The Grothendieck groups of mesh algebras

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September 10th, 2015

Self-introduction

- Researching on the representation theory of finite-dimensional algebras
  - Derived equivalence
  - Stable equivalence for self-inj. algebras
  - Two derived eq. self-inj. algebras are stable eq. [Rickard]
- Classifying fin.-dim. mesh algebras by stable equivalences in recent

I have calculated the Grothendieck groups of the stable categories of all fin.-dim. mesh algebras

- An important invariant under stable eq's
- Today's theme

Notations

- K: a field
- Λ: a fin.-dim. basic self-inj. K-algebra
- $\operatorname{mod}\Lambda$: the cat. of fin. gen. (right) Λ-modules
  - An abelian Frobenius category
- $\operatorname{proj}\Lambda = \{P \in \operatorname{mod}\Lambda \mid P\text{: projective}\}$
- $\underline{\operatorname{mod}}\,\Lambda = \operatorname{mod}\Lambda/\operatorname{proj}\Lambda$: the stable category
  - A triangulated category
- $P_i = e_i\Lambda$, $I_i = D(\Lambda e_i)$, $S_i = \operatorname{top} P_i = \operatorname{soc} I_i$,
  - $1_\Lambda = e_1 + \cdots + e_m$ (primitive orthogonal idempotents)
- ν: Nakayama permutation ($P_i \cong I_{\nu(i)}$)

The Grothendieck group $K_0(C)$

- $K_0(C)$: An abelian group for a triangle cat. C
  - An important invariant under triangle eq's
- Generated by the set of isomorphic classes
- If $X \to Y \to Z \to X[1]$ is a triangle in C, $[X] - [Y] + [Z] = 0$ in $K_0(C)$

- $[S_1], \ldots, [S_m]$: a $\mathbf{Z}$-basis of $K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))$
- $[P_1], \ldots, [P_m]$: a $\mathbf{Z}$-basis of $K_0(K^{\mathrm{b}}(\mathrm{proj}\,\Lambda))$
- $K_0(\underline{\mathrm{mod}}\,\Lambda) \cong \dfrac{K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))}{\langle [P_1], \ldots, [P_m] \rangle}$
  - By $\underline{\mathrm{mod}}\,\Lambda \cong \dfrac{D^{\mathrm{b}}(\mathrm{mod}\,\Lambda)}{K^{\mathrm{b}}(\mathrm{proj}\,\Lambda)}$ [Rickard]

Mesh algebras (categories)

- Constructed by a stable translation quiver
  - (Q, τ) with $Q = (Q_0, Q_1)$ quiver and $\tau \in \operatorname{Aut} Q_0$, such that $\# Q_1(y \to x) = \# Q_1(\tau x \to y)$
- Able to recover many derived categories from their AR quivers

Theorem [Riedtmann]

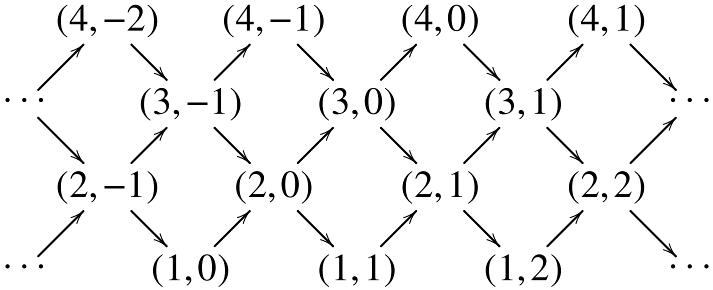
- All fin.-dim. mesh algebras are given by

$$\mathbf{Z}\Delta/G \quad \begin{pmatrix} \Delta\text{: a Dynkin diagram} \\ \mathbf{Z}\Delta\text{: infinite trans. quiver} \\ G \subset \operatorname{Aut} \mathbf{Z}\Delta\text{: an adm. subgroup} \end{pmatrix}$$

and they're all self-inj.

