On the Hochschild cohomology ring modulo nilpotence of the quiver algebra with quantum-like relations

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Introduction

Notation

- K: field, $\mathrm{char}K = 0$,
- A: finite dimensional K-algebra,
- $A^e := A \otimes_K A^{\mathrm{op}}$: enveloping algebra,
- $\mathrm{HH}^n(A) \simeq \mathrm{Ext}^n_{A^e}(A, A)$: n-th Hochschild cohomology group of A,
- $\mathrm{HH}^*(A) \simeq \oplus_{n\geq 0}\mathrm{HH}^n(A)$: Hochschild cohomology ring of A with Yoneda product,
- $\mathcal{N}$: ideal of $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements.
- $\mathrm{HH}^*(A)/\mathcal{N}$: Hochschild cohomology ring of A modulo nilpotence.

Hochschild cohomology group

Hochschild cohomology group

Let $\mathbb{P}$ be the projective A-bimodule resolution of A. Applying $\mathrm{Hom}_{A^e}(-, A)$ to $\mathbb{P}$, we have the following complex $\mathrm{Hom}_{A^e}(\mathbb{P}, A)$:

$$\mathrm{Hom}_{A^e}(P_0, A) \to \mathrm{Hom}_{A^e}(P_1, A) \to \mathrm{Hom}_{A^e}(P_2, A) \to \cdots .$$

Then, the n-th Hochschild cohomology group is given by n-th cohomology of $\mathrm{Hom}_{A^e}(\mathbb{P}, A)$.

$$\mathrm{HH}^n(A) \simeq \mathrm{Ext}^n_{A^e}(A, A) = \mathrm{Ker}\,\mathrm{Hom}_{A^e}(P_{n+1}, A)/\mathrm{Im}\,\mathrm{Hom}_{A^e}(P_n, A).$$

The support variety of an A-module M

- M: A-module.
- ϕ_M: $\mathrm{HH}^*(A) \xrightarrow{-\otimes_A M} \mathrm{Ext}_A^*(M, M)$ is a homomorphism of graded rings for an A-module M.
- $\mathrm{Ext}_A^*(M, M)$ is an $\mathrm{HH}^*(A)$-module.

Definition [[Snashall,Solberg (2004)], Definision 3.3]

The support variety of M is given by

$$V(M) = \{m \in \mathrm{MaxSpec}\,\mathrm{HH}^*(A)/\mathcal{N} | \mathrm{AnnExt}_A^*(M, M) \subseteq m'\}$$

where $\mathrm{AnnExt}_A^*(M, M)$ is the annihilator of $\mathrm{Ext}_A^*(M, M)$ and m' is the preimage in $\mathrm{HH}^*(A)$ of the ideal m in $\mathrm{HH}^*(A)/\mathcal{N}$.

Properties of support varieties

Snashall and Solberg showed the following properties.

Theorem [SnSo(2004)]

1. $V(M_1 \oplus M_2) = V(M_1) \cup V(M_2)$,
2. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then $V(M_{i_1}) \subseteq V(M_{i_2}) \cup V(M_{i_3})$ whenever $\{i_1, i_2, i_3\} = \{1, 2, 3\}$,
3. If $\mathrm{Ext}^i_A(M, M) = (0)$ for $i \gg 0$ or the projective or the injective dimension of M is finite, then $V(M)$ is trivial.

Question [Snashall(2009)]

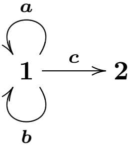
Whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as an algebra?

With respect to sufficient conditions, it is shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra for various classes of algebras by many authors as follows:

- **Any block of a group ring of a finite group (See [Evens(1961)], [Venkov(1959)])**
- **Finite dimensional algebras of finite global dimension (See [Happel(1989)])**
- **Finite dimensional self-injective algebras of finite representation type over an algebraically closed field (See [Green, Snashall, Solberg(2003)])**
- **Finite dimensional monomial algebras (See [Green, Snashall, Solberg(2006)])**
- **A class of special biserial algebras (See [Snashall, Taillefer(2010)])**

Counter example of Snashall-Solberg conjecture [Xu(2008)], [Snashall(2009)]

Let $A = kQ/I$ where Q is the quiver



and $I = \langle a^2, b^2, ab - ba, ac \rangle$. Snashall showed the following Theorem.

