oplus S(e_nR)$$

とする．

Remark 12. R が基本的左原田環のとき，Theorem 11 の設定で，必ずしも fR が入射的でなくても，

$$S(e_1R_R)\oplus\cdots\oplus S(e_nR_R) = S_i({}_RRf)$$

であることが知られている．

4. $\mathcal{H}$ の最小性

$\mathcal{H}$ の最小性については，次の 2 つの Lemma から従う．

Lemma 13 ([2, Proposition 2.15])**.** R を基本的左原田環，fR $(f\in \mathrm{Pi}(R))$ は入射的でないとする．このとき，$(1-f)R(1-f)$ は左原田環となる．

Lemma 14 ([2, Lemma 2.6 and Lemma 2.7])**.** R を基本的原田環，$fR\cong J(eR)$ $(e,f\in \mathrm{Pi}(R))$ とする．$R'=(1-f)R(1-f)$，$\tilde{R}=R'_e$ とおく．

(1) Re が入射的でないとき，$R\cong\tilde{R}$ である．
(2) Re が入射的のとき，ある $i\geq 1$ が存在して $R\cong\tilde{R}/K_i$ である．ただし，$K_i=S_i(\tilde{R}\hat{e})$ は Theorem 11 の通りとする．

左原田環の構造において，これらの Lemma は非常に重要である．Lemma 13 は，与えられた左原田環が QF 環でなければ，右入射的でない直既約射影的加群に対応する原始べき等元を取り除いて「縮小」したものも，再び左原田環であることを示している．Lemma 14 は，それにある原始べき等元を添加すれば，元の左原田環が「復元」できることを示している．これらを用いれば Theorem 4 における $\mathcal{H}$ の最小性が示せるが，その前にこれらの Lemma を例で述べよう．

Example 15. R を許容列 $(4,4,3)$ をもつ serial 環とし，e_1,e_2,e_3 を対応する直交原始べき等元とする．このとき e_1R と e_2R は入射的で，$J(e_2R)\cong e_3R$ である．R は左原田環であるから Lemma 13 が適用できるので，$R'=(1-e_3)R(1-e_3)$ は左原田環 (実際には serial 環) である．$\tilde{R}=R'_{e_2}$ とおけば，Theorem 4 (II) より左原田環 (serial 環) である．Lemma 14 は，

$$R\cong\tilde{R}/(S(e_2\tilde{R})\oplus S(\hat{e}_2\tilde{R}))$$

として，R は $R'=(1-e)R(1-e)$ から復元できることを示している．直既約射影的右加群を図示すると，次の通りである．

$$R_R = \begin{array}[t]{c}1\\2\\3\\1\end{array}\oplus\begin{array}[t]{c}2\\3\\1\\2\end{array}\oplus\begin{array}[t]{c}3\\1\\2\end{array},\quad R'_{R'}=\begin{array}[t]{c}1\\2\\1\end{array}\oplus\begin{array}[t]{c}2\\1\\2\end{array},\quad \tilde{R}_{\tilde{R}}=\begin{array}[t]{c}1\\2\\\hat{2}\\1\end{array}\oplus\begin{array}[t]{c}2\\\hat{2}\\1\\2\\\boxed{\hat{2}}\end{array}\oplus\begin{array}[t]{c}\hat{2}\\1\\2\\\boxed{\hat{2}}\end{array}.$$

実際，$\tilde{R}$ において $\boxed{}$ の部分で割って $\hat{2}$ と 3 を同一視すれば，R に一致する．

それでは Theorem 4 における $\mathcal{H}$ の最小性の証明を与えよう．

Proof (Theorem 4 における $\mathcal{H}$ の最小性) ．$\mathcal{H}'$ を Theorem 4 の性質 (I)–(IV) を満たす環のクラスとする．$\mathcal{H} \subset \mathcal{H}'$ を示せばよい．$R \in \mathcal{H}$ とする．$R \in \mathcal{H}'$ を直交原始べき等元の完全集合に含まれる元の個数 $\#\,\mathrm{Pi}(R)$ に関する帰納法で示す．R が QF であれば，性質 (I) より $R \in \mathcal{H}'$ であるから，R は QF でないと仮定する．

R は QF でないから，$e, f \in \mathrm{Pi}(R)$ で $fR \cong J(eR)$ となるものが存在する．$R' = (1-f)R(1-f)$ とおくと，Lemma 13 より R' は左原田環である．したがって帰納法の仮定より $R' \in \mathcal{H}'$ である．よって $\tilde{R} = R'_e$ とおけば，性質 (II) より $\tilde{R} \in \mathcal{H}'$ である．

Re が入射的でないとき，Lemma 14(1) より $R \cong \tilde{R} \in \mathcal{H}'$ を得る．Re が入射的のとき，Lemma 14(2) より $R \cong \tilde{R}/K_i$ $(i \geq 1)$ である．したがって，Theorem 11 (これは性質 (III), (IV) を繰り返し用いて得られている) より，$R \cong \tilde{R}/K_i \in \mathcal{H}'$ が分かる．

最後に Theorem 4 の一つの応用について述べる．[2] において，筆者はすべての左原田環は**概自己双対性** (almost self-duality) と呼ばれる自己双対性 (self-duality) の一般化をもつことを示した．そこでの証明は，左原田環に関する Lemma 13, 14 を示すことと，概自己双対性のいくつかの性質を示すことによって与えたが，Theorem 4 の観点からは次のように述べることができる．

Example 16 ([2, Theorem 3.2])**.** 概自己双対性をもつ基本的アルチン環のクラス $\mathcal{A}$ は，Theorem 4 の性質 (I)–(IV) を満たす．したがって Theorem 4 の $\mathcal{H}$ の最小性より，$\mathcal{H} \subset \mathcal{A}$ である．すなわち，すべての (基本的) 左原田環は概自己双対性をもつ．

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BASICALIZATION OF KLR ALGEBRAS

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Abstract. We describe an algorithm to basicalize KLR algebras arising from quivers.

1. Preliminaries

Let k be field and A be a finite dimensional or "good" infinite dimensional connected algebra over k. Throughout this paper, an algebra is associative and with an unit element 1_A. Then A is decomposed into indecomposable projective left A-modules P_i as left A-module, where P_i is Ae_i for a complete set of primitive orthogonal idempotents :

(i) $\sum_{i=1}^{n} e_i = 1_A$,
(ii) if $e_i = f + g$ where $fg = gf = 0$, $f^2 = f$, $g^2 = g$ then $f = 0$ or $g = 0$,
(iii) $(e_i)^2 = e_i$,
(iv) $e_i e_j = 0$ for $i \neq j$.

We call A basic if $P_i \ncong P_j$ for $i \neq j$. Even if A is not basic, we can basicalize A like that. Choose some primitive idempotents e_{j_k} to satisfy the following property: for every e_i there exists exactly one r such that $P_i \cong P_{j_r}$. Set e the sum of those idempotents then $A^b := eAe$ is basic algebra. Note that A and A^b are Morita equivalent therefore module categories of those two are equivalent.

Let A be a basic algebra, then we can obtain a connected quiver Q and an admissible ideal I of a path algebra kQ such that $A \cong kQ/I$. Our final destination is to describe an algorithm to obtain such Q and I for KLR algebras.

Let Γ be a finite connected quiver without loops and multiple arrows. Let $\Gamma_0 = \{1, 2, \ldots, n\}$. Let ν be n-tuple $(\nu_1, \nu_2, \ldots, \nu_n)$ of non-negative integers. In general, KLR algebras $R_\Gamma(\nu)$ is defined depend on ν however in this paper we fix $\nu_i = 1$ for every i. Let $I_n = \{\sigma(1, 2, \cdots, n) \,|\, \sigma \in S_n\}$, $s_k = (k, k+1) \in S_n$. For $\mathbf{i} \in I_n$, describe $\mathbf{i}$ as $(i_1, i_2, \ldots, i_n)$.

Definition 1. A *KLR* algebra R_Γ is defined from these generators and relations.

- generators:
  $\{\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n\} \cup \{y_1, \cdots, y_n\} \cup \{\psi_1, \cdots, \psi_{n-1}\}$.
- relations:
  $\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}\mathbf{e}(\mathbf{i})$, $\sum_{\mathbf{i} \in I_n} \mathbf{e}(\mathbf{i}) = 1$,
  $y_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})y_k$, $\psi_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(s_k\mathbf{i})\psi_k$,
  $y_k y_l = y_l y_k$,

The detailed version of this paper will be submitted for publication elsewhere.

$\psi_k y_l = y_l \psi_k \ (l \neq k, k+1)$,
$\psi_k y_{k+1} = y_k \psi_k$, $y_{k+1}\psi_k = \psi_k y_k$,
$\psi_k \psi_l = \psi_l \psi_k \ (|k-l| > 1)$,
$\psi_k \psi_{k+1} \psi_k = \psi_{k+1}\psi_k\psi_{k+1}$,

$$\psi_k^2 \mathbf{e}(\mathbf{i}) = \begin{cases} \mathbf{e}(\mathbf{i}) & (i_k \not\leftrightarrow i_{k+1}) \\ (y_{k+1} - y_k)\mathbf{e}(\mathbf{i}) & (i_k \to i_{k+1}) \\ (y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftarrow i_{k+1}) \\ (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftrightarrow i_{k+1}) \end{cases}.$$

Note that the first (resp. second) equation shows $\mathbf{e}(\mathbf{i})$s are orthogonal (resp. complete). Moreover, R_Γ is $\mathbb{Z}$-graded algebra by $deg(\mathbf{e}(\mathbf{i})) = 0$, $deg(y_k) = 2$, $deg(\psi_k) = 0$ if $i_k \not\leftrightarrow i_{k+1}$, 1 if $i_k \to i_{k+1}$ or $i_k \leftarrow i_{k+1}$, 2 if $i_k \leftrightarrow i_{k+1}$.

2. The starting point

As the first step, we define a class of quiver called gemstone quiver.

Definition 2. A gemstone quiver G_n is defined as follows.

- vertices: $\mathbf{i} \in I^n$.
- arrows:
  - $y_k^{\mathbf{i}} : \mathbf{i} \to \mathbf{i}$ for each $\mathbf{i} \in I_n$ and $1 \leq k \leq n$,
  - $\psi_l^{\mathbf{i}} : \mathbf{i} \to s_l\mathbf{i}$ for each $\mathbf{i} \in I_n$ and $1 \leq l < n$.

Then we obtain following lemma.

Lemma 3. *There exists an epimorphism $kG_n \longrightarrow R_\Gamma$ by $\mathbf{i} \mapsto \mathbf{e}(\mathbf{i})$, $y_k^{\mathbf{i}} \mapsto \mathbf{e}(\mathbf{i})y_k\mathbf{e}(\mathbf{i})$, $\psi_l^{\mathbf{i}} \mapsto \mathbf{e}(\mathbf{i})\psi_l\mathbf{e}(s_l\mathbf{i})$. Moreover, $kG_n/I_\Gamma \cong R_\Gamma$ where I_Γ is an ideal obtained by rewriting relations of R_Γ by the above correspondence.*

Note that I_Γ is not admissible ideal since there are those relations : $\psi_k^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ if $i_k \not\leftrightarrow i_{k+1}$, $(y_{k+1} - y_k)\mathbf{e}(\mathbf{i})$ if $i_k \to i_{k+1}$, $(y_k - y_{k+1})\mathbf{e}(\mathbf{i})$ if $i_k \leftarrow i_{k+1}$. Therefore we need some processes except for some cases. The following corollary is straightforward from the next section.

Corollary 4. *Let Γ be a quiver with 2-cycle for each two vertices. Then G_n and I_Γ present R_Γ.*

3. Processes

We should start from removing this type of relations: $\psi_k^2\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ if $i_k \not\leftrightarrow i_{k+1}$. In fact, that relations are useful to determine an isomorphic class of indecomposable projective modules.

Lemma 5. *All $\mathbf{e}(\mathbf{i})$ are primitive. Therefore $R_\Gamma\mathbf{e}(\mathbf{i})$ is indecomposable.*

Lemma 6. $R_\Gamma\mathbf{e}(\mathbf{i}) \cong R_\Gamma\mathbf{e}(s_k\mathbf{i})$ *if and only if* $i_k \not\leftrightarrow i_{k+1}$

Using this lemma repeatedly, we can obtain the following property.

Let $\bar{G}_n$ be a graph obtained by removing loops and replacing each 2-cycles by edge on G_n. Cut edges between $\mathbf{i}$ and $s_k\mathbf{i}$ if there exists some arrows between i_k and i_{k+1} on Γ,

denote this cut graph G_Γ. Then the followings are equivalent:
(a) $\mathbf{i}$ and $\mathbf{j}$ are on the same connected component on G_Γ,
(b) $R_\Gamma\mathbf{e}(\mathbf{i}) \cong R_\Gamma\mathbf{e}(\mathbf{j})$.
We get a new quiver by identifying the vertices of G_n for each connected components of G_Γ.

To rewrite relations, we should pick up one $\mathbf{i}$ from each connected components. Then vertices $\mathbf{i}$ means $\mathbf{e}(\mathbf{i})$ and loops $y_k^{\mathbf{i}}$ means $\mathbf{e}(\mathbf{i})y_k\mathbf{e}(\mathbf{i})$. However the meaning of two cycles for two vertices $\mathbf{i}$ and $\mathbf{j}$ are bit complicated. Since there are two cycles between them, there exists some paths from $\mathbf{i}$ to $\mathbf{j}$ in G_n constructed from three parts:
(i) a path in connected component with $\mathbf{i}$, from $\mathbf{i}$ to some $\mathbf{i}'$,
(ii) an arrow $\mathbf{i}'$ to $\mathbf{j}'$ where $\mathbf{j}'$ picked from a connected component with $\mathbf{j}$,
(iii) a path in connected component with $\mathbf{j}$ from $\mathbf{j}'$ to $\mathbf{j}$.

We pick two minimal paths for each two cycles between $\mathbf{i}$ and $\mathbf{j}$ to be inverse each other. Then the arrow $\mathbf{i}$ to $\mathbf{j}$ means $\mathbf{e}(\mathbf{i})\psi_\omega\mathbf{e}(\mathbf{j})$, where ψ_ω is a multiplication of ψs in G_n taken as above. Note that only part (ii) has positive degree in that path.

Then relations for this quiver are obtained from G_n by rewriting with the correspondence above. However there still remains a problem from these type of relations:
$\psi_k^2\mathbf{e}(\mathbf{i}) = \pm(y_{k+1} - y_k)\mathbf{e}(\mathbf{i})$ if there exists one arrow between i_k and i_{k+1}.
The problem is on right hand side, it must not be in admissible ideal since it's just a sum of two arrows. Therefore we delete arrows by rewriting relations as follows:
$y_{k+1}\mathbf{e}(\mathbf{i}) = y_k\mathbf{e}(\mathbf{i}) \pm \psi_k^2\mathbf{e}(\mathbf{i})$.
After that process all relations are obtained from a linear combination of paths of length greater than 2. Therefore it's completed.

From the construction above, we can obtain some combinatorial observations such as :

Corollary 7. *The quiver for R_Γ has at least one loop for each vertex.*

4. Cyclotomic case

We can use previous method for cyclotomic case.

Definition 8. For $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$, a cyclotomic ideal I^Λ is generated by $\left\{ y_1^{\lambda_{i_1}}\mathbf{e}(\mathbf{i}) | \mathbf{i} \in I_n \right\}$.
We call a quotient algebra $R_\Gamma^\Lambda = R_\Gamma / I^\Lambda$ a cyclotomic KLR algebra.

Only what we do is adding relations from that generators. However there is $\lambda_k \leq 1$, we need rewrite something. If there is $\lambda_k = 0$, we need to trim some vertices by using following lemma.

Lemma 9. *In R_Γ^Λ, $\mathbf{e}(\mathbf{i}) = 0$ if and only if $\lambda_{i_1} = 0$ or there exists k such that for every $s < k$ there is no arrow between i_s and i_k on Γ.*

We trim $\mathbf{i}$ with $\mathbf{e}(\mathbf{i}) = 0$ and rewrite relations including $\mathbf{i}$.

The remaining problem is about this type of relations: $y_1\mathbf{e}(\mathbf{i}) = 0$. This happens if $\lambda_{i_1} = 1$. To avoid this relation, delete arrows $y_1^{\mathbf{i}}$ and rewrite relations including $\mathbf{i}$. Then it‘s completed.

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THE ARTINIAN CONJECTURE (FOLLOWING DJAMENT, PUTMAN, SAM, AND SNOWDEN)

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ABSTRACT. This note provides a self-contained exposition of the proof of the artinian conjecture, following closely Djament's Bourbaki lecture. The original proof is due to Putman, Sam, and Snowden.

1. INTRODUCTION

This note provides a complete proof of the celebrated artinian conjecture. The proof is due to Putman, Sam, and Snowden [6, 7]. Here, we follow closely the elegant exposition of Djament in [3]. For the origin of the conjecture and its consequences, we refer to those papers and Djament's Bourbaki lecture [4]. In addition, the expository articles by Kuhn, Powell and Schwartz in [5] are recommended.

There are two main result. Fix a locally noetherian Grothendieck abelian category $\mathcal{A}$, for instance, the category of modules over a noetherian ring.

Theorem 1.1. *Let A be a ring whose underlying set is finite. For the category $\mathcal{P}(A)$ of free A-modules of finite rank, the functor category $\mathrm{Fun}(\mathcal{P}(A)^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

This result amounts to the assertion of the artinian conjecture when A is a finite field and $\mathcal{A}$ is the category of A-modules.

The first theorem is a direct consequence of the following.

Theorem 1.2. *For the category Γ of finite sets, the functor category $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

The basic idea for the proof is to formulate finiteness conditions on an essentially small category $\mathcal{C}$ such that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian. This leads to the notion of a Gröbner category. Such finiteness conditions have a 'direction'. For that reason we consider contravariant functors $\mathcal{C} \to \mathcal{A}$, because then the direction is preserved (via Yoneda's lemma) when one passes from $\mathcal{C}$ to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.

2. NOETHERIAN POSETS

Let $\mathcal{C}$ be a poset. A subset $\mathcal{D} \subseteq \mathcal{C}$ is a *sieve* if the conditions $x \leq y$ in $\mathcal{C}$ and $y \in \mathcal{D}$ imply $x \in \mathcal{D}$. The sieves in $\mathcal{C}$ are partially ordered by inclusion.

Definition 2.1. A poset $\mathcal{C}$ is called

(1) *noetherian* if every ascending chain of elements in $\mathcal{C}$ stabilises, and
(2) *strongly noetherian* if every ascending chain of sieves in $\mathcal{C}$ stabilises.

The paper is in a final form and no version of it will be submitted for publication elsewhere.

For a poset $\mathcal{C}$ and $x \in \mathcal{C}$, set $\mathcal{C}(x) = \{t \in \mathcal{C} \mid t \leq x\}$. The assignment $x \mapsto \mathcal{C}(x)$ yields an embedding of $\mathcal{C}$ into the poset of sieves in $\mathcal{C}$.

Lemma 2.2. *For a poset $\mathcal{C}$ the following are equivalent:*

(1) *The poset $\mathcal{C}$ is strongly noetherian.*
(2) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there exists $i \in \mathbb{N}$ such that $x_j \leq x_i$ for infinitely many $j \in \mathbb{N}$.*
(3) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there is a map $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\alpha(i) < \alpha(j)$ and $x_{\alpha(j)} \leq x_{\alpha(i)}$.*
(4) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there are $i < j$ in $\mathbb{N}$ such that $x_j \leq x_i$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathcal{C}$ is strongly noetherian and let $(x_i)_{i\in\mathbb{N}}$ be elements in $\mathcal{C}$. For $n \in \mathbb{N}$ set $\mathcal{C}_n = \bigcup_{i\leq n} \mathcal{C}(x_i)$. The chain $(\mathcal{C}_n)_{n\in\mathbb{N}}$ stabilises, say $\mathcal{C}_n = \mathcal{C}_N$ for all $n \geq N$. Thus there exists $i \leq N$ such that $x_j \leq x_i$ for infinitely many $i \in \mathbb{N}$.

(2) $\Rightarrow$ (3): Define $\alpha\colon \mathbb{N} \to \mathbb{N}$ recursively by taking for $\alpha(0)$ the smallest $i \in \mathbb{N}$ such that $x_j \leq x_i$ for infinitely many $j \in \mathbb{N}$. For $n > 0$ set

$$\alpha(n) = \min\{i > \alpha(n-1) \mid x_j \leq x_i \leq x_{\alpha(n-1)} \text{ for infinitely many } j \in \mathbb{N}\}.$$

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (1): Suppose there is a properly ascending chain $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of sieves in $\mathcal{C}$. Choose $x_n \in \mathcal{C}_{n+1} \setminus \mathcal{C}_n$ for each $n \in \mathbb{N}$. There are $i < j$ in $\mathbb{N}$ such that $x_j \leq x_i$. This implies $x_j \in \mathcal{C}_{i+1} \subseteq \mathcal{C}_j$ which is a contradiction. ☐

3. Functor categories

Let $\mathcal{C}$ be an essentially small category and $\mathcal{A}$ a Grothendieck abelian category. We denote by $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. The morphisms between two functors are the natural transformations. Note that $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is a Grothendieck abelian category.