Example ($\mathbb{Z}A_4$: infin. trans. quiver)

Let A_4 oriented as $1 \to 2 \to 3 \to 4$, then $\mathbb{Z}A_4$ is



$\tau(i, a) = (i, a-1)$

The longest nonzero paths: $(i, a) \rightsquigarrow (5-i, a+i-1)$

(considering the mesh relations)

- The mesh cat. of $\mathbb{Z}A_4$ can recover $D^{\mathrm{b}}(\operatorname{mod} KA_4)$

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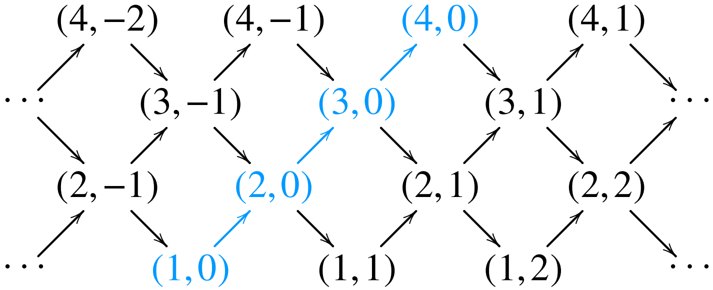

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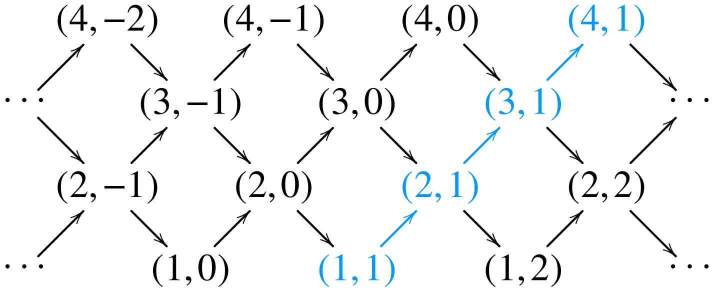

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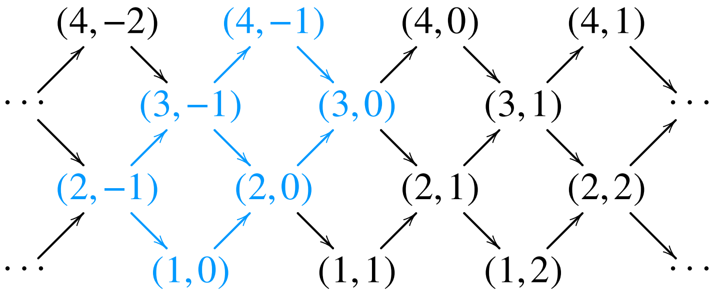

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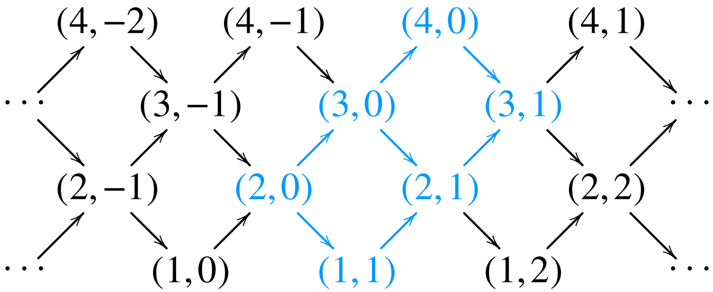

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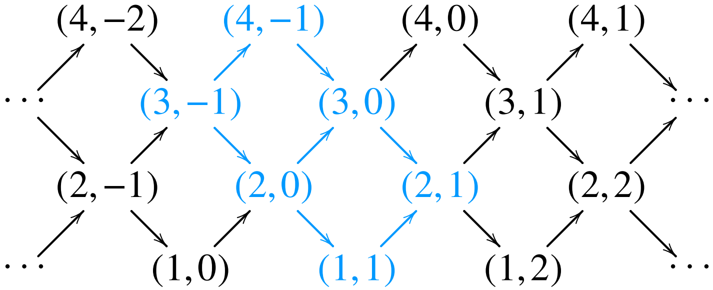