[Sn(2009), Theorem 4.5]

1. $\mathrm{HH}^*(A)/\mathcal{N} \cong \begin{cases} k \oplus k[a,b]b & \text{if } \mathrm{char} k = 2, \\ k \oplus k[a^2, b^2]b^2 & \text{if } \mathrm{char} k \neq 2. \end{cases}$
2. $\mathrm{HH}^*(A)/\mathcal{N}$ is not finitely generated as an algebra.

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Quiver algebra with quantum-like relation

c-th quantum complete intersection [Oppermann(2010)]

Let c and n_i be integers with $c \geq 2$ and $n_i \geq 2$ for $1 \leq i \leq c$. Let I be an ideal in $K\langle x_1, \ldots, x_c \rangle$ generated by

$$x_i^{n_i} \quad \text{for } 1 \leq i \leq c,\ x_j x_i - q_{i,j} x_i x_j \quad \text{for } 1 \leq i < j \leq c,$$

where $q_{i,j}$ is non-zero element in K for $1 \leq i < j \leq c$.
$A = K\langle x_1, \ldots, x_n \rangle / I$ is a quantum complete intersection. Then we have $\mathrm{HH}^*(A)/\mathcal{N}$ as follows.

Theorem [Oppermann(2010) Theorem 5.5]

$\mathrm{HH}^*(A)/\mathcal{N}$ is isomorphic to the following finitely generated K-algebra.

$$\begin{aligned}\mathrm{HH}^*(A)/\mathcal{N} \cong {}_K\langle y_1^{p_1 n_1/2} \cdots y_c^{p_c n_c/2} \in K[y_1, \ldots, y_c] \mid \\ \prod_{j=1}^{c} q_{i,j}^{p_j n_j/2} = 1 \text{ for all } i \text{ with } p_i \text{ even}, \\ \prod_{j=1}^{c} q_{i,j}^{(p_j - 1) n_j/2 + 1} = -1 \text{ and } n_i = 2 \text{ for all } i \text{ with } p_i \text{ odd} \rangle.\end{aligned}$$

where $q_{i,i} = 1$ and $q_{i,j} = q_{j,i}^{-1}$ for $1 \leq j < i \leq c$.
Then $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra.

Projective resolution of a quantum complete intersection

We consider the case of $c = 2$ and $q_{1,2} = 1$.
Let $A_i = K[x_i]/(x_i^{n_i})$ for $1 \leq i \leq 2$. Then the projective bimodule resolution of A_i is

$$\mathbb{P}_i : A_i^e \xleftarrow{d_{(i,1)}} A_i^e \xleftarrow{d_{(i,2)}} A_i^e \xleftarrow{d_{(i,3)}} A_i^e \xleftarrow{d_{(i,4)}} \cdots ,$$

where

$$d_{(i,j)} : 1 \otimes 1 \mapsto 1 \otimes x_i - x_i \otimes 1 \qquad \text{if } j \text{ is odd,}$$
$$1 \otimes 1 \mapsto \sum_{k=0}^{n_i - 1} x_i^k \otimes x_i^{n_i - 1 - k} \qquad \text{if } j \text{ is even.}$$