Given an object $x \in \mathcal{C}$, the evaluation functor

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}) \longrightarrow \mathcal{A}, \quad F \mapsto F(x)$$

admits a left adjoint

$$\mathcal{A} \longrightarrow \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}), \quad M \mapsto M[\mathcal{C}(-,x)]$$

where for any set X we denote by $M[X]$ a coproduct of copies of M indexed by the elements of X. Thus we have a natural isomorphism

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})(M[\mathcal{C}(-,x)], F) \cong \mathcal{A}(M, F(x)). \tag{3.1}$$

Lemma 3.1. *If $(M_i)_{i\in I}$ is a set of generators of $\mathcal{A}$, then the functors $M_i[\mathcal{C}(-,x)]$ with $i \in I$ and $x \in \mathcal{C}$ generate $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.*

Proof. Use the adjointness isomorphism (3.1). ☐

A Grothendieck abelian category $\mathcal{A}$ is *locally noetherian* if $\mathcal{A}$ has a generating set of noetherian objects. In that case an object $M \in \mathcal{A}$ is noetherian iff M is *finitely presented* (that is, the representable functor $\mathcal{A}(M,-)$ preserves filtered colimits); see [8, Chap. V] for details.

Lemma 3.2. *Let $\mathcal{A}$ be locally noetherian. Then* $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian iff* $M[\mathcal{C}(-,x)]$ *is noetherian for every noetherian* $M \in \mathcal{A}$ *and* $x \in \mathcal{C}$.

Proof. First observe that $M[\mathcal{C}(-,x)]$ is finitely presented if M is finitely presented. This follows from the isomorphism (3.1) since evaluation at $x \in \mathcal{C}$ preserves colimits. Now the assertion of the lemma is an immediate consequence of Lemma 3.1. □

4. Noetherian functors

Let $\mathcal{C}$ be an essentially small category and fix an object $x \in \mathcal{C}$. Set

$$\mathcal{C}(x) = \coprod_{t \in \mathcal{C}} \mathcal{C}(t,x).$$

Given $f, g \in \mathcal{C}(x)$, let $\langle f \rangle$ denote the set of morphisms in $\mathcal{C}(x)$ that factor through f, and set $f \leq_x g$ if $\langle f \rangle \subseteq \langle g \rangle$. We identify f and g when $\langle f \rangle = \langle g \rangle$. This yields a poset which we denote by $\bar{\mathcal{C}}(x)$.

A functor is *noetherian* if every ascending chain of subfunctors stabilises.

Lemma 4.1. *The functor* $\mathcal{C}(-,x) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *is noetherian iff the poset* $\bar{\mathcal{C}}(x)$ *is strongly noetherian.*

Proof. Sending $F \subseteq \mathcal{C}(-,x)$ to $\bigcup_{t \in \mathcal{C}} F(t)$ induces an inclusion preserving bijection between the subfunctors of $\mathcal{C}(-,x)$ and the sieves in $\bar{\mathcal{C}}(x)$. □

For a poset $\mathcal{T}$ let $\mathrm{Set} \wr \mathcal{T}$ denote the category consisting of pairs (X, ξ) such that X is a set and $\xi \colon X \to \mathcal{T}$ is a map. A morphism $(X,\xi) \to (X', \xi')$ is a map $f \colon X \to X'$ such that $\xi(a) \leq \xi' f(a)$ for all $a \in X$.

A functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \wr \mathcal{T}$ is given by a pair (F, ϕ) consisting of a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and a map $\phi \colon \coprod_{t \in \mathcal{C}} F(t) \to \mathcal{T}$ such that $\phi(a) \leq \phi(F(f)(a))$ for every $a \in F(t)$ and $f \colon t' \to t$ in $\mathcal{C}$.

Lemma 4.2. *Let $\mathcal{T}$ be a noetherian poset. If $\mathcal{C}(-,x)$ is noetherian, then any functor* $(\mathcal{C}(-,x), \phi) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \wr \mathcal{T}$ *is noetherian.*

Proof. Let $(F_n, \phi_n)_{n \in \mathbb{N}}$ be a strictly ascending chain of subfunctors of (F, ϕ). The chain $(F_n)_{n \in \mathbb{N}}$ stabilises since $\mathcal{C}(-,x)$ is noetherian. Thus we may assume that $F_n = F$ for all $n \in \mathbb{N}$, and we find $f_n \in \coprod_{t \in \mathcal{C}} F(t)$ such that $\phi_n(f_n) < \phi_{n+1}(f_n)$. The poset $\bar{\mathcal{C}}(x)$ is strongly noetherian by Lemma 4.1. It follows from Lemma 2.2 that there is a map $\alpha \colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\alpha(i) < \alpha(j)$ and $f_{\alpha(j)} \leq_x f_{\alpha(i)}$. Thus

$$\phi_{\alpha(n)}(f_{\alpha(n)}) < \phi_{\alpha(n)+1}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n+1)}).$$

This yields a strictly ascending chain in $\mathcal{T}$, contradicting the assumption on $\mathcal{T}$. □

Definition 4.3. A partial order $\leq$ on $\mathcal{C}(x)$ is *admissible* if the following holds:

(1) The order $\leq$ restricted to $\mathcal{C}(t,x)$ is total and noetherian for every $t \in \mathcal{C}$.
(2) For $f, f' \in \mathcal{C}(t,x)$ and $e \in \mathcal{C}(s,t)$, the condition $f < f'$ implies $fe < f'e$.

Fix an admissible partial order $\leq$ on $\mathcal{C}(x)$ and an object M in a Grothendieck abelian category $\mathcal{A}$. Let $\mathrm{Sub}(M)$ denote the poset of subobjects of M and consider the functor

$$\mathcal{C}(-,x) \wr M \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \mathrm{Sub}(M), \quad t \mapsto \big(\mathcal{C}(t,x), (M)_{f \in \mathcal{C}(t,x)}\big).$$

For a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ define a subfunctor $\tilde{F} \subseteq \mathcal{C}(-,x) \wr M$ as follows:

$$\tilde{F}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \mathrm{Sub}(M), \quad t \mapsto \Big(\mathcal{C}(t,x), \big(\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t))\big)_{f\in\mathcal{C}(t,x)}\Big)$$

where $\mathcal{C}(t,x)_f = \{g \in \mathcal{C}(t,x) \mid f \leq g\}$ and $\pi_f\colon M[\mathcal{C}(t,x)_f] \to M$ is the projection onto the factor corresponding to f. For a morphism $e\colon t' \to t$ in $\mathcal{C}$, the morphism $\tilde{F}(e)$ is induced by precomposition with e. Note that

$$\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t)) \subseteq \pi_{fe}(M[\mathcal{C}(t',x)_{fe}] \cap F(t'))$$

since $\leq$ is compatible with the composition in $\mathcal{C}$.

Lemma 4.4. *Suppose there is an admissible partial order on $\mathcal{C}(x)$. Then the assignment which sends a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ to $\tilde{F}$ preserves proper inclusions. Therefore $M[\mathcal{C}(-,x)]$ is noetherian provided that $\mathcal{C}(-,x) \wr M$ is noetherian.*

Proof. Let $F \subseteq G \subseteq M[\mathcal{C}(-,x)]$. Then $\tilde{F} \subseteq \tilde{G}$. Now suppose that $F \neq G$. Thus there exists $t \in \mathcal{C}$ such that $F(t) \neq G(t)$. We have $\mathcal{C}(t,x) = \bigcup_{f\in\mathcal{C}(t,x)} \mathcal{C}(t,x)_f$, and this union is directed since $\leq$ is total. Thus

$$F(t) = \sum_{f\in\mathcal{C}(t,x)} \big(M[\mathcal{C}(t,x)_f] \cap F(t)\big)$$

since filtered colimits in $\mathcal{A}$ are exact. This yields f such that

$$M[\mathcal{C}(t,x)_f] \cap F(t) \neq M[\mathcal{C}(t,x)_f] \cap G(t).$$

Choose $f \in \mathcal{C}(t,x)$ maximal with respect to this property, using that $\leq$ is noetherian. Now observe that the projection π_f induces an exact sequence

$$0 \longrightarrow \sum_{f<g} \big(M[\mathcal{C}(t,x)_g] \cap F(t)\big) \longrightarrow F(t) \longrightarrow \pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \longrightarrow 0$$

since the kernel of π_f equals the directed union $\sum_{f<g} M[\mathcal{C}(t,x)_g]$. For the directedness one uses again that $\leq$ is total. Thus

$$\pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \neq \pi_f\big(M[\mathcal{C}(t,x)_f] \cap G(t)\big)$$

and therefore $\tilde{F} \neq \tilde{G}$. □

Proposition 4.5. *Let $x \in \mathcal{C}$. Suppose that $\mathcal{C}(-,x)$ is noetherian and that $\mathcal{C}(x)$ has an admissible partial order. If $M \in \mathcal{A}$ is noetherian, then $M[\mathcal{C}(-,x)]$ is noetherian.*

Proof. Combine Lemmas 4.2 and 4.4. □

5. Gröbner categories

Definition 5.1. An essentially small category $\mathcal{C}$ is a *Gröbner category* if the following holds:

(1) The functor $\mathcal{C}(-,x)$ is noetherian for every $x \in \mathcal{C}$.
(2) There is an admissible partial order on $\mathcal{C}(x)$ for every $x \in \mathcal{C}$.

Theorem 5.2. *Let $\mathcal{C}$ be a Gröbner category and $\mathcal{A}$ a Grothendieck abelian category. If $\mathcal{A}$ is locally noetherian, then $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

Proof. Combine Lemma 3.1 and Proposition 4.5. ☐

Example 5.3. (1) A strongly noetherian poset (viewed as a category) is a Gröbner category.

(2) The additive monoid $\mathbb{N}$ of natural numbers (viewed as a category with a single object) is a Gröbner category. Let $\mathcal{A}$ be the module category of a noetherian ring A. Then $\operatorname{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{A})$ equals the module category of the polynomial ring in one variable over A. Thus Theorem 5.2 generalises Hilbert's Basis Theorem.

6. Base change

Given functors $F, G\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, we write $F \rightsquigarrow G$ if there is a finite chain

$$F = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = G$$

of epimorphisms and monomorphisms of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

Definition 6.1. A functor $\phi\colon \mathcal{C} \to \mathcal{D}$ is *contravariantly finite*[1] if the following holds:

(1) Every object $y \in \mathcal{D}$ is isomorphic to $\phi(x)$ for some $x \in \mathcal{C}$.
(2) For every object $y \in \mathcal{D}$ there are objects $x_1, \dots, x_n$ in $\mathcal{C}$ such that

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) \rightsquigarrow \mathcal{D}(\phi -, y).$$

The functor ϕ is *covariantly finite* if $\phi^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is contravariantly finite.

Note that a composite of contravariantly finite functors is contravariantly finite.

Lemma 6.2. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a Grothendieck abelian category. Fix $M \in \mathcal{A}$ and suppose that $M[\mathcal{C}(-, x)]$ is noetherian for all $x \in \mathcal{C}$. Then $M[\mathcal{D}(-, y)]$ is noetherian for all $y \in \mathcal{D}$.*

Proof. A finite chain

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = \mathcal{D}(\phi -, y)$$

of epimorphisms and monomorphisms induces a chain

$$\prod_{i=1}^{n} M[\mathcal{C}(-, x_i)] = \bar{F}_0 \twoheadrightarrow \bar{F}_1 \leftarrowtail \bar{F}_2 \twoheadrightarrow \cdots \twoheadrightarrow \bar{F}_{n-1} \leftarrowtail \bar{F}_n = M[\mathcal{D}(\phi -, y)]$$

of epimorphisms and monomorphisms in $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. Thus $M[\mathcal{D}(\phi -, y)]$ is noetherian. It follows that $M[\mathcal{D}(-, y)]$ is noetherian, since precomposition with ϕ yields a faithful and exact functor $\operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A}) \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. ☐

Proposition 6.3. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a locally noetherian Grothendieck abelian category. If the category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian, then $\operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A})$ is locally noetherian.*

Proof. Combine Lemmas 3.2 and 6.2. ☐

[1] The terminology follows that introduced by Auslander and Smalø [1] for an inclusion functor.

7. Categories of finite sets

Let Γ denote the category of finite sets (a skeleton is given by the sets $\mathbf{n} = \{1, 2, \dots, n\}$). The subcategory of finite sets with surjective morphisms is denoted by Γ_{sur}. A surjection $f\colon \mathbf{m} \to \mathbf{n}$ is *ordered* if $i < j$ implies $\min f^{-1}(i) < \min f^{-1}(j)$. We write Γ_{os} for the subcategory of finite sets whose morphisms are ordered surjections. Given a surjection $f\colon \mathbf{m} \to \mathbf{n}$, let $f^!\colon \mathbf{n} \to \mathbf{m}$ denote the map given by $f^!(i) = \min f^{-1}(i)$. Note that $ff^! = \mathrm{id}$, and $gf = f^!g^!$ provided that f and g are ordered surjections.

Lemma 7.1. (1) *The inclusion $\Gamma_{\mathrm{sur}} \to \Gamma$ is contravariantly finite.*
(2) *The inclusion $\Gamma_{\mathrm{os}} \to \Gamma_{\mathrm{sur}}$ is contravariantly finite.*

Proof. (1) For each integer $n \geq 0$ there is an isomorphism

$$\coprod_{\mathbf{m}\hookrightarrow\mathbf{n}} \Gamma_{\mathrm{sur}}(-, \mathbf{m}) \xrightarrow{\sim} \Gamma(-, \mathbf{n})$$

which is induced by the injective maps $\mathbf{m} \to \mathbf{n}$.

(2) For each integer $n \geq 0$ there is an isomorphism

$$\Gamma_{\mathrm{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \xrightarrow{\sim} \Gamma_{\mathrm{sur}}(-, \mathbf{n})$$

which sends a pair (f, σ) to σf. The inverse sends a surjective map $g\colon \mathbf{m} \to \mathbf{n}$ to $(\tau^{-1}g, \tau)$ where $\tau \in \mathfrak{S}_n$ is the unique permutation such that $g^!\tau$ is increasing. □

Fix an integer $n \geq 0$. Given $f, g \in \Gamma(\mathbf{n})$ we set $f \leq g$ if there exists an ordered surjection h such that $f = gh$.

Lemma 7.2. *The poset $(\Gamma(\mathbf{n}), \leq)$ is strongly noetherian.*

Proof. We fix some notation for each $f \in \Gamma(\mathbf{m}, \mathbf{n})$. Set $\lambda(f) = m$. If f is not injective, set

$$\mu(f) = m - \max\{i \in \mathbf{m} \mid \text{there exists } j < i \text{ such that } f(i) = f(j)\}$$

and $\pi(f) = f(m - \mu(f))$. Define $\tilde{f} \in \Gamma(\mathbf{m-1}, \mathbf{n})$ by setting $\tilde{f}(i) = f(i)$ for $i < m - \mu(f)$ and $\tilde{f}(i) = f(i+1)$ otherwise.

Note that $f \leq \tilde{f}$. Moreover, $\mu(f) = \mu(g)$, $\pi(f) = \pi(g)$, and $\tilde{f} \leq \tilde{g}$ imply $f \leq g$.

Suppose that $(\Gamma(\mathbf{n}), \leq)$ is not strongly noetherian. Then there exists an infinite sequence $(f_r)_{r\in\mathbb{N}}$ in $\Gamma(\mathbf{n})$ such that $i < j$ implies $f_j \not\leq f_i$; see Lemma 2.2. Call such a sequence *bad*. Choose the sequence *minimal* in the sense that $\lambda(f_i)$ is minimal for all bad sequences $(g_r)_{r\in\mathbb{N}}$ with $g_j = f_j$ for all $j < i$. There is an infinite subsequence $(f_{\alpha(r)})_{r\in\mathbb{N}}$ (given by some increasing map $\alpha\colon \mathbb{N} \to \mathbb{N}$) such that μ and π agree on all $f_{\alpha(r)}$, since the values of μ and π are bounded by n. Now consider the sequence $f_0, f_1, \dots, f_{\alpha(0)-1}, \tilde{f}_{\alpha(0)}, \tilde{f}_{\alpha(1)}, \dots$ and denote this by $(g_r)_{r\in\mathbb{N}}$. This sequence is not bad, since $(f_r)_{r\in\mathbb{N}}$ is minimal. Thus there are $i < j$ in $\mathbb{N}$ with $g_j \leq g_i$. Clearly, $j < \alpha(0)$ is impossible. If $i < \alpha(0)$, then

$$f_{\alpha(j-\alpha(0))} \leq \tilde{f}_{\alpha(j-\alpha(0))} = g_j \leq g_i = f_i,$$

which is a contradiction, since $i < \alpha(0) \leq \alpha(j - \alpha(0))$. If $i \geq \alpha(0)$, then $f_{\alpha(j-\alpha(0))} \leq f_{\alpha(i-\alpha(0))}$; this is a contradiction again. Thus $(\Gamma(\mathbf{n}), \leq)$ is strongly noetherian. □

Proposition 7.3. *The category Γ_{os} is a Gröbner category.*

Proof. Fix an integer $n \geq 0$. The poset $\bar{\Gamma}_{\mathrm{os}}(\mathbf{n})$ is strongly noetherian by Lemma 7.2, and it follows from Lemma 4.1 that the functor $\Gamma_{\mathrm{os}}(-, \mathbf{n})$ is noetherian.

The admissible partial order on $\Gamma_{\mathrm{os}}(\mathbf{n})$ is given by the lexicographic order. Thus for $f, g \in \Gamma_{\mathrm{os}}(\mathbf{m}, \mathbf{n})$, we have $f < g$ if there exists $j \in \mathbf{m}$ with $f(j) < g(j)$ and $f(i) = g(i)$ for all $i < j$. □

Theorem 7.4. *Let $\mathcal{A}$ be a locally noetherian Grothendieck abelian category. Then the category* $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof. The category Γ_{os} is a Gröbner category by Proposition 7.3. It follows from Theorem 5.2 that $\mathrm{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian. The inclusion $\Gamma_{\mathrm{os}} \to \Gamma$ is contravariantly finite by Lemma 7.1. Thus $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 6.3. □

8. The artinian conjecture

Let A be a ring. We denote by $\mathcal{P}(A)$ the category of free A-modules of finite rank. If A is finite, then the functor $\Gamma \to \mathcal{P}(A)$ sending X to $A[X]$ is a left adjoint of the forgetful functor $\mathcal{P}(A) \to \Gamma$.

Lemma 8.1. *Let A be finite. Then the functor $\Gamma \to \mathcal{P}(A)$ is contravariantly finite.*

Proof. The assertion follows from the adjointness isomorphism

$$\mathcal{P}(A)(A[X], P) \cong \Gamma(X, P).$$

□

Theorem 8.2. *Let A be a finite ring and $\mathcal{A}$ a locally noetherian Grothendieck abelian category. Then the category* $\mathrm{Fun}(\mathcal{P}(A)^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof. Combine Theorem 7.4 with Lemma 8.1 and Proposition 6.3. □

9. FI-modules

The proof of the artinian conjecture yields an alternative proof of the following result due to Church, Ellenberg, Farb, and Nagpal.

Let Γ_{inj} denote the category whose objects are finite sets and whose morphisms are injective maps.

Theorem 9.1 ([2, Theorem A])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck abelian category. Then the category* $\mathrm{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ *is locally noetherian.*

Proof. The following argument has been suggested by Kai-Uwe Bux. Consider the functor $\phi\colon \Gamma_{\mathrm{os}} \to (\Gamma_{\mathrm{inj}})^{\mathrm{op}}$ which is the identity on objects and takes a map $f\colon \mathbf{m} \to \mathbf{n}$ to $f^! \colon \mathbf{n} \to \mathbf{m}$ given by $f^!(i) = \min f^{-1}(i)$. This functor is contravariantly finite, since for each integer $n \geq 0$ the morphism

$$\Gamma_{\mathrm{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \longrightarrow \Gamma_{\mathrm{inj}}(\mathbf{n}, \phi -)$$

which sends a pair (f, σ) to $f^!\sigma$ is an epimorphism.

It follows from Proposition 6.3 that the category $\mathrm{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ is locally noetherian, since $\mathrm{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 7.3 and Theorem 5.2. □

Note added in proof

After completing this paper I found that Theorem 5.2 is precisely the statement of Theorem 3.1 in [G. Richter, Noetherian semigroup rings with several objects, in *Group and semigroup rings (Johannesburg, 1985)*, 231–246, North-Holland Math. Stud., 126, North-Holland, Amsterdam, 1986].

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HALF EXACT FUNCTORS ASSOCIATED WITH GENERAL HEARTS ON EXACT CATEGORIES

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Abstract. We construct a half exact functor from the exact category to the heart of a cotorsion pair. This is analog of the construction of Abe and Nakaoka for triangulated categories. When the cotorsion pair comes from a cluster tilting subcategory, our half exact functor coincides with the canonical quotient functor from the exact category to the quotient category of it by this cluster tilting subcategory. We will also use this half exact functor to find out the relationship between different hearts.