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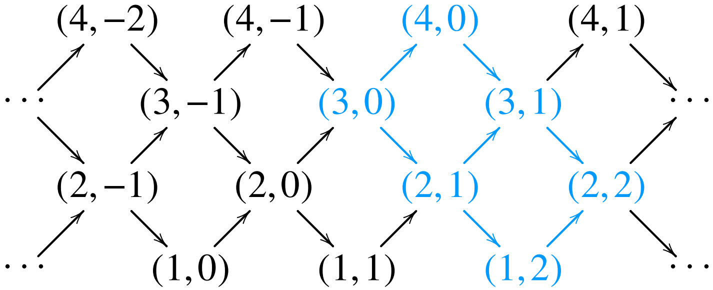

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The longest nonzero paths: $(i, a) \rightsquigarrow (5 - i, a + i - 1)$
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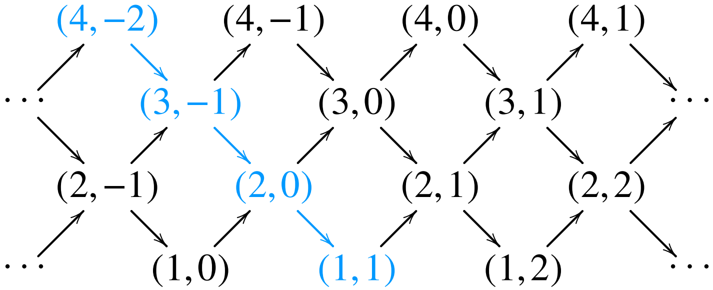

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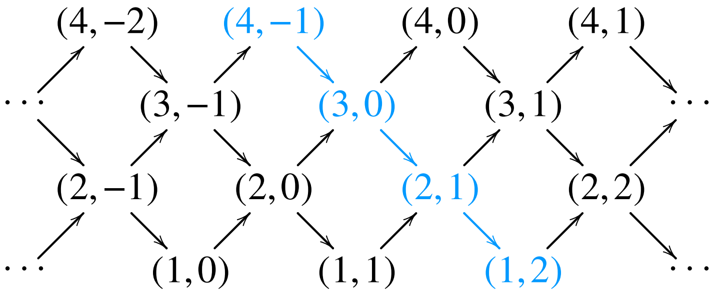

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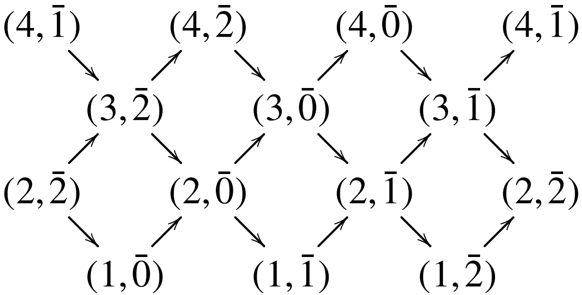
Example ($\mathbb{Z}A_4/\langle\tau^3\rangle$)

Identify the vertices on each τ^3-orbit on $\mathbb{Z}A_4$, then



$$\tau(i, a + 3\mathbb{Z}) = (i, a - 1 + 3\mathbb{Z}),$$
$$\nu(i, a + 3\mathbb{Z}) = (5 - i, a + i - 1 + 3\mathbb{Z})$$

- $\mathbb{Z}A_n/\langle\tau^k\rangle$ gives a fin.-dim. self-inj. mesh alg. with $\nu(i, a + k\mathbb{Z}) = (n + 1 - i, a + i - 1 + k\mathbb{Z})$

Main theorem ([A])

Let

- $Q = Q_{n,k} := \mathbf{Z}A_n/\langle \tau^k \rangle$
- $\Lambda_{n,k}$: the mesh algebra given by $Q_{n,k}$
- $d := \gcd(n+1, k)$
- $r := (n+1)/d$

Then $K_0(\underline{\mathrm{mod}}\, \Lambda_{n,k})$ is isomorphic to

$$\begin{cases} \mathbf{Z}^{(nd-3d+2)/2} \oplus (\mathbf{Z}/2\mathbf{Z})^{d-1} & (r \in 2\mathbf{Z}) \\ \mathbf{Z}^{(nd-2d+2)/2} & (r \notin 2\mathbf{Z}) \end{cases}$$