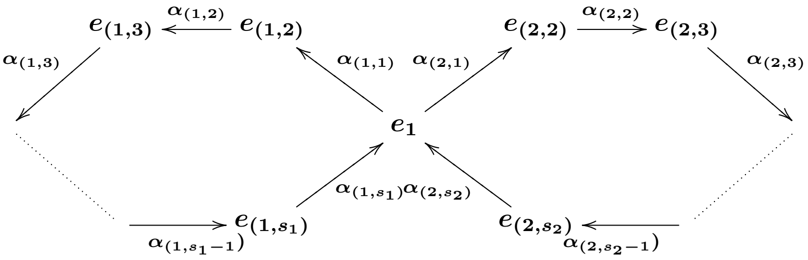
Then the projective bimodule resolution of the quantum complete intersection A is the total complex of the following commutative diagram.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A^e & \xleftarrow{d_{(1,1)}} & A^e & \xleftarrow{d_{(1,2)}} & A^e \longleftarrow \cdots \\
 & & \downarrow{\scriptstyle d_{(2,2)}} & & \downarrow{\scriptstyle -d_{(2,2)}} & & \downarrow{\scriptstyle d_{(2,2)}} \\
 & & A^e & \xleftarrow{d_{(1,1)}} & A^e & \xleftarrow{d_{(1,2)}} & A^e \longleftarrow \cdots \\
 & & \downarrow{\scriptstyle d_{(2,1)}} & & \downarrow{\scriptstyle -d_{(2,1)}} & & \downarrow{\scriptstyle d_{(2,1)}} \\
 & & A^e & \xleftarrow[d_{(1,1)}]{} & A^e & \xleftarrow[d_{(1,2)}]{} & A^e \longleftarrow \cdots \\
 & \swarrow & & & & & \\
A & & & & & &
\end{array}
$$

where A-homomorphisms $d_{(1,j)}$ correspond to the projective resolution $\mathbb{P}_1$, and A-homomorphisms $d_{(2,j)}$ correspond to the projective resolution $\mathbb{P}_2$.

Quiver algebra defined by 2 cycles and a quantum-like relation [Obara(2012)]

Let $s_1, s_2 \geq 2$ be integers. We consider the quiver algebra $A = kQ/I$.
Q: the quiver with $s+t-1$ vertices and $s+t$ arrows as follows:



I: the ideal of kQ generated by

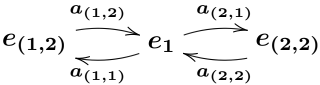
$$X_1^{s_1 n_1}, X_1^{s_1} X_2^{s_2} - q_{1,2} X_2^{s_2} X_1^{s_1}, X_2^{s_2 n_2}$$

where $X_i := \alpha_{(i,1)} + \alpha_{(i,2)} + \cdots + \alpha_{(i,s_1)}$, integers $n_i \geq 2$ for $1 \leq i \leq 2$ and $q_{1,2}$ is non-zero element in K.

Quiver algebra defined by 2 cycles and a quantum-like relation [Obara(2012)]

For simplicity, we consider the case of $s_1 = s_2 = 2$ and $q_{1,2} = 1$. Then $A = kQ/I$.

Q: the quiver with 3 vertices and 4 arrows as follows:



I: the ideal of kQ generated by

$$X_1^{2n_1}, X_1^2X_2^2 - X_2^2X_1^2, X_2^{2n_2}$$

where $X_i := a_{(i,1)} + a_{(i,2)}$ and integers $n_i \geq 2$ for $1 \leq i \leq 2$.

Projective resolution of an algebra defined by 2 cycles and a quantum-like relation

The complex

$$P_0 \xleftarrow{d_{(1,0)}} Q_{(1,0)} \xleftarrow{d_{(2,0)}} Q_{(2,0)} \xleftarrow{d_{(3,0)}} Q_{(3,0)} \xleftarrow{d_{(4,0)}} \cdots \longleftarrow$$

correspond to the projective resolution of Nakayama algebra $KQ_1/\langle X_1^{2n_1}\rangle$ **and**

$$P_0 \xleftarrow{d_{(0,1)}} Q_{(0,1)} \xleftarrow{d_{(0,2)}} Q_{(0,2)} \xleftarrow{d_{(0,3)}} Q_{(0,3)} \xleftarrow{d_{(0,4)}} \cdots \longleftarrow$$

correspond to the projective resolutions of Nakayama $KQ_2/\langle X_2^{2n_2}\rangle$ **where**