Key Words: exact category, abelian category, cotorsion pair, heart, half exact functor.

1. Introduction

Cotorsion pairs play an important role in representation theory (see [2] and see [3] for more examples). In [4], we define hearts $\underline{\mathcal{H}}$ of cotorsion pairs $(\mathcal{U},\mathcal{V})$ on exact categories $\mathcal{B}$ and proved that they are abelian. This is similar as Nakaoka's result on triangulated categories [5]. It is natural to ask whether we can find any relationship between the hearts and the original exact categories. Abe and Nakaoka have already given an answer by constructing a cohomological functor in the case of triangulated categories [1]. In this paper we will construct an associated half exact functor H from the exact category $\mathcal{B}$ to the heart $\underline{\mathcal{H}}$, which is similar as the construction of Abe and Nakaoka.

Let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the full subcategory of projectives (resp. injectives) of $\mathcal{B}$.

We recall the definition of a cotorsion pair on $\mathcal{B}$ [4, Definition 2.3]:

Definition 1. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{B}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V})=0$.

(b) For any object $B\in\mathcal{B}$, there exits two short exact sequences

$$V_B \rightarrowtail U_B \twoheadrightarrow B, \quad B \rightarrowtail V^B \twoheadrightarrow U^B$$

satisfying $U_B, U^B\in\mathcal{U}$ and $V_B, V^B\in\mathcal{V}$.

For any cotorsion pairs $(\mathcal{U},\mathcal{V})$, let $\mathcal{W}:=\mathcal{U}\cap\mathcal{V}$. We denote the quotient of $\mathcal{B}$ by $\mathcal{W}$ as $\underline{\mathcal{B}}:=\mathcal{B}/\mathcal{W}$. For any morphism $f\in\mathrm{Hom}_{\mathcal{B}}(X,Y)$, we denote its image in $\mathrm{Hom}_{\underline{\mathcal{B}}}(X,Y)$ by $\underline{f}$. For any subcategory $\mathcal{C}\supseteq\mathcal{W}$ of $\mathcal{B}$, we denote by $\underline{\mathcal{C}}$ the full subcategory of $\underline{\mathcal{B}}$ consisting of the same objects as $\mathcal{C}$. Let

$$\mathcal{B}^+ := \{B\in\mathcal{B} \mid U_B\in\mathcal{W}\}, \quad \mathcal{B}^- := \{B\in\mathcal{B} \mid V^B\in\mathcal{W}\}.$$

The detailed version of this paper will be submitted for publication elsewhere.

Let

$$\mathcal{H} := \mathcal{B}^+ \cap \mathcal{B}^-.$$

Since $\mathcal{H} \supseteq \mathcal{W}$, we have an additive subcategory $\underline{\mathcal{H}}$ which we call the *heart* of cotorsion pair $(\mathcal{U}, \mathcal{V})$.

Definition 2. A covariant functor F from $\mathcal{B}$ to an abelian category $\mathcal{A}$ is called *half exact* if for any short exact sequence

$$A \rightarrowtail^{f} B \xrightarrow{g} \twoheadrightarrow C$$

in $\mathcal{B}$, the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact in $\mathcal{A}$.

We will prove the following theorem.

Theorem 3. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$, there exists an associated half exact functor*

$$H : \mathcal{B} \to \underline{\mathcal{H}}.$$

The half exact functor we construct gives us a way to find out the relationship between different hearts. Let $k \in \{1, 2\}$, $(\mathcal{U}_k, V_k)$ be a cotorsion pair on $\mathcal{B}$ and $\mathcal{W}_k = \mathcal{U}_k \cap \mathcal{V}_k$. Let $\mathcal{H}_k/\mathcal{W}_k$ be the heart of $(\mathcal{U}_k, V_k)$ and H_k be the associated half exact functor. For $i, j \in \{1, 2\}$ and $i \neq j$, if $H_i(\mathcal{W}_j) = 0$, then H_i induces a functor $\beta_{ji} : \mathcal{H}_j/\mathcal{W}_j \to \mathcal{H}_i/\mathcal{W}_i$. Moreover, we have the following theorem.

Theorem 4. *If $H_j(\mathcal{U}_i) = H_j(\mathcal{V}_i) = 0$ and $H_i(\mathcal{U}_j) = H_i(\mathcal{V}_j) = 0$, then we have an equivalence $\mathcal{H}_i/\mathcal{W}_i \simeq \mathcal{H}_j/\mathcal{W}_j$ between two hearts. More precisely, we have natural isomorphisms $\beta_{ij}\beta_{ji} \simeq \mathrm{id}_{\mathcal{H}_j/\mathcal{W}_j}$ and $\beta_{ji}\beta_{ij} \simeq \mathrm{id}_{\mathcal{H}_i/\mathcal{W}_i}$ of functors.*

2. Notations

For briefly review of the important properties of exact categories, we refer to [4, §2].

Throughout this paper, let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the full subcategory of projectives (resp. injectives) of $\mathcal{B}$.

Definition 5. For any $B \in \mathcal{B}$, we define B^+ as follows:
Take two short exact sequences:

$$V_B \rightarrowtail U_B \xrightarrow{u_B} \twoheadrightarrow B\ , \quad U_B \rightarrowtail^{w'} W^0 \twoheadrightarrow U^0$$

where $U_B, U^0 \in \mathcal{U}$, W^0,$\mathcal{V}_B \in \mathcal{V}$. In fact, $W^0 \in \mathcal{W}$ since $\mathcal{U}$ is closed under extension. We get the following commutative diagram

$$\begin{tikzcd}
V_B \arrow[r, tail] \arrow[d, equal] & U_B \arrow[r, two heads, "u_B"] \arrow[d, tail, "w'"'] & B \arrow[d, tail, "\alpha_B"] \\
V_B \arrow[r, tail] & W^0 \arrow[r, two heads, "w"'] \arrow[d, two heads] & B^+ \arrow[d, two heads] \\
 & U^0 \arrow[r, equal] & U^0
\end{tikzcd}$$

where the upper-right square is both a push-out and a pull-back.

By [4, Lemma 3.2], $B^+ \in \mathcal{B}^+$, and if $B \in \mathcal{B}^-$, then $B^+ \in \mathcal{H}$.

Proposition 6. [4, Proposition 3.3] *For any $B \in \mathcal{B}$ and $Y \in \mathcal{B}^+$,* $\mathrm{Hom}_{\mathcal{B}}(\alpha_B, Y) : \mathrm{Hom}_{\mathcal{B}}(B^+, Y) \to \mathrm{Hom}_{\mathcal{B}}(B, Y)$ *is surjective and* $\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{\alpha_B}, Y) : \mathrm{Hom}_{\underline{\mathcal{B}}}(B^+, Y) \to \mathrm{Hom}_{\underline{\mathcal{B}}}(B, Y)$ *is bijective.*

We define a functor σ^+ from $\underline{\mathcal{B}}$ to $\underline{\mathcal{B}}^+$ as follows:
For any object $B \in \mathcal{B}$, since all the $B^{+'}s$ are isomorphic to each other in $\underline{\mathcal{B}}$ by Proposition 6, we fix a B^+ for B. Let

$$\begin{aligned} \sigma^+ : \underline{\mathcal{B}} &\to \underline{\mathcal{B}}^+ \\ B &\mapsto B^+ \end{aligned}$$

and for any morphism $\underline{f} : B \to C$, we define $\sigma^+(\underline{f})$ as the unique morphism given by Proposition 6

$$\begin{tikzcd}
B \arrow[r, "\underline{f}"] \arrow[d, "\underline{\alpha_B}"'] & C \arrow[d, "\underline{\alpha_C}"] \\
B^+ \arrow[r, dotted, "\sigma^+(\underline{f})"'] & C^+.
\end{tikzcd}$$

Dually, we can define σ^-.
Let $\pi : \mathcal{B} \to \underline{\mathcal{B}}$ be the canonical functor. We denote $\sigma^- \circ \sigma^+ \circ \pi$ by

$$H : \mathcal{B} \to \underline{\mathcal{H}}.$$

3. Main results

Proposition 7. *The functor H has the following properties:*

(a) *For any objects A and B in $\mathcal{B}$, $H(A \oplus B) \simeq H(A) \oplus H(B)$ in $\underline{\mathcal{H}}$.*
(b) $H|_{\mathcal{H}} = \pi|_{\mathcal{H}}$.
(c) *$H(\mathcal{U}) = 0$ and $H(\mathcal{V}) = 0$. In particular, $H(\mathcal{P}) = 0$ and $H(\mathcal{I}) = 0$.*

Theorem 8. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{B}$, the functor*

$$H : \mathcal{B} \to \underline{\mathcal{H}}$$

is half exact. We call H the associated half exact functor to $(\mathcal{U}, \mathcal{V})$.

We have the following general property of half exact functors which H satisfies.

Proposition 9. *Let $\mathcal{A}$ be an abelian category and $F : \mathcal{B} \to \mathcal{A}$ be a half exact functor satisfying $F(\mathcal{P}) = 0$ and $F(\mathcal{I}) = 0$. Then for any short exact sequence*

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

in $\mathcal{B}$, there exist morphisms $h : C \to \Omega^{-}A$ and $h' : \Omega C \to A$ such that the following sequence

$$\cdots \xrightarrow{F(\Omega h')} F(\Omega A) \xrightarrow{F(\Omega f)} F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(h')} F(A) \xrightarrow{F(f)} F(B)$$
$$\xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Omega^{-}A) \xrightarrow{F(\Omega^{-} f)} F(\Omega^{-}B) \xrightarrow{F(\Omega^{-} g)} F(\Omega^{-}C) \xrightarrow{F(\Omega^{-} h)} \cdots$$

is exact in $\mathcal{A}$.

Let $i \in \{1, 2\}$. Let $(\mathcal{U}_i, \mathcal{V}_i)$ be a cotorsion pair on $\mathcal{B}$ and $\mathcal{W}_i = \mathcal{U}_i \cap \mathcal{V}_i$.

(a) $\mathcal{B}_i^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$V_B \rightarrowtail U_B \twoheadrightarrow B$$

where $U_B \in \mathcal{W}_i$ and $V_B \in \mathcal{V}_i$.

(b) $\mathcal{B}_i^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$B \rightarrowtail V^B \twoheadrightarrow U^B$$

where $V^B \in \mathcal{W}_i$ and $U^B \in \mathcal{U}_i$.

Denote

$$\mathcal{H}_i := \mathcal{B}_i^+ \cap \mathcal{B}_i^-.$$

Then $\mathcal{H}_i/\mathcal{W}_i$ is the heart of $(\mathcal{U}_i, \mathcal{V}_i)$. Let $\pi_i : \mathcal{B} \to \mathcal{B}/\mathcal{W}_i$ be the canonical functor and $\iota_i : \mathcal{H}_i/\mathcal{W}_i \hookrightarrow \mathcal{B}/\mathcal{W}_i$ be the inclusion functor.

If $H_2(\mathcal{W}_1) = 0$, then there exists a functor $h_{12} : \mathcal{B}/\mathcal{W}_1 \to \mathcal{H}_2/\mathcal{W}_2$ such that $H_2 = h_{12}\pi_1$.

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\pi_1"] \arrow[dr, "H_2"'] & & \mathcal{B}/\mathcal{W}_1 \arrow[dl, dotted, "h_{12}"] \\ & \mathcal{H}_2/\mathcal{W}_2 & \end{tikzcd}$$

Hence we get a functor $\beta_{12} := h_{12}\iota_1 : \mathcal{H}_1/\mathcal{W}_1 \to \mathcal{H}_2/\mathcal{W}_2$.

Proposition 10. *Let $(\mathcal{U}_1, \mathcal{V}_1)$, $(\mathcal{U}_2, \mathcal{V}_2)$ be cotorsion pairs on $\mathcal{B}$. If $H_2(\mathcal{W}_1) = 0$ and $H_1(\mathcal{U}_2) = 0 = H_1(\mathcal{V}_2)$, then we have a natural isomorphism $\beta_{21}\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$ of functors.*

Moreover, we have the following theorem.

Theorem 11. *If $H_1(\mathcal{U}_2) = H_1(\mathcal{V}_2) = 0$ and $H_2(\mathcal{U}_1) = H_2(\mathcal{V}_1) = 0$, then we have an equivalence $\mathcal{H}_1/\mathcal{W}_1 \simeq \mathcal{H}_2/\mathcal{W}_2$ between two hearts. More precisely, we have natural isomorphisms $\beta_{12}\beta_{21} \simeq \mathrm{id}_{\mathcal{H}_2/\mathcal{W}_2}$ and $\beta_{21}\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$ of functors.*

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SINGULARITY CATEGORIES OF STABLE RESOLVING SUBCATEGORIES

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Abstract. In this article[1] we study resolving subcategories $\mathcal{X}$ of an abelian category from the structure of their associated triangulated categories. More precisely, we investigate the singularity categories

$$\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\underline{\mathcal{X}})/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\,\underline{\mathcal{X}}))$$

of the stable categories $\underline{\mathcal{X}}$ of $\mathcal{X}$. We consider when the stable categories of two resolving subcategories have triangle equivalent singularity categories. Applying this to resolving subcategories of modules over Gorenstein rings, we characterize simple hypersurface singularities of type (A_1) as complete intersections over which the stable categories of resolving subcategories have trivial singularity categories.

1. introduction

Let R be a noetherian ring. The *singularity category* of R is by definition the Verdier quotient

$$\mathsf{D}_{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,R)/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\,R)),$$

where $\mathsf{mod}\,R$ denotes the category of finitely generated R-modules, $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,R)$ the bounded derived category and $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\,R))$ the bounded homotopy category. The singularity category $\mathsf{D}_{\mathsf{sg}}(R)$ is a triangulated category, which has been introduced by Buchweitz [4] by the name of *stable derived category* and connected to the Homological Mirror Symmetry Conjecture by Orlov [10]. A lot of studies on singularity categories have been done in recent years; see [5, 8, 11, 15] for instance.

In this article, we consider the singularity category of a stable resolving category. Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$, and $\underline{\mathcal{X}}$ its stable category. Then the category $\mathsf{mod}\,\underline{\mathcal{X}}$ of finitely presented right $\underline{\mathcal{X}}$-modules is an abelian category with enough projective objects [1]. We take the Verdier quotient of

$$\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) := \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\underline{\mathcal{X}})/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\,\underline{\mathcal{X}})),$$

and call this the *singularity category* of $\underline{\mathcal{X}}$. For two resolving subcategories X, Y we say that $\underline{\mathcal{X}}$, $\underline{\mathcal{Y}}$ are *singulary equivalent* if there is a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) \cong \mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{Y}})$.

The main purpose of this article is to study the following question.

Question 1. Let $\mathcal{A}$ be an abelian category with enough projective objects. Let X, Y be resolving subcategories of $\mathcal{A}$. When are the stable categories $\underline{\mathcal{X}}$, $\underline{\mathcal{Y}}$ singularly equivalent?

[1]The detailed version of this article will be submitted for publication elsewhere.

We give a sufficient condition for two stable resolving subcategories to be singularly equivalent. We also apply it to resolving subcategories of module categories of commutative Gorenstein rings, and characterize the simple hypersurface singularities of type (A_1) in terms of singular equivalence classes.

2. Preliminaries

In this section, we introduce the several notions. Throughout this article, let $\mathcal{A}$ be an abelian category with enough projective objects, and denote by $\mathsf{proj}\,\mathcal{A}$ the full subcategory of projective objects of $\mathcal{A}$.

Definition 2. An object M of $\mathcal{A}$ is said to be *Cohen-Macaulay* if there is an exact sequence

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} \cdots$$

of projectives whose dual by any projective is also exact, such that M is isomorphic to the image of d_0. Denote by $\mathsf{CM}(\mathcal{A})$ the subcategory of $\mathcal{A}$ consisting of Cohen-Macaulay objects and by $\mathsf{CM}_n(\mathcal{A})$ the subcategory of $\mathcal{A}$ consisting objects whose n-th syzygies are Cohen-Macaulay.

In [7], a Cohen-Macaulay object is called a Gorenstein projective object. The category consisting of Cohen-Macaulay objects is a Frobenius category, hence its stable category is a triangulated category.

Next, we recall the definition of the category of finitely presented modules over an additive category.

Definition 3. Let $\mathcal{C}$ be an additive category. Denote by $\mathsf{Mod}\,\mathcal{C}$ the *functor category* of $\mathcal{C}$, that is, the objects are additive contravariant functors from $\mathcal{C}$ to the category $\mathbf{Ab}$ of abelian groups, and the morphisms are natural transformations. An object and a morphism of $\mathsf{Mod}\,\mathcal{C}$ are called a *(right)* $\mathcal{C}$*-module* and a $\mathcal{C}$*-homomorphism*, respectively. A $\mathcal{C}$-module F is said to be *finitely presented* if there is an exact sequence

$$\mathsf{Hom}_{\mathcal{C}}(-, X) \to \mathsf{Hom}_{\mathcal{C}}(-, Y) \to F \to 0$$

in the abelian category $\mathsf{Mod}\,\mathcal{C}$ with $X, Y \in \mathcal{C}$. The full subcategory of $\mathsf{Mod}\,\mathcal{C}$ consisting of finitely presented $\mathcal{C}$-modules is denoted by $\mathsf{mod}\,\mathcal{C}$.

Definition 4. An additive category $\mathcal{C}$ is called *Gorenstein* of dimension at most n if $\Omega^n(\mathsf{mod}\,\mathcal{C}) = \mathsf{CM}(\mathsf{mod}\,\mathcal{C})$.

Example 5. Let Λ be a Gorenstein ring of selfinjective dimension at most n, and denote by $\mathsf{proj}\,\Lambda$ the category of finitely generated Λ-modules. Then $\mathsf{proj}\,\Lambda$ is Gorenstein of dimension at most n.

We introduce the main target in this article.

Definition 6. Let $\mathcal{C}$ be an additive category. The *singularity category* $\mathcal{C}$ is defined as follows:

$$\mathsf{D}_{\mathsf{sg}}(\mathcal{C}) = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathcal{C})/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}(\mathsf{mod}\,\mathcal{C})).$$

Definition 7. Additive categories $\mathcal{C}, \mathcal{C}'$ are *singularly equivalent* if there is a triangule equivalence $\mathsf{D}_{\mathsf{sg}}(\mathcal{C}) \cong \mathsf{D}_{\mathsf{sg}}(\mathcal{C}')$, and then denote this by $\mathcal{C} \overset{sg}{\sim} \mathcal{C}'$.

Let us give the definition of a resolving subcategory, which is mainly studied in this article.

Definition 8. A full subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is *resolving* if:

(1) $\mathcal{X}$ contains all projective objects of $\mathcal{A}$.
(2) $\mathcal{X}$ is closed under direct summands, extensions and syzygies.

Here we recall the definition of a stable category.

Definition 9. Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ containing $\mathsf{proj}\,\mathcal{A}$. Then the quotient category

$$\underline{\mathcal{X}} := \mathcal{X}/\operatorname{\mathsf{proj}}\mathcal{A}$$

is called the *stable category* of $\mathcal{X}$; the objects of $\underline{\mathcal{X}}$ are the same as those of $\mathcal{X}$, and the hom-set $\mathsf{Hom}_{\underline{\mathcal{X}}}(M,N)$ of $M,N \in \underline{\mathcal{X}}$ is defined as follows:

$$\mathsf{Hom}_{\underline{\mathcal{X}}}(M,N) := \mathsf{Hom}_{\mathcal{A}}(M,N)/\mathsf{P}_{\mathcal{A}}(M,N),$$

where $\mathsf{P}_{\mathcal{A}}(M,N)$ consists of all morphisms from M to N that factor through objects in $\mathsf{proj}\,\mathcal{A}$.

Finally, we recall a structure result due to Auslander and Reiten on finitely presented modules over the stable category of a resolving subcategory.

Theorem 10. [1] *If $\mathcal{X}$ is a resolving subcategory of $\mathcal{A}$, then the category $\mathsf{mod}\,\underline{\mathcal{X}}$ of finitely presented right $\underline{\mathcal{X}}$-modules is an abelian category with enough projectives.*

3. Singularity categories and singularly equivalent

In this section, we give a sufficient condition for two resolving subcategories to be singularly equivalent. In particular, there is a natural asking when a resolving subcategory is singularly equivalent to $\mathbf{0}$. We give an answer to this question.

The following result is the key to study singular equivalence.

Theorem 11. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$ such that $\Omega^{-1}\Omega^n\mathcal{X} \subset \Omega^n\mathcal{X} \subset \mathsf{CM}(\mathcal{A})$. Then:*

(1) *$\underline{\mathcal{X}}$ is Gorenstein of dimension at most $3n$.*
(2) *There is a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}})$.*

This theorem gives some characterizations of a singularity category.

Corollary 12. *For each $n \geq 0$ there is a triangle equivalence*

$$\mathsf{D}_{\mathsf{sg}}(\underline{\mathsf{CM}}_n(\mathcal{A})) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathsf{CM}}_n(\mathcal{A})).$$

Corollary 13. *Let R be a local complete intersection. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$. Then there is a triangle equivalence*

$$\mathsf{D}_{\mathsf{sg}}(\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}}).$$

Let $n = 0$ in Theorem 11. Then the following result holds, whose assertion is nothing but [14].