(The results for the other mesh alg's are stated later)

Key lemma ([A])

Let

- A_n numbered as $1 \to 2 \to \cdots \to n$
- $Q_0 = (Q_{n,k})_0 = \{1, \ldots, n\} \times (\mathbf{Z}/k\mathbf{Z})$
- $H, H', H'' \subset K_0(D^{\mathrm{b}}(\mathrm{mod}\, \Lambda_{n,k}))$ be

$$\begin{aligned} H &:= \langle [P_x] \mid x \in Q_0 \rangle \\ H' &:= \langle [S_x] + [S_{\nu\tau^{-1}x}] \mid x \in Q_0 \rangle \\ H'' &:= \langle [P_x] \mid x \in \{1\} \times (\mathbf{Z}/k\mathbf{Z}) \rangle \subset H \end{aligned}$$

Then $H = H' + H''$ and thus

$$K_0(\underline{\mathrm{mod}}\, \Lambda_{n,k}) \cong \frac{K_0(D^{\mathrm{b}}(\mathrm{mod}\, \Lambda_{n,k}))}{H' + H''}$$

Proof of $H' \subset H$

A projective resolution of $\Lambda_{n,k}$-module S_x is given by

$$0 \to S_{\nu\tau^{-1}x} \to P_{\tau^{-1}x} \to \bigoplus_{y \in x^+} P_y \to P_x \to S_x \to 0$$

- x^+: the set of direct successor of x in $Q_{n,k}$
- Induced by a projective resolution of $\Lambda_{n,k}$ as $\Lambda_{n,k}$-$\Lambda_{n,k}$-bimodule [Dugas]
- Also efficient for $\mathbf{Z}\Delta/\langle\tau^k\rangle$ (Δ: Dynkin)

and thus

$$[S_x] + [S_{\nu\tau^{-1}x}] = [P_{\tau^{-1}x}] - \sum_{y \in x^+} [P_y] + [P_x] \in H$$

Proof of $H \subset H' + H''$

Assume $k = 1$ ($Q_0 = \{1, \ldots, n\}$) first

- We'll show $[P_i] \in H' + H''$ by induction on i
- $i = 1$: clear
- $i = 2, \ldots, n$: Put $x = i - 1$ then

$$[S_x] + [S_{\nu\tau^{-1}x}] = [P_{\tau^{-1}x}] - \sum_{y \in x^+} [P_y] + [P_x]$$
$$= [P_{i-1}] - ([P_{i-2}] + [P_i]) + [P_{i-1}]$$

and induction hypothesis leads to $[P_i] \in H' + H''$ ($P_0 := 0$)

A similar proof holds even if $k \neq 1$

The expression of $K_0(\underline{\mathrm{mod}}\,\Lambda_{n,k})$

$$K_0(\underline{\mathrm{mod}}\,\Lambda_{n,k}) \cong \mathrm{Cok}\left(1_{nk} + T_n(X_k) \quad U_n(X_k)\right)$$

- $X_k \in \mathrm{GL}_k(\mathbf{Z})$: the perm. matrix of $(1, 2, \ldots, k)$
- $T_n(x) \in \mathrm{Mat}_{n,n}(\mathbf{Z}[x])$, $U_n(x) \in \mathrm{Mat}_{n,1}(\mathbf{Z}[x])$ are

$$T_n(x) = \begin{pmatrix} & & & x^n \\ & & \cdots & \\ & x^2 & & \\ x & & & \end{pmatrix}, \quad U_n(x) = \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix}$$

- The columns of $1_{nk} + T_n(X_k), U_n(X_k)$ correspond to the generators of H', H''
  - the $(i-1)k + (a+1)$th row $\rightsquigarrow [S_{i,a+k\mathbf{Z}}]$ $(i \in \{1, \ldots, n\}, a \in \{0, \ldots, k-1\})$

Example for $n = 7$

$\begin{pmatrix} 1_7 + T_7(x) & U_7(x) \end{pmatrix}$ is transformed into

$$\begin{pmatrix} 1 & & & & & & x^7 & 1 \\ & 1 & & & & x^6 & & 1 \\ & & 1 & & x^5 & & & 1 \\ & & & 1 + x^4 & & & & 1 \\ & & x^3 & & 1 & & & 1 \\ & x^2 & & & & 1 & & 1 \\ x & & & & & & 1 & 1 \end{pmatrix}$$