Q_1 **:** $e_{(1,2)} \underset{a_{(1,1)}}{\overset{a_{(1,2)}}{\rightleftarrows}} e_1$ **and** Q_2 **:** $e_1 \underset{a_{(2,2)}}{\overset{a_{(2,1)}}{\rightleftarrows}} e_{(2,2)}$ **and** $X_i := a_{(i,1)} + a_{(i,2)}$.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Q_{(0,2)} & \xleftarrow{\delta_{(1,2)}} & Ae_1 \otimes e_1A & \xleftarrow{\delta_{(2,2)}} & Ae_1 \otimes e_1A & \longleftarrow & \cdots \\
\downarrow d_{(0,2)} & & \downarrow -\sigma_{(1,2)} & & \downarrow \sigma_{(2,2)} & & \\
Q_{(0,1)} & \xleftarrow{\delta_{(1,1)}} & Ae_1 \otimes e_1A & \xleftarrow{\delta_{(2,1)}} & Ae_1 \otimes e_1A & \longleftarrow & \cdots \\
\downarrow d_{(0,1)} & & \downarrow -\sigma_{(1,1)} & & \downarrow \sigma_{(2,1)} & & \\
P_0 & \xleftarrow{d_{(1,0)}} & Q_{(1,0)} & \xleftarrow{d_{(2,0)}} & Q_{(2,0)} & \longleftarrow & \cdots
\end{array}
$$

δ correspond to the projective bimodule resolution of Nakayama algebra defined by 1 loop $K[e_1X_1^2e_1]/\langle e_1X_1^{2n_1}e_1\rangle$.
σ correspond to the projective bimodule resolution of Nakayama algebra defined by 1 loop $K[e_1X_2^2e_1]/\langle e_1X_2^{2n_2}e_1\rangle$.
We have the projective bimodule resolution of this algebra as total complex of this commutative diagram.

Projective resolution of an algebra defined by 2 cycles and a quantum-like relation

In fact, we have the A^e-homomorphisms δ and σ as follows.

$$\delta_{(l_1,l_2)} : e_1 \otimes e_1 \mapsto \begin{cases} e_1 \otimes e_1 X_1^2 - X_1^2 e_1 \otimes e_1 & \text{if } l_1 \text{ is odd,} \\ \displaystyle\sum_{k=0}^{n_1-1} X_1^{2k} e_1 \otimes e_1 X_1^{2(n_1-1-k)} & \text{if } l_1 \text{ is even,} \end{cases}$$

$$\sigma_{(l_1,l_2)} : e_1 \otimes e_1 \mapsto \begin{cases} e_1 \otimes e_1 X_2^2 - X_2^2 e_1 \otimes e_1 & \text{if } l_2 \text{ is odd,} \\ \displaystyle\sum_{k=0}^{n_1-1} X_2^{2k} e_1 \otimes e_1 X_2^{2(n_2-1-k)} & \text{if } l_2 \text{ is even.} \end{cases}$$

Hohschild cohomology ring modulo nilpotence

In [Obara(2015)], we determine the Hochoschild cohomology ring modulo nilpotence of a quiver algebra defined by two cycles and a quantum-like relation.

Theorem [Obara(2015)]

If $q_{1,2}$ is a root of unity, then $\mathrm{HH}^*(A)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables.
If $q_{1,2}$ is not a root of unity, $\mathrm{HH}^*(A)/\mathcal{N} \cong K$.

In fact, in the case of $s_1 = s_2 = 2$ and $q_{1,2} = 1$, we have the Hochschild cohomology ring of A modulo nilpotence as follows:

$$\mathrm{HH}^*(A)/\mathcal{N} = k[x, y]$$

where $x = e_1 + e_{(1,2)}$, $y = e_1 + e_{(2,2)} \in \mathrm{HH}^2(A)$.