Corollary 14. *Let $\mathcal{X}$ be a resolving subcateory of $\mathcal{A}$ contained in $\mathsf{CM}(\mathcal{A})$ and closed under cosyzygies. Then $\mathsf{mod}\,\underline{\mathcal{X}} = \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}})$, and hence $\mathsf{mod}\,\underline{\mathcal{X}}$ is a Frobenius category.*

Taking advantage of Theorem 11, we obtain a sufficient condition for singular equivalence.

Theorem 15. *Let $\mathcal{X}, \mathcal{Y}$ be resolving subcategories of $\mathcal{A}$ such that $\Omega^n\mathcal{X} \cup \Omega^{-1}\mathcal{Y} \subseteq \mathcal{Y} \subseteq \mathcal{X} \cap \mathsf{CM}(\mathcal{A})$ for some $n \geq 0$. Then there are triangle equivalences*

$$\mathsf{D_{sg}}(\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}}) \cong \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{Y}}) \cong \mathsf{D_{sg}}(\underline{\mathcal{Y}}).$$

Hence $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ are singularly equivalent.

Sketch of proof. The restriction $F \mapsto F|_{\underline{\mathcal{Y}}}$ makes a covariant exact functor

$$\Phi : \mathsf{Mod}\,\underline{\mathcal{X}} \to \mathsf{Mod}\,\underline{\mathcal{Y}}$$

of abelian categories. This induces an equivalent functor

$$\phi : \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{X}}) \to \underline{\mathsf{CM}}(\mathsf{mod}\,\underline{\mathcal{Y}}).$$

of triangulated categories. ■

Corollary 16. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$ with $\Omega^n\mathcal{X} \subseteq \mathsf{CM}(\mathcal{A}) \subseteq \mathcal{X}$ for some $n \geq 0$. Then $\underline{\mathcal{X}}$ and $\underline{\mathsf{CM}}(\mathcal{A})$ are singularly equivalent. In particular, $\underline{\mathsf{CM}}_p(\mathcal{A})$ and $\underline{\mathsf{CM}}_q(\mathcal{A})$ are singularly equivalent for all $p, q \geq 0$.*

Remark 17. A singular equivalence between $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ does not necessarily imply that $\mathcal{X}$, $\mathcal{Y}$ have an inclusion relation. Indeed, let $(R, \mathfrak{m})$ be a Gorenstein local domain of dimension at least 2. Set

$$\mathcal{X} = \{M \in \mathsf{mod}\,R \mid \mathfrak{m} \notin \mathsf{Ass}M\},$$
$$\mathcal{Y} = \{M \in \mathsf{mod}\,R \mid \mathsf{Ass}M \subseteq \{0, \mathfrak{m}\}\}.$$

These are resolving subcategories of $\mathsf{mod}\,R$ containing $\mathsf{CM}(R)$. Hence $\underline{\mathcal{X}} \overset{sg}{\approx} \mathsf{CM}(R) \overset{sg}{\approx} \underline{\mathcal{Y}}$. However, $\mathcal{X}$ and $\mathcal{Y}$ have no inclusion relation.

In the proof of our last theorem, the following two lemmas are necessary.

Lemma 18. *Let R be a Gorenstein complete local ring. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{CM}(R)$ and closed under cosyzygies. Assume that there exists a nonsplit exact sequence*

$$\sigma : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of R-modules with $X, Y, Z \in \mathcal{X}$ such that X, Z are indecomposable. If $\underline{\mathcal{X}}$ is singularly equivalent to $\mathbf{0}$, then Y is free, and X is isomorphic to ΩZ.

Lemma 19. *Let R and S be Gorenstein complete local rings. Let $\Phi : \underline{\mathsf{CM}}(R) \to \underline{\mathsf{CM}}(S)$ be a triangle equivalence. If f is an irreducible homomorphism of nonfree indecomposable MCM R-modules and g is a homomorphism of S-modules such that $\Phi(\underline{f}) = \underline{g}$, then g is an irreducible homomorphism of nonfree indecomposable MCM S-modules.*

Let R be a local ring. Recall that M is said to have *complexity* c, denoted by $\mathsf{cx}_R\, M = c$, if c is the least nonnegative integer n such that there exists a real number r satisfying the inequality $\beta_i^R(M) \leq ri^{n-1}$ for all $i \gg 0$. It is known that if R is a complete intersection, then the codimension of R is the maximum of the complexities of R-modules. For details on the complexity of a module, we refer the reader to [2, §4.2].

Let R be a d-dimensional Gorenstein local ring with algebraically closed residue field k of characteristic zero. Then R contains a field isomorphic to k, and it is known that R has finite CM-representation type if and only if R is a *simple (hypersurface) singularity* [13, §8], namely, R is isomorphic to a hypersurface

$$k[[x_0, \dots, x_d]]/(f),$$

where f is one of the following.

$$\begin{array}{ll} (\mathsf{A}_n) & x_0^2 + x_1^{n+1} + x_2^2 + \cdots + x_d^2, \\ (\mathsf{D}_n) & x_0^2 x_1 + x_1^{n-1} + x_2^2 + \cdots + x_d^2, \\ (\mathsf{E}_6) & x_0^3 + x_1^4 + x_2^2 + \cdots + x_d^2, \\ (\mathsf{E}_7) & x_0^3 + x_0 x_1^3 + x_2^2 + \cdots + x_d^2, \\ (\mathsf{E}_8) & x_0^3 + x_1^5 + x_2^2 + \cdots + x_d^2. \end{array}$$

For each $\mathsf{T} \in \{\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8\}$, a simple hypersurface singularity of type (T) is shortly called a (T)-*singularity.*

We give a characterization of the (A_1)-singularities in terms of singular equivalence.

Theorem 20. *Let R be a d-dimensional nonregular complete local ring with algebraically closed residue field k of characteristic 0. Then the following conditions are equivalent;*
(1) *R is a Gorenstein ring, and $\underline{\mathsf{CM}}(R)$ is singularly equivalent to* $\mathbf{0}$.
(2) *R is a complete intersection, and $\underline{\mathcal{X}}$ is singularly equivalent to $\mathbf{0}$ for every resolving subcategory $\mathcal{X}$ of* $\mathsf{mod}\, R$.
(3) *R is a complete intersection, and $\underline{\mathcal{X}}$ is singularly equivalent to $\mathbf{0}$ for some resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ that containing a module of maximal complexity.*
(4) *R is an (A_1)-singularity.*

Sketch of proof. (1) $\Rightarrow$ (4): Using Lemma 18, we can show that R has finite CM representation type. By [13, Corollary 8.16] R is a simple singularity. The classification of the Auslander-Reiten quivers of the MCM modules over simple singularities [13, Chapters 8–12] together with Lemma 19 implies that the only simple singularities R where $\mathsf{CM}(R)$ possesses such an Auslander-Reiten quiver are (A_1)-singularities. ■

Let R be a simple hypersurface singularity. Theorem 20 especially says that $\underline{\mathsf{CM}}(R)$ is not singularly equivalent to $\mathbf{0}$ unless R is an (A_1)-singularity. One can actually confirm this for a 1-dimensional (A_2)- singularity by direct calculation.

Example 21. Let k be an algebraically closed field of characteristic 0. Let R be an (A_2)-singularity of dimension 1 over k. Then there is a triangle equivalence

$$\mathsf{D_{sg}}(\underline{\mathsf{CM}}(R)) \cong \mathsf{D_{sg}}(k[t]/(t^2)).$$

In particular, $\underline{\mathsf{CM}}(R)$ is not singularly equivalent to $\mathbf{0}$.

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JACOBIAN ALGEBRAS AND DEFORMATION QUANTIZATIONS

IZURU MORI

Abstract. Let V be a 3-dimensional vector space over an algebraically closed field k of characteristic 0. In this paper, we study the following two classes of algebras: (1) the Jacobian algebra $J(\omega)$ of a potential $0 \neq \omega \in V^{\otimes 3}$, and (2) the algebra S_f^λ induced by the deformation quantization of the polynomial algebra $S := S(V) = k[x, y, z]$ in three variables whose semi-classical limit has a quadratic unimodular Poisson bracket on S determined by $f \in S_3$. It is known that every noetherian quadratic Calabi-Yau algebra of dimension 3 is of the form $J(\omega)$, however, it is not easy to see for which potential $0 \neq \omega \in V^{\otimes 3}$, $J(\omega)$ is a Calabi-Yau algebra of dimension 3. In this paper, we try to answer this question by relating $J(\omega)$ to S_f^λ.

1. Jacobian Algebras

This is a report on a joint work in progress with S. Paul Smith. Throughout this paper, let k be an algebraically closed field of characteristic 0, and V a finite dimensional vector space over k. We denote by $T(V)$ the tensor algebra and $S(V)$ the symmetric algebra.

We define the action of $\theta \in \mathfrak{S}_m$ on $V^{\otimes m}$ by

$$\theta(v_1 \otimes \cdots \otimes v_m) := v_{\theta(1)} \otimes \cdots \otimes v_{\theta(m)}.$$

Specializing to the m-cycle $\phi \in \mathfrak{S}_m$, we define

$$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{m-1} \otimes v_m) := v_m \otimes v_1 \otimes \cdots \otimes v_{m-2} \otimes v_{m-1}.$$

We define linear maps $c, s, a : V^{\otimes m} \to V^{\otimes m}$ by

$$c(\omega) := \frac{1}{m} \sum_{i=0}^{m-1} \phi^i(\omega)$$

$$s(\omega) := \frac{1}{m!} \sum_{\theta \in \mathfrak{S}_m} \theta(\omega)$$

$$a(\omega) := \frac{1}{m!} \sum_{\theta \in \mathfrak{S}_m} (\operatorname{sgn} \theta)\theta(\omega).$$

We define the following subspaces of $V^{\otimes m}$:

$$\operatorname{Sym}^m V := \{\omega \in V^{\otimes m} \mid \theta(\omega) = \omega \text{ for all } \theta \in \mathfrak{S}_m\}$$

$$\operatorname{Alt}^m V := \{\omega \in V^{\otimes m} \mid \theta(\omega) = (\operatorname{sgn} \theta)\omega \text{ for all } \theta \in \mathfrak{S}_m\}.$$

It is easy to see that $\operatorname{Sym}^m V = \operatorname{Im} s$ and $\operatorname{Alt}^m V = \operatorname{Im} a$.

The following is a key lemma in this paper.

The detailed version of this paper will be submitted for publication elsewhere.

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Lemma 1. *Suppose that* $\dim V = 3$. *For every choice of a basis* x, y, z *for* V, $\mathrm{Alt}^3 V = k\omega_0$ *where*

$$\omega_0 = 2a(xyz) = c(xyz - zyx) = \frac{1}{3}(xyz + zxy + yzx - zyx - xzy - yxz).$$

By Lemma 1, we can define a linear map $\mu : V^{\otimes 3} \to k$ by the formula $a(\omega) = \mu(\omega)\omega_0$ when $\dim V = 3$.

We define three kinds of derivatives: Choose a basis $x_1, \dots, x_n$ for V so that $S(V) = k[x_1, \dots, x_n]$ and $T(V) = k\langle x_1, \dots, x_n\rangle$. For $f \in k[x_1, \dots, x_n]$, the usual partial derivative of f with respect to x_i is denoted by f_{x_i}. For a monomial $\omega = x_{i_1}x_{i_2}\cdots x_{i_{m-1}}x_{i_m} \in k\langle x_1, \dots, x_n\rangle_m$ of degree m, we define

$$x_i^{-1}\omega := \begin{cases} x_{i_2}\cdots x_{i_{m-1}}x_{i_m} & \text{if } i_1 = i, \\ 0 & \text{if } i_1 \neq i, \end{cases} \quad \textit{and}$$

$$\partial_{x_i}(\omega) := m x_i^{-1} c(\omega).$$

We extend the map $\partial_{x_i} : k\langle x_1, \dots, x_n\rangle \to k\langle x_1, \dots, x_n\rangle$ by linearity. We call ∂_{x_i} the cyclic derivative with respect to x_i.

Definition 2. The Jacobian algebra of $\omega \in k\langle x_1, \dots, x_n\rangle$ is the algebra of the form

$$J(\omega) := k\langle x_1, \dots, x_n\rangle/(\partial_{x_1}\omega, \dots, \partial_{x_n}\omega).$$

We call ω the potential of $J(\omega)$.

It is easy to see that the Jacobian algebra is independent of the choice of a basis $x_1, \dots, x_n$ for V. Note that if ω is homogeneous, then $J(\omega)$ is a graded algebra. In this paper, we focus on the case that $\dim V = 3$ and $0 \neq \omega \in V^{\otimes 3}$. In this case, $J(\omega) = T(V)/(R)$ is a quadratic algebra where $R \subset V \otimes V$.

A Calabi-Yau algebra defined below plays an important role in many branches of mathematics. For an algebra A, we denote by $A^e := A \otimes A^{op}$ the enveloping algebra of A.

Definition 3. An algebra A is called Calabi-Yau of dimension d (d-CY for short) if

(1) A has a resolution of finite length consisting of finitely generated projective A^e-modules, and

(2) $\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$ as A^e-modules.

Bocklandt [3] showed that every graded Calabi-Yau algebra is a Jacobian algebra. Specializing to the noetherian quadratic case, we have the following result, which is the main motivation of this paper.

Theorem 4. [3] *Every noetherian quadratic Calabi-Yau algebra of dimension 3 is of the form* $J(\omega)$ *where* $\dim V = 3$ *and* $0 \neq \omega \in V^{\otimes 3}$.

By the above theorem, it is interesting to know for which potential $0 \neq \omega \in V^{\otimes 3}$, $J(\omega)$ is a Calabi-Yau algebra of dimension 3. Some criteria were given by [4], [2], however, these criteria are difficult to check in practice. The purpose of this paper is to give a more effective criterion by using geometry.

2. Deformation Quantizations

Let A be a commutative algebra.

Definition 5. A Poisson algebra is an algebra A together with a bilinear map $\{-,-\}: A \times A \to A$, called the Poisson bracket, satisfying the following axioms:

(1) $\{a,b\} = -\{b,a\}$ for all $a,b \in A$.
(2) $\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0$ for all $a,b,c \in A$.
(3) $\{a,bc\} = \{a,b\}c + b\{a,c\}$ for all $a,b,c \in A$.

Definition 6. A formal deformation of A is a $k[[t]]$-algebra $A[[t]]$ with the multiplication $\varphi : A[[t]] \times A[[t]] \to A[[t]]$ of the form $\varphi = \sum_{i\in\mathbb{N}} \varphi_i t^i$ where $\varphi_0 : A \times A \to A$ is the original multiplication of A and each $\varphi_i : A \times A \to A$ is a k-bilinear map extended to be $k[[t]]$-bilinear.

Since A is commutative, for all $a,b \in A$, $\varphi_0(a,b) = \varphi_0(b,a)$, so

$$\begin{aligned}\varphi(a,b) - \varphi(b,a) &= \sum_{i\in\mathbb{N}} \varphi_i(a,b)t^i - \sum_{i\in\mathbb{N}} \varphi_i(b,a)t^i \\ &= \sum_{i\in\mathbb{N}} (\varphi_i(a,b) - \varphi_i(b,a))t^i \\ &= (\varphi_1(a,b) - \varphi_1(b,a))t + O(t^2).\end{aligned}$$

It is easy to see that $(A, \{-,-\}_\varphi)$ where $\{a,b\}_\varphi := \varphi_1(a,b) - \varphi_1(b,a)$ for $a,b \in A$ is a Poisson algebra. We call $(A, \{-,-\}_\varphi)$ the semi-classical limit of $(A[[t]], \varphi)$. It is not easy to see which Poisson algebra can be realized as a semi-classical limit of a formal deformation. If this is the case, we call it a deformation quantization.

Definition 7. Let $(A, \{-,-\})$ be a Poisson algebra. A formal deformation $(A[[t]], \varphi)$ of A is called a deformation quantization of $(A, \{-,-\})$ if $\{-,-\} = \{-,-\}_\varphi$.

We now focus on the case $A = S(V)$. For $m \geq 2$, $S(V)_m = V^{\otimes m} / \sum_{i+j=m-2} V^i \otimes R \otimes V^j$ is the quotient space where $R = \{u \otimes v - v \otimes u \in V \otimes V \mid u,v \in V\}$. We denote the quotient map by $\overline{(-)} : V^{\otimes m} \to S(V)_m$. Since $s(\omega) = 0$ for every $\omega \in V^i \otimes R \otimes V^j$, the linear map $s : V^{\otimes m} \to V^{\otimes m}$ induces a linear map $\widetilde{(-)} : S(V)_m \to V^{\otimes m}$, called the symmetrization map.

Lemma 8. *The linear maps* $\overline{(-)} : V^{\otimes m} \to S(V)_m$ *and* $\widetilde{(-)} : S(V)_m \to V^{\otimes m}$ *induce isomorphisms* $\overline{(-)} : \operatorname{Sym}^m V \to S(V)_m$ *and* $\widetilde{(-)} : S(V)_m \to \operatorname{Sym}^m V$ *inverses to each other.*

For the rest of the paper, we assume that $\dim V = 3$ and we write $S = S(V) = k[x,y,z]$. In this case, every Poisson bracket on S is uniquely determined by $\{y,z\}, \{z,x\}, \{x,y\} \in S$. A Poisson algebra $(S, \{-,-\})$ is called quadratic if $\{y,z\}, \{z,x\}, \{x,y\} \in S_2$.

Theorem 9. [5] *If* $(S, \{-,-\})$ *is a quadratic Poisson algebra, then*

$$k[[t]]\langle x,y,z\rangle / ([y,z] - t\widetilde{\{y,z\}}, [z,x] - t\widetilde{\{z,x\}}, [x,y] - t\widetilde{\{x,y\}})$$

is a deformation quantization of $(S, \{-,-\})$.

For every $f \in S$,

$$\{y, z\}_f := f_x,\ \{z, x\}_f := f_y,\ \{x, y\}_f := f_z$$

defines a Poisson bracket on S. In fact, it is known that $\{-,-\}$ is a unimodular Poisson bracket on S if and only if $\{-,-\} = \{-,-\}_f$ for some $f \in S$. If $f \in S_3$, then $(S, \{-,-\}_f)$ is a quadratic Poisson algebra, so

$$k[[t]]\langle x, y, z\rangle/([y, z] - t\widetilde{f_x}, [z, x] - t\widetilde{f_y}, [x, y] - t\widetilde{f_z})$$

is a deformation quantization of $(S, \{-,-\}_f)$ by Theorem 9. For $f \in S_3$ and $\lambda \in k$, we define the algebra induced by the above deformation quantization as

$$S_f^\lambda := k\langle x, y, z\rangle/([y, z] - \lambda\widetilde{f_x}, [z, x] - \lambda\widetilde{f_y}, [x, y] - \lambda\widetilde{f_z}).$$

The next two results show that Jacobian algebras and deformation quantizations are strongly ralated.

Theorem 10. *For every $f \in S_3$ and every $\lambda \in k$, $S_f^\lambda = J\left(\omega_0 - \lambda\widetilde{f}\right)$.*

Theorem 11. *For $J(\omega) = T(V)/(R)$ where $0 \neq \omega \in V^{\otimes 3}$ and $R \subset V \otimes V$, the following are equivalent:*

(1) *$J(\omega) = S_f^\lambda$ for some $f \in S_3, \lambda \in k$.*
(2) *$R \cap \operatorname{Sym}^2 V = \{0\}$.*
(3) *$R \not\subset \operatorname{Sym}^2 V$.*
(4) *$c(\omega) \notin \operatorname{Sym}^3 V$.*
(5) *$a(\omega) \neq 0$.*
(6) *$\mu(\omega) \neq 0$.*

If any of the above equivalent condition holds, then $J(\omega) = S_{\overline{\omega}}^{-1/\mu(\omega)}$.

The above theorem shows that majority of Jacobian algebras are induced by deformation quantizations.

3. A Criterion for the Calabi-Yau Property

In this section, we will give a criterion for which potential $0 \neq \omega \in V^{\otimes 3}$, $J(\omega)$ is 3-CY. By the previous section, we divide into two cases (1) $a(\omega) \neq 0$ (majority), and (2) $a(\omega) = 0$ (minority).

Let $H(f) := \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$ be the Hessian of $f \in S$. Since $H(f) \in S$, we can define $H^{i+1}(f) := H(H^i(f))$ for every $i \in \mathbb{N}$. The classification of cubic divisors in $\mathbb{P}^2$ is well-known. There are eight singular ones and one family of smooth ones (elliptic curves) up to isomorphisms. The Hessian gives a rough classification of cubic divisors in $\mathbb{P}^2$.

Lemma 12. *For $0 \neq f \in S_3$, the exactly one of the following occurs:*

(1) *$H(f) = 0$. In this case,* $\operatorname{Proj} S/(f)$ *is either triple lines, the union of double line and a line, or the union of three lines meeting at one point.*
(2) *$H(f) \neq 0$, but $H^2(f) = 0$. In this case,* $\operatorname{Proj} S/(f)$ *is either the union of a conic and a line meeting at one point, or a cuspidal curve.*

(3) *$H^i(f) \neq 0$ for every $i \in \mathbb{N}$, In this case,* $\operatorname{Proj} S/(f)$ *is either a triangle, the union of a conic and a line meeting at two points, a nodal curve or an elliptic curve.*

Recall that $a(\omega) \neq 0$ if and only if $J(\omega) = S_f^\lambda$ for some $f \in S_3$ and $\lambda \in k$ by Theorem 11, so it is essential to ask which S_f^λ is 3-CY.