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& 1 & & & & & & \\
& & 1 & & & & & \\
& & & 1 + x^4 & & & & 1 \\
& & & & 1 - x^8 & & & 1 - x^3 \\
& & & & & 1 - x^8 & & 1 - x^2 \\
& & & & & & 1 - x^8 & 1 - x
\end{pmatrix}$$

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Example for $n = 7$

$\begin{pmatrix} 1_7 + T_7(x) & U_7(x) \end{pmatrix}$ is transformed into

$$\begin{pmatrix}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & & 0 & & & & 1 \\
 & & & & 1-x^8 & & & \\
 & & & & & 1-x^8 & & \\
 & & & -(1-x)(1+x^4) & & & 1-x^8 & 0
\end{pmatrix}$$

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& & & & & 1 - x^8 & & \\
& & & -(1-x)(1+x^4) & & & 0 & 0
\end{pmatrix}$$

Thus $\mathrm{Cok} \begin{pmatrix} 1_{7k} + T_7(X_k) & U_7(X_k) \end{pmatrix}$ is isomorphic to

$$(\mathrm{Cok}(1 - X_k^8))^2 \oplus \mathrm{Cok}((1 - X_k)(1 + X_k^4))$$

Example for $n = 6$

$\begin{pmatrix} 1_6 + T_6(x) & U_6(x) \end{pmatrix}$ is transformed into

$$\begin{pmatrix}
1 & & & & & x^6 & 1 \\
 & 1 & & & x^5 & & 1 \\
 & & 1 & x^4 & & & 1 \\
 & & x^3 & 1 & & & 1 \\
 & x^2 & & & 1 & & 1 \\
x & & & & & 1 & 1
\end{pmatrix}$$

Example for $n = 6$

$\begin{pmatrix} 1_6 + T_6(x) & U_6(x) \end{pmatrix}$ is transformed into

$$\begin{pmatrix} 1 & & & & & & & & & & & \\ & 1 & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ & & & 1-x^7 & & & & & & & & 1-x^3 \\ & & & & 1-x^7 & & & & & & & 1-x^2 \\ & & & & & 1-x^7 & & & & & & 1-x \end{pmatrix}$$

Example for $n = 6$

$\begin{pmatrix} 1_6 + T_6(x) & U_6(x) \end{pmatrix}$ is transformed into

$$\begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 - x^7 &&& \\ &&&& 1 - x^7 && \\ &&&&& 1 - x^7 & 1 - x \end{pmatrix}$$

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Thus $\operatorname{Cok} \begin{pmatrix} 1_{6k} + T_6(X_k) & U_6(X_k) \end{pmatrix}$ is isomorphic to

$$(\operatorname{Cok}(1 - X_k^7))^2 \oplus \operatorname{Cok}(1 - X_k)$$

Conclusion

For $n \geq 2$, $\mathrm{Cok}\left(1_{nk} + T_n(X_k) \quad U_n(X_k)\right)$ is iso. to

$$\begin{cases} (\mathrm{Cok}(1_k - X_k^{n+1}))^{\frac{n-3}{2}} \oplus \mathrm{Cok}((1_k - X_k)(1_k + X_k^{\frac{n+1}{2}})) & (n \notin 2\mathbf{Z}) \\ (\mathrm{Cok}(1_k - X_k^{n+1}))^{\frac{n-2}{2}} \oplus \mathrm{Cok}(1_k - X_k) & (n \in 2\mathbf{Z}) \end{cases}$$

and the main theorem is proved by using

$$\mathrm{Cok}(1_k - X_k) \cong \mathbf{Z}, \quad \mathrm{Cok}(1_k - X_k^{n+1}) \cong \mathbf{Z}^d$$

and if $n \notin 2\mathbf{Z}$,

$$\mathrm{Cok}((1_k - X_k)(1_k + X_k^{\frac{n+1}{2}})) \cong \begin{cases} \mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})^{d-1} & (r \in 2\mathbf{Z}) \\ \mathbf{Z}^{\frac{d+2}{2}} & (r \notin 2\mathbf{Z}) \end{cases}$$