Conjecture

Now, we have the following conjecture.

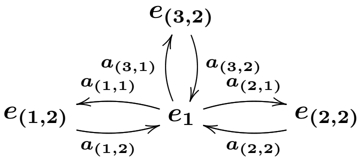
Conjecture

The projective bimodule resolution of the finite dimensional algebra with quantum-like relations is given by a total complex of projective bimodule resolutions depending on each relation.

With respect to this conjecture, we have the projective bimodule resolutions of the following algebras.

Example 1: Quiver algebra defined by 3 cycles and quantum-like relations 1

Let Q be the quiver as follows:



I: the ideal of KQ generated by

$$X_i^{2n_i} \text{ for } 1 \leq i \leq 3, \qquad X_i^2 X_j^2 - X_j^2 X_i^2 \text{ for } 1 \leq i < j \leq 3.$$

where $X_i := a_{(i,1)} + a_{(i,2)}$, n_i are integers with $n_i \geq 2$ for $1 \leq i \leq 3$.
We consider the quiver algebra $A = KQ/I$.

Projective resolution

We have the projective bimodule resolution of this algebra as total complex of the following commutative diagram. The complex

$$P_0 \overset{d_{(1,0)}}{\longleftarrow} Q_{(1,0)} \overset{d_{(2,0)}}{\longleftarrow} Q_{(2,0)} \overset{d_{(3,0)}}{\longleftarrow} Q_{(3,0)} \overset{d_{(4,0)}}{\longleftarrow} \cdots \longleftarrow$$

correspond to the projective resolutions of the quiver algebra $KQ_1/\langle X_1^{2n_1}, X_2^{2n_2}, X_1^2X_2^2 - X_2^2X_1^2\rangle$ and

$$P_0 \overset{d_{(0,1)}}{\longleftarrow} Q_{(0,1)} \overset{d_{(0,2)}}{\longleftarrow} Q_{(0,2)} \overset{d_{(0,3)}}{\longleftarrow} Q_{(0,3)} \overset{d_{(0,4)}}{\longleftarrow} \cdots \longleftarrow$$

correspond to the projective resolutions of Nakayama algebra $KQ_2/\langle X_3^{2n_3}\rangle$

whereQ_1 : $e_{(1,2)} \underset{a_{(1,1)}}{\overset{a_{(1,2)}}{\rightleftarrows}} e_1 \underset{a_{(2,2)}}{\overset{a_{(2,1)}}{\rightleftarrows}} e_{(2,2)}$ and Q_2 : $e_1 \underset{a_{(3,2)}}{\overset{a_{(3,1)}}{\rightleftarrows}} e_{(3,2)}$ where $X_i := a_{(i,1)} + a_{(i,2)}$.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Q_{(0,2)} & \xleftarrow{\delta_{(1,2)}} & Q_{(1,2)} & \xleftarrow{\delta_{(2,2)}} & Q_{(2,2)} & \longleftarrow & \cdots \\
\downarrow d_{(0,2)} & & \downarrow -\sigma_{(1,2)} & & \downarrow \sigma_{(2,2)} & & \\
Q_{(0,1)} & \xleftarrow{\delta_{(1,1)}} & Q_{(1,1)} & \xleftarrow{\delta_{(2,1)}} & Q_{(2,1)} & \longleftarrow & \cdots \\
\downarrow d_{(0,1)} & & \downarrow -\sigma_{(1,1)} & & \downarrow \sigma_{(2,1)} & & \\
P_0 & \xleftarrow{d_{(1,0)}} & Q_{(1,0)} & \xleftarrow{d_{(2,0)}} & Q_{(2,0)} & \longleftarrow & \cdots
\end{array}$$

δ is the projective bimodule resolution depending on the relations $X_1^{2n_1}$, $X_2^{2n_2}$, $X_1^2X_2^2 - X_2^2X_1^2$ and σ is the projective bimodule resolution depending on the relation $X_3^{2n_3}$ as follows.