Theorem 13. *Let $f \in S_3$.*

(1) *If $H^2(f) = 0$, then S_f^λ is 3-CY for every $\lambda \in k$.*
(2) *If $H^2(f) \neq 0$ and* $\operatorname{Proj} S/(f)$ *is singular, then S_f^λ is 3-CY except for exactly two values of $\lambda \in k$ for each $f \in S_3$.*
(3) *If $H^2(f) \neq 0$ and* $\operatorname{Proj} S/(f)$ *is smooth, then S_f^λ is 3-CY for every $\lambda \in k$.*

The above theorem shows that majority of S_f^λ is 3-CY. In fact, there are only three exceptions up to isomorphisms.

Theorem 14. *Let $f \in S_3$ and $\lambda \in k$. If S_f^λ is not 3-CY, then it is isomorphic to one of the following algebras:*

- $k\langle x, y, z\rangle/(yz, zx, xy)$.
- $k\langle x, y, z\rangle/(yz + x^2, zx, xy)$.
- $k\langle x, y, z\rangle/(yz + x^2, zx + y^2, xy)$.

On the other hand, if $a(\omega) = 0$, then there are not much choice for ω (minority), so we can show the following theorem by case-by-case analysis.

Theorem 15. *Let $0 \neq \omega \in V^{\otimes 3}$ such that $a(\omega) = 0$.*

(1) *If $H^2(\overline{\omega}) = 0$, then $J(\omega)$ is not 3-CY.*
(2) *If $H^2(\overline{\omega}) \neq 0$ and* $\operatorname{Proj} S/(\overline{\omega})$ *is singular, then $J(\omega)$ is 3-CY.*
(3) *If $H^2(\overline{\omega}) \neq 0$ and* $\operatorname{Proj} S/(\overline{\omega})$ *is smooth, then $J(\omega)$ is 3-CY if and only if the j-invariant of* $\operatorname{Proj} S/(\overline{\omega})$ *is not 0.*

There are six exceptions up to isomorphisms.

Theorem 16. *Let $0 \neq \omega \in V^{\otimes 3}$ such that $a(\omega) = 0$. If $J(\omega)$ is not 3-CY, then it is isomorphic to one of the following algebras:*

- $k\langle x, y, z\rangle/(x^2)$.
- $k\langle x, y, z\rangle/(xy + yx, x^2)$.
- $k\langle x, y, z\rangle/(y^2, x^2)$.
- $k\langle x, y, z\rangle/(xz + zx + y^2, xy + yx, x^2)$.
- $k\langle x, y, z\rangle/(xz + zx, y^2, x^2)$.
- $k\langle x, y, z\rangle/(z^2, y^2, x^2)$.

These nine exceptional algebras in Theorem 14 and Theorem 16 are in one-to-one correspondence with eight singular cubics together with the elliptic curve of j-invariant 0. By [1], every noetherian quadratic Calabi-Yau algebra of dimension 3 is a domain. On the other hand, none of the nine exceptional algebras above is a domain, so we have a rather surprising result:

Theorem 17. *Let $0 \neq \omega \in V^{\otimes 3}$. Then $J(\omega)$ is 3-CY if and only if it is a domain.*

The point scheme is an essential ingredient to study noetherian quadratic Calabi-Yau algebras of dimension 3 in noncommutative algebraic geometry.

Theorem 18. *Let $f \in S_3$ and $\lambda \in k$. If S_f^λ is 3-CY, then the point scheme of S_f^λ is given by* $\operatorname{Proj} S/(24\lambda f + \lambda^3 H(f))$.

It follows that, for a generic choice of $f \in S_3$ and $\lambda \in k$, the point scheme of S_f^λ parameterizes 0-dimensional symplectic leaves for the unimodular Poisson structure on $\mathbb{P}^2 = \operatorname{Proj} S$ induced by f.

A few more calculations for minority show the following theorem:

Theorem 19. *Let $0 \neq \omega \in V^{\otimes 3}$. If $J(\omega)$ is 3-CY, then the point scheme of $J(\omega)$ is given by* $\operatorname{Proj} A/(24\mu(\omega)^2\overline{\omega} + H(\overline{\omega}))$.

4. Examples

We claim that the criterion given in this paper is effective. In fact, given $\omega \in V^{\otimes 3}$, it is routine to calculate $a(\omega)$. Moreover, given $f \in S_3$, it is routine to calculate $H^2(f)$, and it is easy to check if $\operatorname{Proj} S/(f)$ is singular or smooth because $\operatorname{Proj} S/(f)$ is singular if and only if the system of polynomial equations $f_x = f_y = f_z = 0$ has a non-trivial solution. Alternately, by sketching the curve, we can fit $\operatorname{Proj} S/(f)$ into one of the cubic divisors in the classification. Then we can see if it is singular or smooth and we can determine if $H^2(f) = 0$ or not by Lemma 12.

Example 20. If $f = x^2z + xy^2$, then it is easy to see that $\operatorname{Proj} S/(f)$ is the union of a conic and a line meeting at one point, so $H^2(f) = 0$ by Lemma 12, hence S_f^λ is 3-CY for every $\lambda \in k$ by Theorem 13.

Example 21. If $f = xyz + (1/3)x^3 \in S_3$, then it is easy to see that $\operatorname{Proj} S/(f)$ is the union of a conic and a line meeting at two points, so $H^2(f) \neq 0$ by Lemma 12. Since $\operatorname{Proj} S/(f)$ is singular, S_f^λ is 3-CY except for exactly two values of $\lambda \in k$ by Theorem 13. These exceptional values can also be determined by a geometric condition as follows. Since

$$H(f) = \begin{vmatrix} 2x & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix} = 2(xyz - x^3),$$

if S_f^λ is 3-CY, then the point scheme of S_f^λ is $\operatorname{Proj} S/(g)$ where

$$g = 24\lambda f + \lambda^3 H(f) = 2\lambda\{(12 + \lambda^2)xyz + (4 - \lambda^2)x^3\}$$

by Theorem 18. It is easy to see that

$$\operatorname{Proj} S/(g) = \begin{cases} \text{the union of a conic and a line meeting at two points} & \text{if } \lambda^2 \neq 0, -12, 4, \\ \mathbb{P}^2 & \text{if } \lambda = 0, \\ \text{a triple line} & \text{if } \lambda^2 = -12, \\ \text{a triangle} & \text{if } \lambda^2 = 4. \end{cases}$$

We can show that S_f^λ is 3-CY if and only if $\operatorname{Proj} S/(g)$ is not a triangle. In fact, the defining relations of S_f^λ are

$$\begin{aligned}
[y,z]-\lambda\widetilde{f_x} &= yz-zy-\lambda\left(\frac{yz+zy}{2}+x^2\right)=\frac{2-\lambda}{2}yz-\frac{2+\lambda}{2}zy-\lambda x^2\\
[z,x]-\lambda\widetilde{f_y} &= zx-xz-\lambda\left(\frac{zx+xz}{2}\right)=\frac{2-\lambda}{2}zx-\frac{2+\lambda}{2}xz\\
[x,y]-\lambda\widetilde{f_z} &= xy-yx-\lambda\left(\frac{xy+yx}{2}\right)=\frac{2-\lambda}{2}yx-\frac{2+\lambda}{2}yx,
\end{aligned}$$

so if $\lambda=\pm 2$, then S_f^λ is not a domain, hence it is not 3-CY.

Example 22. If $\omega = x^3+y^3+z^3+(3\alpha/2)(xyz+zyx)\in V^{\otimes 3}$ where $\alpha\in k$, then it is easy to see that $a(\omega)=0$, so we apply Theorem 15 to this example. Since $f:=\overline{\omega}=x^3+y^3+z^3+3\alpha xyz\in S_3$, it is well-known that

$$\operatorname{Proj} S/(f)=\begin{cases}\text{a triangle} & \text{if } \alpha^3=-1,\\ \text{an elliptic curve} & \text{if } \alpha^3\neq -1,\end{cases}$$

so $H^2(f)\neq 0$ in either case by Lemma 12. If $\alpha^3=-1$, then $\operatorname{Proj} S/(f)$ is singular, so $J(\omega)$ is 3-CY by Theorem 15. On the other hand, if $\alpha^3\neq -1$, then $\operatorname{Proj} S/(f)$ is smooth (an elliptic curve) and the j-invariant of $\operatorname{Proj} S/(f)$ is given by the formula

$$\frac{\alpha^3(8-\alpha^3)}{(1+\alpha^3)^3},$$

so $J(\omega)$ is 3-CY if and only if $\alpha^3\neq 0,8$ by Theorem 15.

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ON THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE OF THE QUIVER ALGEBRA DEFINED BY c CYCLES AND A QUANTUM-LIKE RELATION

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ABSTRACT. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Osaka City University, Japan, 13–15 September 2014.

In this paper, we consider the quiver algebra A over a field K defined by c cycles and a quantum-like relation. We describe the minimal projective bimodule resolution of A, and determine the ring structure of the Hochschild cohomology ring of A modulo nilpotence. And we give some examples of the support variety of A-modules.

1. INTRODUCTION

Let K be a field and A an indecomposable finite dimensional algebra over K. We denote by A^e the enveloping algebra $A \otimes_K A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The n-th Hochschild cohomology group is given by $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$ and the Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \oplus_{n \geq 0} \mathrm{HH}^n(A, A)$ with Yoneda product. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. In this paper, we consider the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

The Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ was used in [5] to define a support variety for any finitely generated module over a finite dimensional algebra A. In [5], Snashall and Solberg defined the support variety $V(M)$ of an A-module M by

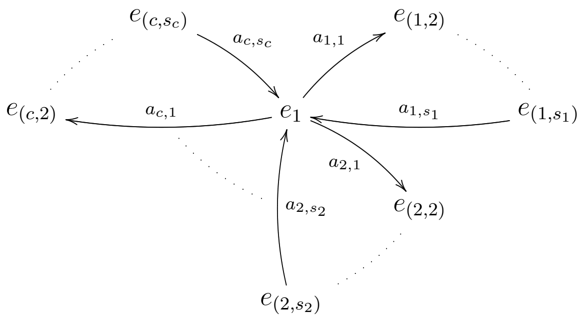
$$V(M) = \{m \in \mathrm{MaxSpecHH}^*(A)/\mathcal{N} |\, \mathrm{Ann}\, \mathrm{Ext}^*_A(M, A/\mathrm{rad}\, A) \subseteq m'\}.$$

where m' is the inverse image of m in $\mathrm{HH}^*(A)$.

Let c be an integer with $c \geq 2$ and $q_{i,j} \in K$ nonzero elements for $1 \leq i < j \leq c$. We consider the quiver algebra KQ/I defined by c cycles and a quantum-like relation where

The detailed version of this paper will be submitted for publication elsewhere.

Q is the following quiver:



where $1 \leq j \leq c$ and $s_j \geq 2$, and where I is the ideal of KQ generated by

$$X_i^{n_i} \text{ for } 1 \leq i \leq c,$$
$$X_i X_j - q_{i,j} X_j X_i \text{ for } 1 \leq i < j \leq c.$$

where $X_i := (\sum_{k_i=1}^{s_i} a_{i,k_i})^{s_i}$ and n_i are integers with $n_i \geq 2$ for $1 \leq i \leq c$.

In the case $c = 2$, we determined the Hochschild cohomology ring of A modulo nilpotence in [2] and [3]. In the case $s_i = 1$ for $1 \leq i \leq c$, the Hochschild cohomology ring of A modulo nilpotence was described by Oppermann in [4]. In this paper, we describe the minimal projective bimodule resolution of A, and determine explicitly the ring structure of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ by giving the K-basis and the multiplication.

2. Precedent results

In this section, we introduce the precedent results about the quiver algebra A. In the case of $s_i = 1$ for $1 \leq i \leq c$, A is called a quantum complete intersection. In this case, the projective bimodule resolution of A and the Hochschild cohomology ring modulo nilpotence of A was given by Oppermann in [4] as follows.

Theorem 1. [4] *In the case of $s_i = 1$ for $1 \leq i \leq c$, the projective bimodule resolution of A is total complex* $\mathrm{Tot}(\mathbb{P}_1 \otimes \mathbb{P}_2 \otimes \cdots \otimes \mathbb{P}_c)$ *where $\mathbb{P}_i$ is the projective bimodule resolution of $A_i = K[\alpha_i]/(\alpha_i^{n_i})$:*

$$\mathbb{P}_i : A_i^e \xleftarrow{1 \otimes x - x_i \otimes 1} A_i^e \xleftarrow{\sum_{k=0}^{n_i-1} x_i^k \otimes x_i^{n_i-1-k}} A_i^e \xleftarrow{1 \otimes x - x_i \otimes 1} A_i^e \longleftarrow \cdots .$$

Theorem 2. [4] $\mathrm{HH}^*(A)/\mathcal{N}$ *is isomorphic to the following finitely generated K-algebra.*

$$_K\langle y_1^{p_1 n_1/2} \cdots y_c^{p_c n_c/2} \in K[y_1, \ldots, y_c] \,|\, \textstyle\prod_{j=1}^c q_{i,j}^{p_j n_j/2} = 1 \textit{ for all } i \textit{ with } p_i \textit{ even},$$
$$\textstyle\prod_{j=1}^c q_{i,j}^{(p_j-1)n_j/2+1} = -1 \textit{ and } n_i = 2 \textit{ for all } i \textit{ with } p_i \textit{ odd}\rangle.$$

where $q_{i,i} = 1$ and $q_{i,j} = q_{j,i}^{-1}$ for $1 \leq j < i \leq c$.

In the case of $c = 2$, we determined the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ in [2] and [3] as follows.

Theorem 3. *Let r be an integer with $r > 0$. In the case of $c = 2$, if $q_{1,2}$ is a primitive r-th root of unity, then $\mathrm{HH}^*(A)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables*:

$$\mathrm{HH}^*(A)/\mathcal{N} \cong \begin{cases} K[x^{2r}, y^{2r}] & \textit{if } n_1, n_2 \not\equiv 0 \bmod r, \\ K[x^2, y^{2r}] & \textit{if } n_1 \equiv 0 \bmod r, n_i \not\equiv 0 \bmod r, \\ K[x^{2r}, y^2] & \textit{if } n_1 \not\equiv 0 \bmod r, n_2 \equiv 0 \bmod r, \\ K[x^2, y^2] & \textit{if } n_1, n_2 \equiv 0 \bmod r, \end{cases}$$

where $x^n = \sum_{k_1=1}^{s_1} e_{(1,k_1)}$, $y^n = \sum_{k_2=1}^{s_2} e_{(2,k_2)}$ *in* $\mathrm{HH}^n(A)$.

Theorem 4. *In the case of $c = 2$, if $q_{1,2}$ is not a root of unity, then* $\mathrm{HH}^*(A)/\mathcal{N} \cong K$.

3. Projective bimodule resolution of A

In this section, we describe the minimal projective bimodule resolution of the quiver algebra $A = KQ/I$ defined by c cycles and a quantum-like relation.

Let c and n be integers with $c \geq 2$ and $n \geq 1$. We set

$$L_n = \{(l_1, l_2, \dots, l_c) \in (\mathbb{N} \cup \{0\})^c \mid \sum_{k=1}^{c} l_k = n\} \text{ for any integer } n \geq 1.$$

We define projective left A^e-modules, equivalently A-bimodules:

$$P_0 = A\varepsilon_0^0 A \oplus \coprod_{i=1}^{c} \coprod_{k_i=2}^{s_i} A\varepsilon_{(i,k_i)}^0 A \text{ and,}$$

$$Q_{(l_1,\dots,l_c)}^n = \begin{cases} \coprod_{k_i=1}^{s_i} A\varepsilon_{(i,k_i)}^n A & \text{if } l_i = n \text{ for some } 1 \leq i \leq c, \\ A\varepsilon_{(l_1,\dots,l_c)}^n A & \text{if } l_i < n \text{ for all } 1 \leq i \leq c, \end{cases}$$

for $(l_1, \dots, l_c) \in L_n$, where $\varepsilon_{(l_1,\dots,l_c)}^n = e_1 \otimes e_1$ and

$$\varepsilon_{(i,k_i)}^n = \begin{cases} e_{(i,k_i)} \otimes e_{(i,k_i)} & \text{if } n \text{ is even,} \\ e_{(i,k_i+1)} \otimes e_{(i,k_i)} & \text{if } n \text{ is odd.} \end{cases}$$

Then, we have the minimal projective A-bimodule resolution of A as the total complex of the following complexes.

Lemma 5. *Let n be an integer with $n \geq 1$ and $E_{i,k_i}^n = \sum_{l=0}^{s_i-1} x_i^l \varepsilon_{(i,k_i-l)}^n x_i^{s_i-1-l}$ for $1 \leq i \leq c$ and $0 \leq k_i \leq s_i - 1$. For $(l_1, \dots, l_c) \in L_n$, we set the integers μ_i by*

$$\mu_i = \begin{cases} n_i(l_i - 1)/2 + 1 & \textit{if } l_i \textit{ is odd,} \\ n_i l_i/2 & \textit{if } l_i \textit{ is even,} \end{cases} \quad \textit{for } 1 \leq i \leq c.$$

Then, we have the following complexes.

(1) *For* $(l_1,\dots,l_c)\in L_n$ *such that* $l_i=n$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),i}:Q^n_{(l_1,\dots,l_c)}\to Q^{n-1}_{(l_1,\dots,l_i-1,\dots,l_c)}$ *by*

$$\partial^n_{(l_1,\dots,l_c),i}:\varepsilon^n_{(i,k_i)}\mapsto\begin{cases}\varepsilon^{n-1}_{(i,k_i+1)}x_i-x_i\varepsilon^{n-1}_{(i,k_i)} & \text{if } n \text{ is odd,}\\ \displaystyle\sum_{l=0}^{n_i-1}X_i^lE^{n-1}_{i,k_i-1}X_i^{n_i-1-l} & \text{if } n \text{ is even,}\end{cases}\quad \text{for } 1\le k_i\le s_i.$$

Then, since $\partial^n_{(l_1,\dots,l_c),i}\circ\partial^{n+1}_{(l_1,\dots,l_i+1,\dots,l_c),i}=0$, *we have the complex* $\mathbb{P}_i$:

$$P_0\xleftarrow{\partial^1_{(0,\dots,1,\dots,0),i}}Q^1_{(0,\dots,1,\dots,0)}\xleftarrow{\partial^2_{(0,\dots,2,\dots,0),i}}\cdots\xleftarrow{\partial^n_{(0,\dots,n,\dots,0),i}}Q^n_{(0,\dots,n,\dots,0)}\leftarrow\cdots.$$

(2) *Let* $m=\min\{i\,|\,l_i>0\}$ *for* $(l_1,\dots,l_c)\in L_n$. *For* $m\le j\le c$ *and* $(l_1,\dots,l_c)\in L_n$ *such that* $l_i<n-1$ *for* $1\le i\le c$ *and* $l_j\neq0$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),j}:Q^n_{(l_1,\dots,l_c)}\to Q^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}$ *as follows*:

$$\partial^n_{(l_1,l_2,\dots,l_c),j}:\varepsilon^n_{(l_1,\dots,l_c)}\mapsto$$
$$\begin{cases}(-1)^{\sum_{k=j+1}^c l_k}\Big(\displaystyle\prod_{h_1=1}^{c-j}q^{\mu_{j+h_1}}_{j,j+h_1}\varepsilon^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}X_j-\prod_{h_2=1}^{j-1}q^{\mu_{h_2}}_{h_2,j}X_j\varepsilon^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}\Big)\\ \hfill\text{if } l_j \text{ is odd,}\\ (-1)^{\sum_{k=j+1}^c l_k}\displaystyle\sum_{k_j=0}^{n_j-1}\prod_{h_1=1}^{c-j}q^{\mu_{j+h_1}(n_j-1-k_j)}_{j,j+h_1}\prod_{h_2=1}^{j-1}q^{\mu_{h_2}k_j}_{h_2,j}X_j^{k_j}\varepsilon^{n-1}_{(l_1,\dots,l_j-1,\dots,l_c)}X_j^{n_j-1-k_j}\\ \hfill\text{if } l_j \text{ is even}(\neq0).\end{cases}$$

For $(l_1,\dots,l_c)\in L_n$ *such that* $l_m=n-1$ *and* $l_j=1$ *for* $m\le j\le c$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),j}$ *by*

$$\partial^n_{(l_1,\dots,l_c),j}:\varepsilon^n_{(l_1,\dots,l_c)}\mapsto\begin{cases}E^{n-1}_{m,0}X_j-q^{\mu_m}_{m,j}X_jE^{n-1}_{m,0} & \text{if } n \text{ is even,}\\ \varepsilon^{n-1}_{(m,1)}X_j-q^{\mu_m}_{m,j}X_j\varepsilon^{n-1}_{(m,1)} & \text{if } n \text{ is odd,}\end{cases}$$

For $(l_1,\dots,l_c)\in L_n$ *such that* $l_m=1$ *and* $l_j=n-1$ *for* $m\le j\le c$, *we define the left* A^e*-homomorphisms* $\partial^n_{(l_1,\dots,l_c),j}$ *by*

$$\partial^n_{(l_1,\dots,l_c),j}:\varepsilon^n_{(l_1,\dots,l_c)}\mapsto\begin{cases}E^{n-1}_{j,0}X_m-q^{\mu_j}_{m,j}X_mE^{n-1}_{j,0} & \text{if } n \text{ is even,}\\ \varepsilon^{n-1}_{(j,1)}X_m-q^{\mu_j}_{m,j}X_m\varepsilon^{n-1}_{(j,1)} & \text{if } n \text{ is odd,}\end{cases}$$

Then, since $\partial^n_{(l_1,\dots,l_c),j}\circ\partial^{n+1}_{(l_1,\dots,l_j+1,\dots,l_c),j}=0$, *for* $(l_1,\dots,l_c)\in L_n$ *such that* $l_j=0$, *we have the complex* $\mathbb{Q}_{(l_1,\dots,l_c),j}$:

$$Q^n_{(l_1,\dots,0,\dots,l_c)}\xleftarrow{\partial^{n+1}_{(l_1,\dots,1,\dots,l_c),j}}Q^{n+1}_{(l_1,\dots,1,\dots,l_c)}\leftarrow\cdots\xleftarrow{\partial^{n+n'}_{(l_1,\dots,n',\dots,l_c),j}}Q^{n+n'}_{(l_1,\dots,n',\dots,l_c)}\leftarrow\cdots.$$

Theorem 6. *The following total complex* $\mathbb{P}$ *is the minimal projective resolution of the left* A^e*-module* A.

$$\mathbb{P}:0\leftarrow A\xleftarrow{\pi}P_0\xleftarrow{d_1}P_1\leftarrow\cdots\xleftarrow{d_n}P_n\leftarrow\cdots$$

where π is the multiplication map and

$$P_n = \coprod_{(l_1,\dots,l_c)\in L_n} Q^n_{(l_1,\dots,l_c)} \text{ and } d_n = \sum_{j=1}^{c} \sum_{(l_1,\dots,l_c)\in L_n} \partial^n_{(l_1,\dots,l_c),j},$$

for $n \geq 1$, where $\partial^n_{(l_1,\dots,l_c),j}$ are the A^e-homomorphisms given in Lemma 5*.*

Now we consider the complex $\mathbb{P} \otimes_A A/\operatorname{rad} A$. We can prove that $\mathbb{P}$ is exact, by the following Lemma.