(Recall $d = \gcd(n+1, k)$, $r = (n+1)/d$)

Results for A_n ([A])

quiver	condition	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/4$
$\mathbf{Z}A_n/\langle\tau^k\rangle$	r: even	$(nd-3d+2)/2$	$d-1$	
	r: odd	$(nd-2d+2)/2$		
$\mathbf{Z}A_n/\langle\tau^k\psi\rangle$	$r/2$: even	$(nd-3d)/2$	$d-1$	1
	$r/2$: odd		$nd-2d+1$	
	r: odd	$(nd-d)/4$		
$\mathbf{Z}A_n/\langle\tau^k\varphi\rangle$			$nd-2d+1$	

- $\mathbf{Z}A_n/\langle\tau^k\psi\rangle$ (n: odd, $\psi^2 = \mathrm{id}$)
- $\mathbf{Z}A_n/\langle\tau^k\varphi\rangle$ (n: even, $\varphi^2 = \tau^{-1}$)
- $d = \begin{cases} \gcd(n+1, 2k-1)/2 & (\mathbf{Z}A_n/\langle\tau^k\varphi\rangle) \\ \gcd(n+1, k) & (\text{otherwise}) \end{cases}$
- $r = (n+1)/d$

Results for D_n ([A])

quiver	condition	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/r$
$\mathbf{Z}D_n/\langle\tau^k\rangle$	k: even, r: even	$d-1$	$nd-3d$	1
	k: even, r: odd	$(nd-d-2)/2$		1
	k: odd, $r/2$: even	d	$nd-3d$	
	k: odd, $r/2$: odd		$nd-d-1$	
$\mathbf{Z}D_n/\langle\tau^k\psi\rangle$	k: even, $r/2$: even	d	$nd-3d$	
	k: even, $r/2$: odd		$nd-d-1$	
	k: even, r: odd	$(nd-2d)/2$		
	k: odd	$d-1$	$nd-3d$	1

- $\mathbf{Z}D_n/\langle\tau^k\psi\rangle$ ($\psi^2 = \mathrm{id}$)
- $d = \gcd(2n-2, k)$
- $r = (2n-2)/d$

Results for D_4, E_6 ([A])

quiver	condition	$\mathbf{Z}$	$\mathbf{Z}/2$
$\mathbf{Z}D_4/\langle\tau^k\chi\rangle$	k: even	4	
	k: odd		4

- $\mathbf{Z}D_4/\langle\tau^k\chi\rangle$ ($\chi^3 = \mathrm{id}$)

quiver	condition	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/4$
$\mathbf{Z}E_6/\langle\tau^k\rangle$	$d = 1, 3$	$d + 1$	$d + 1$	$d - 1$
	$d = 2, 6$	$(3d + 2)/2$	$(3d + 2)/2$	
	$d = 4, 12$	$(9d + 12)/4$		
$\mathbf{Z}E_6/\langle\tau^k\psi\rangle$	$d = 1, 3$	$2d$	$d + 1$	
	$d = 2, 6$		$(9d + 6)/2$	
	$d = 4, 12$	$(3d + 4)/2$		

- $\mathbf{Z}E_6/\langle\tau^k\psi\rangle$ ($\psi^2 = \mathrm{id}$)
- $d = \gcd(12, k)$

Results for E_7, E_8 ([A])

quiver	condition	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/3$
$\mathbf{Z}E_7/\langle\tau^k\rangle$	$d = 1$		6	
	$d = 3, 9$		$6d + 2$	
	$d = 2$	6		1
	$d = 6, 18$	$3d + 2$		

- $d = \gcd(18, k)$

quiver	condition	$\mathbf{Z}$	$\mathbf{Z}/2$
$\mathbf{Z}E_8/\langle\tau^k\rangle$	$d = 1, 3, 5$		$8d$
	$d = 15$		112
	$d = 2, 6, 10$	$4d$	
	$d = 30$	112	

- $d = \gcd(30, k)$

Thank you for your attention.