Let $\varepsilon_{(i,j),(l_1,l_2)} = e_1 \otimes e_1$ for $i, j \geq 1$ and $l_1, l_2 \geq 0$ such that $l_1 + l_2 = i$.

$$Q_{(i,j)} = \coprod_{\substack{i,\, j \geq 1 \\ l_1,\, l_2 \geq 0 \\ l_1 + l_2 = i}} A\varepsilon_{(i,j),(l_1,l_2)}A.$$

Projective resolution

$$\delta_{(i,j)} : \varepsilon_{(i,j),(l_1,l_2)} \mapsto$$

$$\begin{cases}
\varepsilon_{(i-1,j),(l_1-1,l_2)}X_1^2 - X_1^2\varepsilon_{(i-1,j),(l_1-1,l_2)} \\
\quad + \varepsilon_{(i-1,j),(l_1,l_2-1)}X_2^2 - X_2^2\varepsilon_{(i-1,j),(l_1,l_2-1)} & \text{if } l_1, l_2 \text{ are odd,} \\
\sum_{k=0}^{n_1-1} X_1^{2k}\varepsilon_{(i-1,j),(l_1-1,l_2)}X_1^{2(n_1-1-k)} \\
\quad + \sum_{k'=0}^{n_2-1} X_2^{2k'}\varepsilon_{(i-1,j),(l_1,l_2-1)}X_2^{2(n_2-1-k')} & \text{if } l_1, l_2 \text{ are even,} \\
\varepsilon_{(i-1,j),(l_1-1,l_2)}X_1^2 - X_1^2\varepsilon_{(i-1,j),(l_1-1,l_2)} \\
\quad + \sum_{k'=0}^{n_2-1} X_2^{2k'}\varepsilon_{(i-1,j),(l_1,l_2-1)}X_2^{2(n_2-1-k')} & \text{if } l_1 \text{ is odd and } l_2 \text{ are even,} \\
\sum_{k=0}^{n_1-1} X_1^{2k}\varepsilon_{(i-1,j),(l_1-1,l_2)}X_1^{2(n_1-1-k)} \\
\quad + \varepsilon_{(i-1,j),(l_1,l_2-1)}X_2^2 - X_2^2\varepsilon_{(i-1,j),(l_1,l_2-1)} & \text{if } l_1 \text{ is even and } l_2 \text{ is odd,}
\end{cases}$$

Hochschild cohomology ring modulo nilpotence

$$\sigma_{(i,j)} : \varepsilon_{(i,j),(l_1,l_2)} \mapsto \begin{cases} \varepsilon_{(i,j-1),(l_1,l_2)} X_3^2 - X_3^2 \varepsilon_{(i,j),(l_1,l_2)} & \text{if } j \text{ is odd,} \\ \sum_{k=0}^{n_1-1} X_3^{2k} \varepsilon_{(i,j-1),(l_1,l_2)} X_3^{2(n_3-1-k)} & \text{if } j \text{ is even.} \end{cases}$$

Then the Hochschild cohomology ring of A modulo nilpotence is the polynomial ring of 3 variables.

$$\mathrm{HH}^*(A)/\mathcal{N} \simeq K[x_1, x_2, x_3] \text{ where } x_i = e_1 + e_{(i,2)} \in \mathrm{HH}^2(A).$$

Moreover, in general, we have the following result.

Theorem [Obara]

The Hochschild cohomology ring modulo nilpotence of a quiver algebra defined by c cycles and quantum-like relations correspond with that of c-th quantum complete intersection.

In fact, we have the K-basis elements of $\mathrm{HH}^*(A)/\mathcal{N}$ as follows, and these elements form a K-basis of $\mathrm{HH}^*(A)/\mathcal{N}$.