Lemma 7. [1] *If $\mathbb{P} \otimes_A A/\operatorname{rad} A$ is exact sequence then $\mathbb{P}$ is also exact sequence.*

We can prove that $\mathbb{P}\otimes_A A/\operatorname{rad} A$ is exact, that is $\dim_k \operatorname{Im} d_n \otimes_A \operatorname{id}_{A/\operatorname{rad} A} + \dim_k \operatorname{Im} d_{n+1} \otimes_A \operatorname{id}_{A/\operatorname{rad} A} = \dim_k P_n \otimes_A A/\operatorname{rad} A$ by the following Lemma.

Lemma 8. *Let $(l_1,\dots,l_c) \in L_n$ such that $l_i < n-1$ for $1 \leq i \leq c$, and $m = min\{i \,|l_i > 0\}$ for $(l_1,\dots,l_c) \in L_n$.*

(1) *If l_m is even, then the left A-module $AX_m d_n \otimes_A \operatorname{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_c)})$ is generated by*
$d_n \otimes_A \operatorname{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_m+1,\dots,l_j-1,\dots,l_c)})$ *for $m+1 \leq j \leq c$ such that $l_j \neq 0$.*

(2) *If l_m is odd, then the left A-module $AX_m^{n_m-1}\widehat{d}_n(e_{(l_1,\dots,l_c)})$ is generated by*
$d_n \otimes_A \operatorname{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_m+1,\dots,l_j-1,\dots,l_c)})$ *for $m+1 \leq j \leq c$ such that $l_j \neq 0$.*

(3) *For $1 \leq i \leq m-1$, the left A-module $AX_i\widehat{d}_n(e^n_{(l_1,\dots,l_c)})$ is generated by*
$d_n \otimes_A \operatorname{id}_{A/\operatorname{rad} A}(e^n_{(l_1,\dots,l_i+1,\dots,l_j-1,\dots,l_c)})$ *for $m+1 \leq j \leq c$ such that $l_j \neq 0$.*

4. The Hochschild cohomology ring modulo nilpotence

In this section, we give a K-basis of the Hochschild cohomology ring modulo nilpotence. Applying the functor $\operatorname{Hom}_{A^e}(-, A)$ to the A^e-projective resolution $\mathbb{P}$ given in Theorem 6, we have the following complex:

$$\mathbb{P}^* : 0 \to P_0^* \xrightarrow{d_1^*} P_1^* \to \cdots \to P_{n-1}^* \xrightarrow{d_n^*} P_n^* \to \cdots,$$

where

$$P_n^* = \coprod_{(l_1,\dots,l_c)\in L_n} \operatorname{Hom}_{A^e}(Q^n_{(l_1,\dots,l_c)}, A) \text{ and } d_n^* = \sum_{i=1}^{c} \sum_{(l_1,\dots,l_c)\in L_n} \operatorname{Hom}_{A^e}(\partial^n_{(l_1,\dots,l_c),i}, A),$$

for $n \geq 1$. Then we have the following isomorphisms:

$$P_0^* = \operatorname{Hom}_{A^e}(P_0, A) \simeq e_1Ae_0^0 \oplus \coprod_{i=1}^{c} \coprod_{k_i=2}^{s_i} e_{i,k_i}Ae^0_{(i,k_i)},$$

$$\operatorname{Hom}_{A^e}(Q^n_{(l_1,\dots,l_c)}, A) \simeq \begin{cases} \coprod_{k_i=1}^{s_i} e_{(i,k_i)}Ae^n_{(i,k_i)} & \text{if } n \text{ is even and } l_i = n, \\ \coprod_{k_i=1}^{s_i} e_{(i,k_i+1)}Ae^n_{(i,k_i)} & \text{if } n \text{ is odd and } l_i = n, \\ e_1Ae^n_{(l_1,\dots,l_c)} & \text{if } l_i < n \text{ for } 1 \leq i \leq c, \end{cases}$$

for $(l_1,\dots,l_c)\in L_n$. Since we give the Hochschild cohomology ring modulo nilpotence, we only consider the elements, which are trivial passes in A, in $\mathrm{HH}^n(A)=\mathrm{Ker}\,d^*_{n+1}/\mathrm{Im}\,d^*_n$. Now, we give the image of $e^n_{(l_1,\dots,l_c)}$ in P^*_n by $\partial^{n+1*}_{(l_1,\dots,l_j+1,\dots,l_c),j}$ for $(l_1,\dots,l_j+1,\dots,l_c)\in L_{n+1}$ and $1\le j\le c$.

$$\mathrm{Hom}_{A^e}(\partial^{n+1}_{(l_1,\dots,l_j+1,\dots,l_c),j},A):$$
$$\begin{cases}
e^n_{(i,k_i)}\mapsto x_ie^{n+1}_{(i,k_i-1)}-x_ie^{n+1}_{(i,k_i)} \quad \text{for } 1\le k_i\le s_i & \text{if } n \text{ is even}, l_i=n \text{ and } i=j,\\
e^n_{(i,1)}\mapsto (1-q^{\mu_i}_{i,j})X_je^{n+1}_{(l_1,\dots,l_j+1,\dots,l_c)} & \text{if } l_i=n \text{ and } i<j,\\
e^n_{(i,1)}\mapsto (-1)^n(q^{\mu_i}_{j,i}-1)X_je^{n+1}_{(l_1,\dots,l_j+1,\dots,l_c)} & \text{if } l_i=n \text{ and } i>j,\\
e^n_{(l_1,\dots,l_c)}\mapsto & \\
\begin{cases}
(-1)^{\sum_{k=j+1}^c l_k}\Big(\prod_{h_1=1}^{c-j}q^{\mu_{j+h_1}}_{j,j+h_1}-\prod_{h_2=1}^{j-1}q^{\mu_{h_2}}_{h_2,j}\Big)X_je^{n+1}_{(l_1,\dots,l_j+1,\dots,l_c)} & \text{if } l_j \text{ is even},\\
(-1)^{\sum_{k=j+1}^c l_k}\sum_{k_j=0}^{n_j-1}\prod_{h_1=1}^{c-j}q^{\mu_{j+h_1}(n_j-1-k_j)}_{j,j+h_1}\prod_{h_2=1}^{j-1}q^{\mu_{h_2}k_j}_{h_2,j}X_j^{n_j-1}e^{n+1}_{(l_1,\dots,l_j+1,\dots,l_c)} & \text{if } l_j \text{ is odd},
\end{cases} & \\
& \text{if } l_i<n \text{ for } 1\le i\le c,
\end{cases}$$

For homogeneous elements $\eta\in\mathrm{HH}^m(A)$ and $\theta\in\mathrm{HH}^n(A)$, we have the Yoneda product $\eta\theta=\eta\sigma_m\in\mathrm{HH}^{m+n}(A)$ where σ_m is a lifting of θ in the following commutative diagram of A-bimodules.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{m+n} \arrow[r, "d_{m+n}"] \arrow[d, "\sigma_m"] & \cdots \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\sigma_1"] & P_n \arrow[d, "\sigma_0"] \arrow[dr, "\theta"] & & \\
\cdots \arrow[r] & P_m \arrow[r, "d_m"] & \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\pi"] & A \arrow[r] & 0.
\end{tikzcd}$$

Proposition 9. *Let $(l_1,\dots,l_c)\in L_n$, $(l'_1,\dots,l'_c)\in L_{n'}$. Then we have the lifting of $e^n_{(l_1,\dots,l_c)}$ as follows.*

$$\sigma_{n'}:\varepsilon^{n+n'}_{(l_1+l'_1,\dots,l_c+l'_c)}\mapsto \sum_{\substack{0\le k_j\le n_j-2\\ 1\le j\le c\\ \text{such that}\\ l_j,l'_j \text{ are odd}}} Q\prod_{\substack{1\le j\le c\\ \text{such that}\\ l_j,l'_j \text{ are odd}}}X_j^{k_j}\varepsilon^{n'}_{(l'_1,\dots,l'_c)}\prod_{\substack{1\le j\le c\\ \text{such that}\\ l_j,l'_j \text{ are odd}}}X_j^{n_j-2-k_j},$$

for $n'\ge 0$ where $Q\in K$ depending on $(l_1+l'_1,\dots,l_c+l'_c)\in L_{n+n'}$ and integers k_j.

By Proposition 9, if n is odd or l_j is odd for some $1\le j\le c$, $e^n_{(l_1,\dots,l_c)}$ is nilpotence.

By the complex $\mathbb{P}^*$ and Yoneda product given by Proposition 9, we have the K-basis of the Hochschild cohomology ring of A modulo nilpotence as follows.

Theorem 10. *Let $q_{i,j}=q^{-1}_{j,i}$ for $1\le j<i\le c$. The following elements form a K-basis of $\mathrm{HH}^*(A)/\mathbb{N}$.*

(1) $\sum_{k_i=1}^{s_i}e^n_{(i,k_i)}\in\mathrm{HH}^n(A)/\mathcal{N}$ *for the even integer n and the integer i with $1\le i\le c$ which satisfy the following conditions*:

$$q^{n_in/2}_{i,j}=1 \quad \textit{for } 1\le j\le c.$$

(2) $e^n_{(l_1,\dots,l_c)} \in \mathrm{HH}^n(A)/\mathcal{N}$ *for the even integer* n *and* $(l_1,\dots,l_c) \in L_n$ *which satisfy the following conditions*:

$$l_i \text{ is even for } 1 \le i \le c,$$
$$\prod_{h=1}^{c} q_{j,h}^{n_h l_h/2} = 1 \quad \text{for } 1 \le j \le c \text{ such that } l_j \ne 0,$$

Remark 11. In the case of $n_i > 2$ for $1 \le i \le c$, the K-basis elements of $\mathrm{HH}^*(A)/\mathcal{N}$ given in Theorem 10 coincide with those of given in Theorem 2.

5. Examples of the support variety

In this section, we give the examples of the support variety of an A-module. In [5], Snashall and Solberg defined the support variety $V(M)$ of a A-module M by

$$V(M) = \{m \in \mathrm{MaxSpecHH}^*(A)/\mathcal{N} \mid \mathrm{Ann}\,\mathrm{Ext}^*_A(M, A/\mathrm{rad}\,A) \subseteq m'\}.$$

where m' is the inverse image of m in $\mathrm{HH}^*(A)$ and $\mathrm{Ann}\,\mathrm{Ext}^*_A(M, A/\mathrm{rad}\,A)$ is annihilator of $\mathrm{Ext}^*_A(M, A/\mathrm{rad}\,A)$.

Let K be an algebraically closed filed and $r \in \mathbb{N}$. We consider the case $c = 2$, $s_1 = s_2 = 1$,$q_{1,2}$ is a primitive r-th root of unity and $n_1, n_2 \not\equiv 0 \bmod r$ ([2]). Then we have

$$\mathrm{HH}^*(A)/\mathcal{N} = K[X, Y].$$

where $X = \sum_{k_1=1}^{s_1} e_{(1,k_1)}$, $Y = \sum_{k_2=1}^{s_2} e_{(2,k_2)}$ in $\mathrm{HH}^{2r}(A)$.

Example 12. Let $M_1 = Ax_1^{s_1 t} e_1$. We have $\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A)$ and the annihilator of $\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A)$ as follows:

$$\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A) = \textstyle\coprod_{n\ge 0} K e^n_{(1,1)},$$
$$\mathrm{Ann}\,\mathrm{Ext}^*_A(M_1, A/\mathrm{rad}\,A) = (Y).$$

And we have the support variety of M_1 as follows:

$$V(M_1) = \{(a_1, a_2) \in K^2 \mid a_2 = 0\} \text{ as an affine algebraic set.}$$

Example 13. Let $M_2 = AX_1^{s_1 t_1} X_2^{s_2 t_2}$ and $M_3 = AX_1^{s_1 t_1} e_1 + AX_2^{s_2 t_2} e_1$. We have the annihilator of $\mathrm{Ext}^*_A(M_i, A/\mathrm{rad}\,A)$ for $i = 2, 3$ as follows:

$$\mathrm{Ann}\,\mathrm{Ext}^*_A(M_i, A/\mathrm{rad}\,A) = 0.$$

And we have the support variety of M_i for $2 \le i \le 3$ as follows:

$$V(M_i) = K^2 \text{ as an affine plane.}$$

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REMARKS ON A CATEGORICAL DEFINITION OF DEGENERATION IN TRIANGULATED CATEGORIES

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Abstract. This work reports on joint research with Manuel Saorin. For an algebra A over an algebraically closed field k the set of A-module structures on k^d forms an affine algebraic variety. The general linear group $Gl_d(k)$ acts on this variety and isomorphism classes correspond to orbits under this action. A module M degenerates to a module N if N belongs to the Zariski closure of the orbit of M. Yoshino gave a scheme-theoretic characterisation, and Saorin and Zimmermann generalise this concept to general triangulated categories. We show that this concept has an interpretation in terms of distinguished triangles, analogous to the Riedtmann-Zwara characterisation for modules. In this manuscript we report on these results and study the behaviour of this degeneration concept under functors between triangulated categories.

1. Introduction

Already very early in representation theory of algebras a geometric interpretation of representations of an algebra was given, cf e.g. work of Gabriel [3]. For an algebraically closed field k, a finite-dimensional k-algebra A and some integer $d > 0$ the set of A-module structures on k^d forms an affine algebraic variety $mod(A, d)$. The general linear group $Gl_d(k)$ acts on this variety and two A-module structures on k^d are isomorphic if and only if they belong to the same orbit. One says that an A-module M degenerates to the module N if N belongs to the Zariski closure of the orbit of M. Much work was done to explain the geometric structure of the orbit closures. Riedtmann [10] and Zwara [16] prove that M degenerates to N if and only if there is an A-module Z and an embedding of Z into $M \oplus Z$ such that N is isomorphic to the cokernel of this embedding. We refer to Section 2 for more details on this part of the theory.

Yoshino studies in [12, 13, 14] degeneration for more general algebras, including maximal Cohen-Macaulay modules over a local Gorenstein k-algebra, and for this purpose he gave a scheme-theoretic definition of this concept. Using this concept Yoshino studies stable categories of maximal Cohen-Macaulay modules over a local Gorenstein algebra. We refer to Section 3 for more details.

Yoshino's concept is then suitable for general triangulated categories. In joint work with Saorin [11] we define a degeneration concept for general triangulated k-categories with splitting idempotents. We then show that this concept implies in a very general setting that if an object M degenerates to an object N, then there is an object Z and a distinguished triangle $Z \xrightarrow{\binom{v}{u}} Z \oplus M \to N \to Z[1]$ in the triangulated category with nilpotent endomorphism v of Z. We then write $M \leq_\Delta N$. For algebraic triangulated

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categories, and an additional technical hypothesis, we prove the converse. We study in the present paper the question what happens if two objects M and N belonging to a triangulated category $\mathcal{T}_1$ such that $\mathcal{T}_1$ is a full triangulated subcategory of a triangulated category $\mathcal{T}_2$. Under the hypotheses on $\mathcal{T}_2$ which we need for the converse of the main theorem of [11] as mentioned above, we show that then the degeneration concepts coincide.

We finally mention that our degeneration concept applies to the bounded derived category of a finite dimensional algebra, and to the stable category of a selfinjective algebra. For the bounded derived category over an algebra Jensen, Su and Zimmermann gave an alternative definition in [5]. Also in [5] we showed that degeneration there is equivalent to the existence of a distinguished triangle as above. However, this concept is very closely linked to the specific situation of derived categories of bounded complexes over a finite dimensional algebra. Moreover, no clear relation to degeneration of the homology can be seen. In a subsequent approach Jensen, Madsen and Su [4] used A_∞ algebras to define a degeneration by means of the homology of a complex. Again, this is not done for general triangulated categories.

In the classical theory degeneration of modules provides a partial order on the isomorphism classes of objects. In [6] Jensen, Su and Zimmermann study when the degeneration given by the existence of a distinguished triangle $Z \stackrel{\binom{v}{u}}{\to} Z \oplus M \to N \to Z[1]$ gives a partial order. This happens to be the case when some finiteness conditions are assumed, in particular morphism spaces in the triangulated category should be k-modules of finite length for all objects in the triangulated category. Moreover, for two objects X, Y we ask that we may find a shift $n_{X,Y}$ such that there is no non-zero morphism from X to $Y[n_{X,Y}]$.

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2. Classical degeneration concepts

Degeneration between modules over a fixed algebra is a relatively classical subject in representation theory of finite dimensional algebras, and was used in many different ways.

Let k be an algebraically closed field and let A be a finite dimensional k-algebra. Then an A-module of k-dimension d is an algebra homomorphism $A \stackrel{\varphi}{\to} End_k(k^d)$. Hence, if $a_1, \cdots, a_m$ are algebra generators of A, then for each $i \in \{1, \cdots, m\}$ each of the $\varphi(a_i) =: M_i$ is a square matrix of size d. Moreover, A is finitely presented, in the way that there is a finite set $\rho_1(X_1, \cdots, X_m), \cdots, \rho_s(X_1, \cdots, X_m)$ of relations such that if $k\langle X_1, \ldots, X_m\rangle$ denotes the free algebra in m variables $X_1, \cdots, X_m$, then as an algebra we get

$$A \simeq k\langle X_1, \cdots, X_m\rangle/(\rho_1, \cdots, \rho_s).$$

The points of the affine algebraic variety $mod(A, d)$ defined by the $m \cdot d^2$ variables given by the coefficients of the matrices $M_1, \cdots, M_m$ modulo the relations given by the polynomial equations $\rho_1, \cdots, \rho_s$ parameterise A-module structures on k^d. Two modules N_1 and N_2 corresponding to the points n_1, n_2 of $mod(A, d)$ are isomorphic if and only if the corresponding matrices $M_1(n_1), \cdots, M_m(n_1)$ are simultaneously conjugate with the matrices $M_1(n_2), \cdots, M_m(n_2)$. Otherwise said, $G := GL_d(k)$ acts on $mod(A, d)$ by matrix conjugation and the points n_1 and n_2 correspond to isomorphic modules if and only if n_1 and n_2 belong to the same orbit $G \cdot n_1 = G \cdot n_2$ under this action. In general orbits are not Zariski closed, and we denote by $\overline{G \cdot n}$ the Zariski closure of the orbit $G \cdot n$. Now, we say that N_1 *degenerates to* N_2 if and only if $n_2 \in \overline{G \cdot n_1}$. In this case we note $N_1 \leq_{deg} N_2$.