1. If n is even, and i with $1 \leq i \leq c$ satisfy the following conditions, then $\sum_{k_i=1}^{s_i} e^n_{(i,k_i)} \in \mathrm{HH}^n(A)$ is K-basis element of $\mathrm{HH}^*(A)/\mathcal{N}$.

$$q_{i,j}^{n_i n/2} = 1 \text{ for } 1 \leq j \leq c \text{ such that } j > i,$$
$$q_{j,i}^{n_i n/2} = 1 \text{ for } 1 \leq j \leq c \text{ such that } j < i.$$

2. If $n_1, \ldots, n_c$ and $(l_1, \ldots, l_c) \in L_n$ satisfy the following conditions, then $e^n_{(l_1,\ldots,l_c)} \in \mathrm{HH}^n(A)$ is K-basis element of $\mathrm{HH}^*(A)/\mathcal{N}$.

l_i is even or l_i is odd and $n_i = 2$ for $1 \leq i \leq c$,

$$\prod_{h_1=1}^{c-j} q_{j,j+h_1}^{n_{j+h_1} l_{j+h_1}/2} \prod_{h_2=1}^{j-1} q_{h_2,j}^{-n_{h_2} l_{h_2}/2} = 1 \text{ for } 1 \leq j \leq c \text{ s.t. } l_j\text{: even}(\neq 0),$$
$$\prod_{h_1=1}^{c-j} q_{j,j+h_1}^{n_{j+h_1} l_{j+h_1}/2} \prod_{h_2=1}^{j-1} q_{h_2,j}^{-n_{h_2} l_{h_2}/2} = -1 \quad \text{for } 1 \leq j \leq c \text{ s.t. } l_j \text{ is odd},$$

And applying the functor $\mathrm{Hom}_{A^e}(Ae_1 \otimes e_1 A, -)$ to the projective bimodule resolution of A, we have the projective bimodule resolution of a quantum complete intersection $e_1 A e_1$. Then we have the K-basis elements of $\mathrm{HH}^*(e_1 A e_1)/\mathcal{N}$ as follows, and these elements form a K-basis of $\mathrm{HH}^*(e_1 A e_1)/\mathcal{N}$.

1. If n is even, and i with $1 \leq i \leq c$ satisfy the following conditions, then $e_1 + \sum_{k_i=2}^{s_i} (e_1 X_i e_1)_{k_i} \in \mathrm{HH}^n(e_1 A e_1)$ is K-basis element of $\mathrm{HH}^*(e_1 A e_1)/\mathcal{N}$.

$$q_{i,j}^{n_i n/2} = 1 \text{ for } 1 \leq j \leq c \text{ such that } j > i,$$
$$q_{j,i}^{n_i n/2} = 1 \text{ for } 1 \leq j \leq c \text{ such that } j < i.$$

2. $e^n_{(l_1,\ldots,l_c)} \in \mathrm{HH}^n(e_1 A e_1)$ is K-basis element of $\mathrm{HH}^*(e_1 A e_1)/\mathcal{N}$ in the same condition in above page.

Example 2: Quiver algebra with 3 cycles and quantum-like relations

Let Q be the quiver as follows:

$$e_{(1,2)} \underset{a_{(1,2)}}{\overset{a_{(1,1)}}{\leftrightarrows}} e_{(1,1)} \underset{a_{(2,2)}}{\overset{a_{(2,1)}}{\rightleftarrows}} e_{(2,2)} \underset{a_{(3,2)}}{\overset{a_{(3,1)}}{\rightleftarrows}} e_{(3,2)}$$

I: the ideal of KQ generated by

$$X_i^{2n_i} \text{ for } 1 \leq i \leq 3, \quad X_i^2 X_2^2 - X_2^2 X_i^2 \text{ for } i = 1, 3,$$
$$a_{(1,2)} a_{(2,1)} X_2^{2l_1} a_{(3,1)}, a_{(3,2)} a_{(2,2)} X_2^{2l_2} a_{(1,1)} \text{ for } 0 \leq l_1, l_2 \leq n_2 - 1.$$

where $X_i := a_{(i,1)} + a_{(i,2)}$, n_i are integers with $n_i \geq 2$ for $1 \leq i \leq 3$.
We consider the quiver algebra $A = KQ/I$.