An algebraic classification of degenerations was subject of intensive research. It is relatively easy to see that $N_2 \simeq N/N_1$ implies $N \leq_{deg} N_1 \oplus N_2$. Moreover, $N_1 \leq_{deg} N_2$ implies $N_3 \oplus N_1 \leq_{deg} N_3 \oplus N_2$ for all A-modules N_1, N_2, N_3. The converse is not true, as may be shown by an example due to Jon Carlson (cf [10, § 3.1]); another example was given with different methods by Yoshino [14, Proposition 3.3]. Further, if $N_1 \leq_{deg} N_2$, then

$$\dim_k(Hom_A(X, N_1)) \leq \dim_k(Hom_A(X, N_2))$$

and

$$\dim_k(Hom_A(N_1, X)) \leq \dim_k(Hom_A(N_2, X))$$

for all X. This property implies actually that $\leq_{deg}$ is a partial order, as was shown by Auslander [1]. An independent proof was later given by Bongartz [2], and an adaption of this proof was used in [6] to show that $\leq_\Delta$ is a partial order under some reasonable hypotheses. Riedtmann showed in [10] that if there is an A-module Z and a short exact sequence

$$0 \to Z \to Z \oplus N_1 \to N_2 \to 0$$

then $N_1 \leq_{deg} N_2$, and Zwara showed in [16] the converse in this generality. The above relations on the dimension of Hom-spaces is an easy consequence, though it was proved earlier by different methods.

3. On Yoshino's degeneration concept

The fact that we only deal with finite dimensional algebras in Section 2 is in some sense unsatisfying. In order to be able to cover a greater generality, Yoshino changed the classical degeneration $\leq_{deg}$ to a scheme theoretic concept which is well-suited for us. We explain Yoshino's results here.

Let k be a field and let A be a k-algebra. Yoshino developed in a series of papers a degeneration concept which is well-suited for the purpose of commutative algebra. By the symbol (V, tV, k) we denote a discrete valuation ring V with radical tV and residue field k. An algebra which is a discrete valuation ring is a discrete valuation k-algebra.

Definition 1. (Yoshino [13]) Let A be a k-algebra and let M and N be two finitely generated A-modules. We say M *degenerates to* N *along a discrete valuation ring*, and we write in this case $M \leq_{dvr} N$, if there is a discrete valuation k-algebra (V, tV, k) and an $A \otimes_k V$-module Q, which is

- flat as V-module,
- such that $M \otimes_k V[\frac{1}{t}] \simeq Q \otimes_V V[\frac{1}{t}]$ as $A \otimes_k V[\frac{1}{t}]$-modules and
- such that $N \simeq Q/tQ$ as A-modules.

The interpretation of this notion is that there is an affine line, presented by V, and a point Q that moves along V. The algebra V is a discrete valuation algebra since we are only interested in the neighbourhood of the parameter $t = 0$. Now, at the value $t = 0$ the moving point Q becomes Q/tQ, which is assumed to be isomorphic to N, and generically, outside $t = 0$, the moving point looks like M. This last fact is expressed by the condition $M \otimes_k V[\frac{1}{t}] \simeq Q \otimes_V V[\frac{1}{t}]$.

Of course, Yoshino's concept $M \leq_{dvr} N$ immediately generalises to the stable category. The only thing to do is to replace the isomorphisms in the module category by isomorphisms in the stable categories. Yoshino formulated this concept for stable categories of maximal Cohen-Macaulay modules over local Gorenstein rings. A local commutative ring A with residue field k is a Gorenstein ring if A is Noetherian with finite injective dimension. In this case an A-module M is Cohen-Macaulay if $Ext^i_A(M, A) = 0$ for all $i > 0$. It is well-known that the stable category of maximal Cohen-Macaulay modules over a local Gorenstein k-algebra is triangulated.

Definition 2. (Yoshino) [14] Let k be a field, and let $(A, \mathfrak{m}, k)$ be a local Gorenstein k-algebra and let M and N be two A-modules. We say M *stably degenerates to* N *along a discrete valuation ring* if there is a discrete valuation k-algebra (V, tV, k) and a maximal Cohen-Macaulay $A \otimes_k V$-module Q, such that

- $M \otimes_k V[\frac{1}{t}] \simeq Q \otimes_V V[\frac{1}{t}]$ in the stable category of maximal Cohen-Macaulay $A \otimes_k V[\frac{1}{t}]$ modules and
- $N \simeq Q/tQ$ in the stable category of maximal Cohen-Macaulay A-modules.

In this case we write $M \leq_{stdvr} N$.

Now, the most striking fact is that this concept implies, and is in some cases actually equivalent to an analogue of Riedtmann-Zwara's characterisation in terms of short exact sequences.

Definition 3.
- Let $\mathcal{A}$ be an abelian category. We say that an object M degenerates to an object N if there is an object Z and a short exact sequence

$$0 \to Z \stackrel{\binom{v}{u}}{\to} Z \oplus M \to N \to 0$$

with a nilpotent endomorphism v of Z. We write $M \leq_{RZ} N$ in this case.
- Let $\mathcal{T}$ be a triangulated category with suspension functor denoted by T. We say that an object M degenerates to an object N if there is an object Z and a distinguished triangle

$$Z \stackrel{\binom{v}{u}}{\to} Z \oplus M \to N \to Z[1]$$

with a nilpotent endomorphism v of Z. We write $M \leq_{\Delta} N$ in this case

Riedtmann and Zwara considered this degeneration for modules M and N over a finite dimensional algebras A over a field k. In this case Fitting's lemma implies that the hypothesis on v to be nilpotent is not necessary, and actually these authors do not assume that v is nilpotent. Up to my knowledge the importance of this nilpotence hypothesis was first observed by Yoshino [13].

Yoshino gave as a main theorem of [13, 14] the following result. Recall that the stable category of maximal Cohen-Macaulay modules over a local Gorenstein k-algebra is triangulated. In particular $\leq_{stdvr}$ and $\leq_{\Delta}$ are both defined for this category.

Theorem 4. • [13] *Let k be a field and let A be a k-algebra. Let M and N be finitely generated A-modules. Then*

$$(M \leq_{dvr} N) \Leftrightarrow (M \leq_{RZ} N)\,.$$

- [14] *Let k be a field, and let $(R, \mathfrak{m}, k)$ be a local Gorenstein k-algebra. Let $\mathcal{T}$ be the stable category of maximal Cohen Macaulay R-modules and let M and N be two objects of $\mathcal{T}$. Then*

$$(\exists_{m,n\in\mathbb{N}} R^m \oplus M \leq_{dvr} R^n \oplus N) \Rightarrow (M \leq_{\Delta} N) \Rightarrow (M \leq_{stdvr} N)\,.$$

Moreover, these three conditions are equivalent if A is artinian.

The implications may be strict in general. Yoshino gave explicit examples for the first implication.

4. The categorical degeneration

We shall now give a generalisation of Yoshino's degeneration concept Definition 2 for the stable category of maximal Cohen-Macaulay modules.

Definition 5. Let k be a commutative ring and let $\mathcal{C}_k^\circ$ be a k-linear triangulated category with split idempotents. A *degeneration data* for $\mathcal{C}_k^\circ$ is given by

- a triangulated category $\mathcal{C}_k$ with split idempotents and a fully faithful embedding $\mathcal{C}_k^\circ \to \mathcal{C}_k$,
- a triangulated category $\mathcal{C}_V$ with split idempotents and a full triangulated subcategory $\mathcal{C}_V^\circ$,
- triangulated functors $\uparrow_k^V : \mathcal{C}_k \to \mathcal{C}_V$ and $\Phi : \mathcal{C}_V^\circ \to \mathcal{C}_k$, such that $(\mathcal{C}_k^\circ) \uparrow_k^V \subseteq \mathcal{C}_V^\circ$, when we view $\mathcal{C}_k^\circ$ as a full subcategory of $\mathcal{C}_k$,
- a natural transformation $\mathrm{id}_{\mathcal{C}_V} \xrightarrow{t} \mathrm{id}_{\mathcal{C}_V}$ of triangulated functors.

These triangulated categories and functors should satisfy the following axioms:

(1) For each object M of $\mathcal{C}_k^\circ$ the morphism $\Phi(M \uparrow_k^V) \xrightarrow{\Phi(t_{M\uparrow_k^V})} \Phi(M \uparrow_k^V)$ is a split monomorphism in $\mathcal{C}_k$.
(2) For all objects M of $\mathcal{C}_k^\circ$ we get $\Phi(\mathrm{cone}(t_{M\uparrow_k^V})) \simeq M$.

All throughout the paper, whenever we have a degeneration data for $\mathcal{C}_k^\circ$ as above, we will see $\mathcal{C}_k^\circ$ as a full subcategory of $\mathcal{C}_k$.

Definition 6. Given two objects M and N of $\mathcal{C}_k^\circ$ we say that M *degenerates to N in the categorical sense* if there is a degeneration data for $\mathcal{C}_k^\circ$ and an object Q of $\mathcal{C}_V^\circ$ such that

$$p(Q) \simeq p(M \uparrow_k^V) \text{ in } \mathcal{C}_V^\circ[t^{-1}] \text{ and } \Phi(\mathrm{cone}(t_Q)) \simeq N,$$

where $p : \mathcal{C}_V^\circ \to \mathcal{C}_V^\circ[t^{-1}]$ is the canonical functor. In this case we write $M \leq_{cdeg} N$.

Remark 7. The functor $\uparrow_k^V$ models $V \otimes_k -$ from Yoshino's attempt. The functor Φ models the forgetful functor which is the identity on objects, i.e. an $A \otimes_k V$-module M is considered as an A-module only. Of course, in the classical situation considered by Yoshino M is not finitely generated anymore. This is the reason why we need to consider the categories $\mathcal{C}_k^\circ$ inside $\mathcal{C}_k$, and $\mathcal{C}_V^\circ$ inside $\mathcal{C}_V$.

The concept $\leq_{cdvr}$ is appropriate, as we shall see in the following result. It gives under certain conditions an equivalence between the geometric, or scheme-theoretical, degeneration $\leq_{cdeg}$ and the algebraic notion of degeneration $\leq_\Delta$.

Theorem 8. *(Saorin and Zimmermann [11]) Let k be a commutative ring.*

(1) *Let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents and let M and N be two objects of $\mathcal{C}_k^\circ$. Then $M \leq_{cdeg} N$ implies that $M \leq_\Delta N$.*
(2) *Suppose that k is a field. Let $\mathcal{C}_k^0$ be the category of compact objects of an algebraic compactly generated triangulated k-category. If $M \leq_\Delta N$, then $M \leq_{cdeg} N$.*

We observe that this is indeed a generalisation of Yoshino's Theorem 4. Moreover, both parts of the theorem are valid for the bounded derived category of A-modules for A being a finite dimensional k-algebra.

The reason why we need the additional hypotheses for the second part of the theorem is to be able to apply a result due to Keller [7, 8]. This result implies that then $\mathcal{C}_k^\circ$ is the subcategory of compact objects of the derived category of a differential graded k-category. For the first part a main difficulty is first to construct the object Z of $\mathcal{C}_k$. This is done by tricky applications of octahedral axioms. Another main difficulty is to show that the object Z is actually in $\mathcal{Z}_k^\circ$, and not only in $\mathcal{C}_k$. This is shown using a result due to May (cf e.g.[15, Lemma 3.4.5] or [9]).

5. Categorial degeneration and triangle functors

We see immediately that if the triangulated category $\mathcal{C}$ is equivalent to the triangulated category $\mathcal{D}$, given by some functor F, then

$$[M \leq_{cdeg} N \Leftrightarrow F(M) \leq_{cdeg} F(N)] \text{ and } [M \leq_\Delta N \Leftrightarrow F(M) \leq_\Delta F(N)].$$

However, if F is not an equivalence the situation is much less clear.

5.1. **The Zwara-like degeneration defined by triangles.** Consider the degeneration $\leq_\Delta$ given by distinguished triangles. Then, it is not difficult to show that this degeneration concept is well-behaved with respect to the image under a triangle functor.

Lemma 9. *Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories and let*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

be a functor of triangulated categories. In particular F sends distinguished triangles to distinguished triangles. Then for all objects M and N we get

$$M \leq_\Delta N \Rightarrow F(M) \leq_\Delta F(N).$$

Proof. Indeed, suppose $M \leq_\Delta N$. Then there exists an object Z such that

$$Z \stackrel{\binom{v}{u}}{\to} Z \oplus M \to N \to Z[1]$$

is a distinguished triangle. We apply F to this triangle, using that the hypothesis on F implies that F preserves finite direct sums, and using again the hypothesis on F, we obtain that

$$F(Z) \stackrel{\binom{F(v)}{F(u)}}{\to} F(Z) \oplus F(M) \to F(N) \to F(Z)[1]$$

is a distinguished triangle by hypothesis on F. Since v is assumed to be nilpotent, $F(v)$ is also nilpotent. Therefore $F(M) \leq_\Delta F(N)$ as claimed. $\square$

However, if $\mathcal{C}$ is a full triangulated subcategory of $\mathcal{D}$, then $M \leq_\Delta N$ in $\mathcal{D}$ does not necessarily imply that $M \leq_\Delta N$ in $\mathcal{C}$. Indeed, the object Z, which is needed for the construction does not need to lie in $\mathcal{C}$.

5.2. **The Yoshino-like degeneration defined by degeneration data.** Contrary to the situation for $\leq_\Delta$ we get for the geometrically inspired degeneration that $M \leq_{cdeg} N$ does not imply necessarily $F(M) \leq_{cdeg} F(N)$.

The degeneration $\leq_{cdeg}$ is well-behaved with respect to a fully faithful embedding of triangulated categories.

Lemma 10. *Let k be a commutative ring, let $\mathcal{C}_k^\circ$ be a triangulated k-category and let M and N be objects of $\mathcal{C}_k^\circ$. Suppose that $\mathcal{D}_k^\circ$ is a triangulated k-category and suppose that $F : \mathcal{D}_k^\circ \to \mathcal{C}_k^\circ$ is a full embedding of triangulated categories. Then $F(M) \leq_{cdeg} F(N)$ implies that $M \leq_{cdeg} N$.*

Proof. By definition we have a degeneration data $\mathcal{C}_k \xrightarrow{\uparrow_k^V} \mathcal{C}_V$ restricting to $\mathcal{C}_k^\circ \xrightarrow{\uparrow_k^V} \mathcal{C}_V^\circ$ and $\mathcal{C}_V^\circ \xrightarrow{\Phi} \mathcal{C}_k$ with an element $t : \mathrm{id}_{\mathcal{C}_V} \to \mathrm{id}_{\mathcal{C}_V}$ in the centre of $\mathcal{C}_V^\circ$. Moreover, we get an object Q of $\mathcal{C}_V^\circ$ such that $\Phi(\mathrm{cone}(t_Q)) \simeq N$ in $\mathcal{C}_k^\circ$ and $p(Q) \simeq p(M \uparrow_k^V)$. Here $\mathcal{C}_V^\circ \xrightarrow{p} \mathcal{C}_V^\circ[t^{-1}]$ is the canonical functor.

But now, we may replace $\uparrow_k^V : \mathcal{C}_k^V \to \mathcal{C}_V^\circ$ by the composition $\uparrow_k^V \circ F : \mathcal{D}_k^\circ \to \mathcal{C}_V^\circ$ and obtain this way a degeneration data for $\mathcal{D}_k^\circ$, maintaining all the other data. The object Q still serves for degeneration in $\mathcal{D}_k^\circ$. $\square$

Interesting is the case when Theorem 8 fully applies, combined with the above lemmas.

Proposition 11. *Let k be a field and let $\mathcal{C}_k^\circ$ be the category of compact objects in an algebraic compactly generated triangulated k-category. If $\mathcal{D}_k^\circ$ is a full triangulated subcategory of $\mathcal{C}_k^\circ$, then for all objects M and N of $\mathcal{D}_k^\circ$ we get that $M \leq_{cdeg} N$ with respect to $\mathcal{D}_k^\circ$ if and only if $M \leq_{cdeg} N$ in $\mathcal{C}_k^\circ$.*

Proof. Suppose $M \leq_{cdeg} N$ with respect to $\mathcal{D}_k^\circ$. Let $\mathcal{D}_k^\circ \xrightarrow{F} \mathcal{C}_k^\circ$ be the embedding functor. Then $M \leq_\Delta N$ in $\mathcal{D}_k^\circ$ by Theorem 8, item 1. By Lemma 9 we obtain $FM \leq_\Delta FN$ in $\mathcal{C}_k^\circ$. But by Theorem 8, item 2 we get that $FM \leq_{cdeg} FN$ with respect to $\mathcal{C}_k^\circ$.

Suppose $FM \leq_{cdeg} FN$ with respect to $\mathcal{C}_k^\circ$. Then Lemma 10 directly gives that $M \leq_{cdeg} N$ with respect to $\mathcal{D}_k^\circ$ $\square$

6. Partial order

A very important property of $\leq_{deg}$ is that it is a partial order on the set of isomorphism classes of finite dimensional A-modules. Yoshino showed that also $\leq_{stdvr}$ has a partial order property. The question if $\leq_\Delta$ is a partial order is not easy, and finiteness conditions are necessary. This is work due to Jensen, Su and Zimmermann [5]. The antisymmetricity in particular uses that if $\mathcal{T}$ is an R-linear triangulated category for a commutative ring R such that $Hom_{\mathcal{T}}(X, Y)$ is of finite length as R-modules for all objects X and Y, then

$M \leq_\Delta N$ implies that $\mathrm{length}_R(Hom_{\mathcal{T}}(X,M)) \leq \mathrm{length}_R(Hom_{\mathcal{T}}(X,N))$ for all objects X, and likewise for $Hom_{\mathcal{T}}(-,X)$. If in addition there is n such that $Hom_{\mathcal{T}}(M,N[n]) = 0$, then $\mathrm{length}_R(Hom_{\mathcal{T}}(X,M)) = \mathrm{length}_R(Hom_{\mathcal{T}}(X,N))$ for all X implies that $M \simeq N$. The proof of this result is an adaption of Bongartz proof in [2].

Theorem 12. *(Jensen, Su, Zimmermann [5]) Let R be a commutative ring and let $\mathcal{T}$ be an R-linear skeletally small triangulated category with split idempotents satisfying for any two objects X, Y of $\mathcal{T}$*

- *we get $\mathrm{length}_R(Hom_{\mathcal{T}}(X,Y)) < \infty$*
- *there is $n_{X,Y} \in \mathbb{Z} \setminus \{0\}$ such that $Hom_{\mathcal{T}}(X, Y[n_{X,Y}]) = 0$*

Then $\leq_\Delta$ is a partial order relation on the set of isomorphism classes of objects in $\mathcal{T}$.

As a last remark I want to mention that in [11] we generalised this result slightly. The price we have to pay there is that we need to consider the transitive hull of $\leq_\Delta$.

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BATALIN-VILKOVISKY ALGEBRAS, TAMARKIN-TSYGAN CALCULUS AND ALGEBRAS WITH DUALITY; THE CASE OF FROBENIUS ALGEBRAS

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Abstract. This note reports on joint work with Thierry Lambre and Guodong Zhou. Let A be a Frobenius algebra with diagonalisable Nakayama automorphism. We exhibit a Tamarkin-Tsygan calculus on the Hochschild cohomology of A and Hochschild homology of A with values in the Nakayama twisted bimodule. Since this pair is an algebra with duality, as introduced by Lambre, these structures define a Batalin-Vilkovisky structure on the cohomology ring of A. We further give an easy and practical criterion when a Frobenius algebra has diagonalisable Nakayama automorphism.

1. Introduction

Hochschild cohomology $HH^*(A)$ and Hochschild homology $HH_*(A, M)$ with values in a bimodule M of an algebra has a very rich structure. First, the Hochschild cohomology is a graded commutative $\mathbb{N}$-graded algebra. Then, Gerstenhaber showed in [9] that the Hochschild cohomology algebra carries a graded Lie algebra structure, where the Lie bracket is graded in the sense $[\ ,\ ] : H^{n+1}(A) \times H^{m+1}(A) \to H^{n+m+1}(A)$. Moreover, these two structures are compatible in the sense that $[\alpha, -]$ is a graded derivation of the multiplicative structure. Structures of this kind are called Gerstenhaber algebras.

The Gerstenhaber bracket is somewhat mysterious and has been determined in only few cases. A nice description in terms of coderivations was given by Stasheff in [21]. If there is a differential Δ of degree -1 of a Gerstenhaber algebra such that the Gerstenhaber bracket is the obstruction of Δ to be a graded derivation of the Hochschild cohomology, then the Gerstenhaber algebra is called a Batalin-Vilkovisky algebra. This structure comes from theoretical physics, more precisely from quantum field theories as explained in e.g. [10].

In representation theory the Batalin-Vilkovisky structure was popularised by Ginzburg [11], where he proves that the Hochschild cohomology of a Calabi-Yau algebra A is a Batalin-Vilkovisky algebra. This result was generalised by Kowalzig and Krähmer to twisted Calabi-Yau algebras, i.e. there is n, such that the n-th syzygy of A as $A \otimes A^{op}$-module is ${}_1A_\alpha$ for some automorphism α of A, provided the twisting automorphism is diagonalisable. In a parallel development Tradler [23] showed that for symmetric algebras (i.e. k-algebras such that the k-linear dual of A is isomorphic to A as $A-A$-bimodule) the Hochschild cohomology also caries the structure of a Batalin-Vilkovisky algebra. In [17] Lambre, Zhou and Zimmermann show that the Hochschild cohomology ring of a

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Frobenius algebra is Batalin-Vilkovisky provided that the Nakayama automorphism is diagonalisable.

We shall report in this note about the various steps to the proof of this result. We will also give a short criterion which implies that the Nakayama automorphism of a Frobenius algebra is diagonalisable.

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2. Batalin-Vilkovisky algebras

We first give the precise definition of a Batalin-Vilkovisky algebra.

Definition 1.
- A *Gerstenhaber algebra* over a field k is the data $(\mathcal{H}^*, \cup, [\ ,\ ])$, where $\mathcal{H}^* = \oplus_{n\in\mathbb{Z}}\mathcal{H}^n$ is a graded k-vector space equipped with two bilinear maps

$$\cup\colon \mathcal{H}^n \times \mathcal{H}^m \to \mathcal{H}^{n+m}, \quad (\alpha,\, \beta) \mapsto \alpha \cup \beta$$

$$[\ ,\ ]\colon \mathcal{H}^{n+1} \times \mathcal{H}^{m+1} \to \mathcal{H}^{n+m+1}, \quad (\alpha,\, \beta) \mapsto [\alpha,\, \beta]$$

  called the cup product $\cup$, and the Lie bracket $[\ ,\ ]$ respectively such that
  - $(\mathcal{H}^*, \cup)$ is a graded commutative associative algebra with unit $1 \in \mathcal{H}^0$,
  - $(\mathcal{H}^*[-1], [\ ,\ ])$ is a graded Lie algebra,
  - for each homogeneous element $\alpha \in \mathcal{H}^*[-1]$ the map $[\alpha, -]$ is a graded derivation of the algebra $(\mathcal{H}^*, \cup)$.
- A Gerstenhaber algebra $(\mathcal{H}^*, \cup, [\ ,\ ])$ is a *Batalin–Vilkovisky algebra* (BV algebra for short) if there is an operator $\Delta\colon \mathcal{H}^* \to \mathcal{H}^{*-1}$ of degree -1 (called a generator of the Gerstenhaber bracket $[\ ,\ ]$) such that $\Delta \circ \Delta = 0$, $\Delta(1) = 0$, and $[\ ,\ ]$ is the obstruction for Δ to be a graded derivation of $(\mathcal{H}^*, \cup)$, i.e.

$$[\alpha, \beta] = (-1)^{|\alpha|+1}(\Delta(\alpha \cup \beta) - \Delta(\alpha) \cup \beta - (-1)^{|\alpha|}\alpha \cup \Delta(\beta)),$$

  for homogeneous elements $\alpha, \beta \in \mathcal{H}^*$.

Remark 2. Batalin-Vilkovisky algebras appeared in mathematical physics. As explained in [27] and [13] the Batalin-Vilkovisky algebra formalism is fully used in the closed string theory. As explained in [13] the Batalin-Vilkovisky structure gives an additional rigidity to the string theory, and a certain number of choices which have to be made in this theory respect this additional structure. More precisely, in string field theory one first chooses a conformal field theory [10, Definition 3.1]. This field theory defines a vector space, the state space, and a field is an element in this vector field. A string field theory action is written as a formal power series with values in the string field. Then, certain choices have to be made, linked to Feynman rules, and the physical observables are independent of these choices. [13] show that the relation between two string field actions arises from field transformations that are canonical with respect to the Lie bracket.

Some algebras have Hochschild cohomology rings which are Batalin-Vilkovisky algebras.

Theorem 3. *(Ginzburg* [11, Theorem 3.4.3]*) Let A be a Calabi-Yau algebra of dimension d. Then the Hochschild cohomology of A has the structure of a Batalin-Vilkovisky algebra.*

Ginzburg is actually much more precise. He constructs the map Δ explicitly, and obtains Δ from the dual of Connes' B-operator on the Hochschild homology complex, and conjugation by the isomorphism $HH^n(A) \simeq HH_{d-n}(A)$ which is deduced from the Calabi-Yau property. He also exhibits already there a connection to a Tamarkin-Tsygan calculus, in the same way as we will explain in Section 3.

In a parallel development Tradler considered more the case of finite dimensional algebras and proved that the Hochschild cohomology of symmetric algebras is a Batalin-Vilkovisky algebra.

Theorem 4. *(Tradler [23]) Let k be a field and let A be a finite dimensional symmetric k-algebra. Then $HH^*(A)$ is a Batalin-Vilkovisky algebra.*

The operator Δ is in this case the k-linear dual of Connes' B-operator, using that for symmetric algebras A we have $HH^n(A) \simeq Hom_k(HH_n(A), k)$ for all $n \in \mathbb{N}$. Note that the isomorphism uses the symmetrising form.

A next step was given by Kowalzig and Krähmer [15]. They generalise Ginzburg's result to a twisted version. For an automorphism α of an algebra A we denote by ${}_1A_\alpha$ the $A-A$-bimodule which is the regular A-module as left-module, but where the action of $a \in A$ from the right is given by multiplication with $\alpha(a)$. An algebra is twisted Calabi-Yau of dimension d if there is a class $\omega \in H_d(A, {}_1A_\alpha)$ such that $\omega_A \cap - : H^*(A, M) \to H_{d-*}(A, {}_1A_\alpha \otimes_A M)$ is an isomorphism (cf [12, Definition 3.6]).

Theorem 5. *(Kowalzig and Krähmer [15]) Let A be a twisted Calabi-Yau algebra of dimension d and twist α. If α acts as diagonalisable automorphism on the vector space A, then $HH^*(A)$ is a Batalin-Vilkovisky algebra.*

Kowalzig and Krähmer obtain in [15] a twisted version of Connes' map B, and use this twisted version to obtain Δ as its dual.

In joint work with Lambre and Zhou we shall be concerned with Frobenius algebras. These play the same role for symmetric algebras as twisted Calabi-Yau algebras do for Calabi-Yau algebras. Indeed, for Frobenius algebras we get an $A-A$-bimodule isomorphism $Hom_k(A, k) \simeq {}_1A_\nu$ for some automorphism ν of A, the Frobenius automorphism. Therefore, the k-linear dual of $HH^n(A)$ is not isomorphic to $HH_n(A)$, but rather to $HH_n(A, {}_1A_\nu)$, where ν is the Nakayama automorphism of A. For more ample details on Frobenius algebras see [26, Sections 1.10 and 4.5].

3. Twisting by automorphisms, the Tamarkin-Tsygan calculus

We shall not give directly the map Δ. Instead we shall prove that some parts of the Hochschild cohomology, together with the Hochschild homology, of a Frobenius algebra carries another important structure: It is a Tamarkin-Tsygan calculus, sometimes also called differential calculus.

Definition 6. A *Tamarkin-Tsygan calculus* is the data of $\mathbb{Z}$-graded vector spaces $\mathcal{H}^*$ and $\mathcal{H}_*$ together with graded bilinear inner laws $\cup$ and $[\ ,\ ]$ of $\mathcal{H}^*$ an a graded operation map $\cap$ of $(\mathcal{H}^*, \cup)$ on $\mathcal{H}_*$ such that

- $(\mathcal{H}^*, \cup, [\ ,\ ])$ is a Gerstenhaber algebra;

- $\mathcal{H}_*$ is a graded module over $(\mathcal{H}^*, \cup)$ via the map $\cap : \mathcal{H}_r \otimes \mathcal{H}^p \to \mathcal{H}_{r-p}, z \otimes \alpha \mapsto z \cap \alpha$ for $z \in \mathcal{H}_r$ and $\alpha \in \mathcal{H}^p$. That is, if we denote $\iota_\alpha(z) = (-1)^{rp} z \cap \alpha$, then $\iota_{\alpha \cup \beta} = \iota_\alpha \iota_\beta$;
- There is a map $B : \mathcal{H}_* \to \mathcal{H}_{*+1}$ such that $B^2 = 0$ and we have
$$L_\alpha \circ \iota_\beta - (-1)^{|\beta|} \iota_\beta \circ L_\alpha = \iota_{[\alpha,\beta]}$$
where we denote $L_\alpha = B \circ \iota_\alpha - (-1)^{|\alpha|} \iota_\alpha \circ B$.

It is not surprising to learn that [8] prove that Hochschild homology and cohomology give a Tamarkin-Tsygan calculus with the natural Gerstenhaber structure and a $\cap$ operation given by evaluation of the first terms of the Hochschild complex by some Hochschild cocycle. This coincides with the classical $\cap$-product well known in Hochschild theory. We note that the $\cap$ product can be defined as well on the action of $HH^*(A)$ on $HH_*(A, M)$ for any $A-A$-bimodule M, but it is not this Tamarkin-Tsygan structure that we use.

Remark 7. It would be nice to extend Stasheff's description [21] of the Gerstenhaber bracket by coderivations to the Tamarkin-Tsygan calculus on Hochschild (co-)homology.

Let α be an automorphism of the algebra A. We now develop the following very general construction. Recall the bar resolution $\mathbb{B}A$. Its degree n homogeneous component is $A^{\otimes n+2}$ and its differential b is given by $b_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}$. It is well-known that this is a free $A \otimes A^{op}$-module resolution of A (cf e.g. [26]). The complex $Hom_{A \otimes A^{op}}(\mathbb{B}A, A)$ has homology $HH^*(A)$ and the homology of $\mathbb{B}A \otimes_{A \otimes A^{op}} {}_1A_\alpha$ is $HH_*(A, {}_1A_\alpha)$.

Observe that the degree n homogeneous component of $\mathbb{B}A \otimes_{A \otimes A^{op}} {}_1A_\alpha$ is isomorphic to $A^{\otimes n}$ and α acts diagonally on this space. Likewise, the degree n homogeneous component of $Hom_{A \otimes A^{op}}(\mathbb{B}A, A)$ is isomorphic to $Hom_k(A^{\otimes n}, k)$.

Since α is an algebra automorphism, $\alpha(1) = 1$ and so 1 is an eigenvalue of the action of α on A. It is easy to see that the eigenspace for the value 1 of the action of α on $Hom_{A \otimes A^{op}}(\mathbb{B}A, A)$, and on $\mathbb{B}A \otimes_{A \otimes A^{op}} {}_1A_\alpha$ respectively, are actually subcomplexes of $Hom_{A \otimes A^{op}}(\mathbb{B}A, A)$, and $\mathbb{B}A \otimes_{A \otimes A^{op}} {}_1A_\alpha$ respectively. Let $HH^*_{(1)}(A)$, respectively $HH^{(1)}_*(A, {}_1A_\alpha)$, be the corresponding homologies of these subcomplexes.

The structural maps $\cup$, $\cap$, $[\ ,\ ]$ do restrict to $HH^*_{(1)}(A)$ and to $HH^{(1)}_*(A, {}_1A_\alpha)$, which can be verified by an easy computations in a few lines.

Theorem 8. *(Lambre-Zhou-Zimmermann* [17]*) With the notation above, there is a degree 1 map β_α of $HH^*_{(1)}(A)$ such that $(HH^*_{(1)}(A), \cup, [\ ,\ ], HH^{(1)}_{-*}(A, {}_1A_\alpha), \cap, \beta_\alpha)$ is a Tamarkin-Tsygan calculus.*

We note that we need to use negative degrees for the homology part in order to get a formally correct calculus. The map β_α is much more tricky to obtain. It is an adaption of Kowalzig-Krähmer's map used in their proof.

4. Algebras with duality; the main result

The proofs we mentioned so far to prove that Hochschild cohomology is a Batalin-Vilkovisky algebra always used both, the Hochschild cohomology and the Hochschild homology, as well as some duality between them. Lambre formalised this in his concept of an algebra with duality.

Definition 9. (Lambre) An algebra with duality is given by $(\mathcal{H}^*, \cup, \mathcal{H}_*, \partial)$, where

- $(\mathcal{H}^*, \cup)$ is a graded commutative unitary algebra with unit $1 \in \mathcal{H}^0$,
- $\mathcal{H}_*$ is a graded vector space and c is an element of $\mathcal{H}_d$,
- ∂ is an isomorphism of vector spaces $\partial : \mathcal{H}_* \to \mathcal{H}^{d-*}$ satisfying $\partial(c) = 1$.

Observe that it is not really necessary to explicitly mention c. The third axiom implicitly defines it as image of 1 under ∂. Now, we come to the link between Tamarkin-Tsygan calculi and Batalin-Vilkovisky structures.

Proposition 10. *Let $(\mathcal{H}^*, \cup, \mathcal{H}_*, c, \partial)$ be an algebra with duality.*

(1) *We suppose that*
  (a) *$(\mathcal{H}^*, \cup, [\ ,\ ], \mathcal{H}_*, \cap, B)$ is a Tamarkin-Tsygan calculus,*
  (b) *the duality ∂ is a homomorphism of $\mathcal{H}^*$-right modules, i.e. we have the relation $\partial(z \cap \alpha) = \partial(z) \cup \alpha$.*

  Then the Gerstenhaber algebra $(\mathcal{H}^*, \cup, [\ ,\ ])$ is a BV-algebra with generator $\Delta = \partial \circ B \circ \partial^{-1}$.

(2) *We suppose that $(\mathcal{H}^*, \cup, [\ ,\ ], \Delta)$ is a BV-algebra with generator Δ. Then posing $B := \partial^{-1} \circ \Delta \circ \partial$ and $z \cap \alpha := \partial^{-1}(\partial(z) \cup \alpha)$, the data $(\mathcal{H}^*, \cup, [\ ,\ ], \mathcal{H}_*, \cap, B)$ is a Tamarkin-Tsygan calculus.*

If α acts as diagonalisable automorphism on A, then $Hom_{A\otimes A^{op}}(\mathbb{B}A, A)$ and $\mathbb{B}A \otimes_{A\otimes A^{op}} {}_1A_\alpha$ both decompose as a direct sum of eigenspace subcomplexes. Note however that we may get eigenvalues for the complexes which do not occur as eigenvalues for the action on A. This comes from the fact that if $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ is an eigenspace decomposition, then

$$A^{\otimes n} = \bigoplus_{(\lambda_1,\cdots,\lambda_n)\in\Lambda^n} A_{\lambda_1} \otimes \cdots \otimes A_{\lambda_n}.$$

The automorphism α acts on $A_{\lambda_1} \otimes \cdots \otimes A_{\lambda_n}$ with the eigenvalue $\lambda_1 \cdot \cdots \cdot \lambda_n$. Therefore if Λ is the set of eigenvalues of α, then the Hochschild complex decomposes as direct sum of subcomplexes which are eigenspaces for some $\lambda \in \langle\Lambda\rangle$, where $\langle\Lambda\rangle$ is the submonoid of the multiplicative group $k^\times$ of the base field generated by Λ. This decomposition is also the point where we use that α acts on A as diagonalisable automorphism.

Moreover, we get the most important formula on $\mathbb{B}A \otimes_{A\otimes A^{op}} {}_1A_\alpha$:

$$b \circ \beta_\alpha + \beta_\alpha \circ b = 1 - T$$

where T is the diagonal map of α on $A^{\otimes n}$ for each n, where b denotes the Hochschild differential and where β_α is defined in Theorem 8. Hence, only for the eigenspace of α for the eigenvalue 1 the corresponding subcomplex is not homotopic to 0. This shows

Proposition 11. *If α is diagonalisable, then $HH_*^{(1)}(A, {}_1A_\alpha) = HH_*(A, {}_1A_\alpha)$.*

We are almost done. Now suppose that A is a Frobenius algebra with Nakayama automorphism ν and consider the case $\alpha = \nu$. Then Theorem 8 and Proposition 11 provide a Tamarkin-Tsygan calculus on the Hochschild cohomology of a Frobenius algebra and the homology with values in the Nakayama twisted bimodule. Since

$$Hom_k(HH_n(A, {}_1A_\nu), k) \simeq HH^n(A)$$

we easily get an algebra with duality satisfying the hypotheses of the first part of Proposition 10. This shows

Theorem 12. *(Lambre, Zhou, Zimmermann* [17]*) Let k be a field and let A be a Frobenius k-algebra with diagonalisable Nakayama automorphism. Then $HH^*(A)$ is a Batalin-Vilkovisky algebra.*

Remark 13. Volkov obtained in [24] independently and at the same time a similar result by exhibiting the operator Δ by explicit computation on the Hochschild cocycles.

Remark 14. Let $\overline{k}$ be the algebraic closure of k and let $\overline{A} := \overline{k} \otimes_k A$. If A is a Frobenius k-algebra, then $\overline{A}$ is a Frobenius $\overline{k}$-algebra. We actually only need that the Nakayama automorphism of $\overline{A}$ acts as diagonalisable automorphism on $\overline{A}$.

5. Diagonalisable Nakayama automorphism

We are left with the question how we may verify when a Nakayama automorphism is diagonalisable. There is an easy case: If A is a Frobenius k-algebra and ν is of finite order n. Then the action of ν on A is a representation of the cyclic group of order n, and if n is invertible in k, then this group ring is semisimple. Hence, for large enough fields k with $nk = k$ we have that the action of ν is diagonalisable. This happens for example for finite dimensional Hopf algebras by a result of Radford [19] in combination with a result by Larson-Sweedler [18]. Also preprojective algebras of Dynkin type have this property. For quantum complete intersections it can be shown by a direct computation that there also we get a diagonalisable Nakayama automorphism.

What about more general basic Frobenius algebras? Consider basic algebras and let hence $A = kQ/I$ be a finite dimensional Frobenius algebra given by quiver with relations. We can choose a basis $\mathcal{B}$ of A consisting of paths which also contains a basis for the socle of each indecomposable projective A-module. Then by [14, Proposition 2.8], there is a natural choice of the defining bilinear form $\langle a, b\rangle = tr(ab)$ for $a, b \in A$ induced by the trace map

$$tr : A \to k, \quad p \in \mathcal{B} \mapsto \begin{cases} 1 & \text{if } p \in soc(A) \cap \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

Then we show the following useful

Proposition 15. *(Lambre, Zhou, Zimmermann* [17]*) Assume that the basis $\mathcal{B}$ satisfies two further conditions:*

1. *for arbitrary two paths $p, q \in \mathcal{B}$, there exist another path $r \in \mathcal{B}$ and a constant $\lambda \in k$ such that $p \cdot q = \lambda r \in A$*
2. *for each path $p \in \mathcal{B}$, there exists a unique element $p^* \in \mathcal{B}$ such that $0 \neq p \cdot p^* \in soc(A)$*

If k is an algebraically closed field of characteristic 0 or of characteristic p with p strictly bigger than the number of arrows of Q. Then the two conditions (1) and (2) imply that the Nakayama automorphism of A is semisimple and the Hochschild cohomology of A is a BV algebra.

By a classification result of Asashiba [1] we get

Lemma 16. *Each self-injective algebra of finite representation type is Morita equivalent to an algebra kQ/I given by a quiver Q modulo admissible relations I verifying the conditions (1) and (2).*

An alternative proof can be given by the fact that each representation-finite algebra has a multiplicative basis (cf. [2]).

Lemma 17. *Basic special biserial algebras satisfy the hypotheses of Proposition 15.*

Finally, we were looking at algebras of polynomial growth. These were studied by Holm, Skowroński, Bocian, Białkowsky for a classification up to derived equivalences, and by Zhou and Zimmermann [25] up to stable equivalences, clearing also a few remaining cases in the derived equivalence classification. Also there we can show that almost all the cases satisfy the hypotheses of Proposition 15. The few remaining situations can be done by an elementary computation on the quiver, using the construction of Holm-Zimmermann [14] mentioned above.

We finish by mentioning that an easy computation shows that for a field k of characteristic 2 the self-injective Nakayama algebra with two simples and Loewy length 4 does not have a semisimple Nakayama automorphism action. The quiver of this Nakayama algebra has two arrows such that Lemma 16 shows that the hypothesis in Proposition 15 on the characteristic of the base field is indeed necessary.

Remark 18. I want to mention that the formula for the Frobenius bilinear form given by [14] was originally used to classify deformed preprojective algebras ([4], see also [5] for a rectification in case of type E) of type L_n up to derived equivalence. This was done using the so-called Külshammer structure, an additional structure on the degree 0 Hochschild homology of an algebra [3], linked to the p-power map. In joint work with Sorlin [20] we extended the classification to deformed preprojective algebras of type D_n. For the precise and somewhat technical definition of the deformation parameter see [4, Proposition 6.2] or [5, Example 10.6]. We computed the degree 0 Hochschild homology of deformed preprojective algebras of type D_n and showed that over an algebraically closed field the deformed preprojective algebra is never derived equivalent to the non deformed preprojective algebra. Indeed, the dimension of the degree 0 Hochschild homology of the deformed preprojective algebra with deformation parameter k is at most $n+2+k$ for $k \leq n-3$ whereas this dimension is $3n$ in the non-deformed case.

The preprojective algebras of generalised Dynkin type are also interesting with respect to the Tamarkin-Tsygan structure on the Hochschild (co-)homology. Ching-Hwa Eu computed this explicitly (cf [6, 7]).

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