Projective resolution

We have the projective bimodule resolution of this algebra as total complex of the following commutative diagram. The complex

$$P_0 \xleftarrow{d_{(1,0)}} Q_{(1,0)} \xleftarrow{d_{(2,0)}} Q_{(2,0)} \xleftarrow{d_{(3,0)}} Q_{(3,0)} \xleftarrow{d_{(4,0)}} \cdots \longleftarrow$$

correspond to the projective resolution of the quiver algebra $KQ_1/\langle X_1^{2n_1}, X_2^{2n_2}, X_1^2X_2^2 - X_2^2X_1^2\rangle$ and

$$P_0 \xleftarrow{d_{(0,1)}} Q_{(0,1)} \xleftarrow{d_{(0,2)}} Q_{(0,2)} \xleftarrow{d_{(0,3)}} Q_{(0,3)} \xleftarrow{d_{(0,4)}} \cdots \longleftarrow$$

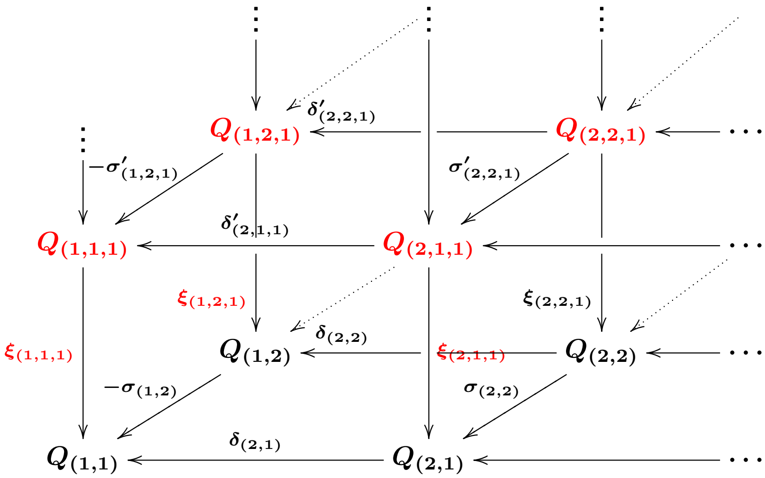
correspond to the projective resolution of Nakayama algebra $KQ_2/\langle X_3^{2n_3}\rangle$

where $Q_1 :\ e_{(1,2)} \underset{a_{(1,1)}}{\overset{a_{(1,2)}}{\rightleftarrows}} e_{(1,1)} \underset{a_{(2,2)}}{\overset{a_{(2,1)}}{\rightleftarrows}} e_{(2,2)}$ and $Q_2 :\ e_{(2,2)} \underset{a_{(3,2)}}{\overset{a_{(3,1)}}{\rightleftarrows}} e_{(3,2)}$

where $X_i := a_{(i,1)} + a_{(i,2)}$.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Q_{(0,2)} & \xleftarrow{\delta_{(1,2)}} & Q_{(1,2)} & \xleftarrow{\delta_{(2,2)}} & Q_{(2,2)} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d_{(0,2)}} & & \downarrow{\scriptstyle -\sigma_{(1,2)}} & & \downarrow{\scriptstyle \sigma_{(2,2)}} & & \\
Q_{(0,1)} & \xleftarrow{\delta_{(1,1)}} & Q_{(1,1)} & \xleftarrow{\delta_{(2,1)}} & Q_{(2,1)} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d_{(0,1)}} & & \downarrow{\scriptstyle -\sigma_{(1,1)}} & & \downarrow{\scriptstyle \sigma_{(2,1)}} & & \\
P_0 & \xleftarrow{d_{(1,0)}} & Q_{(1,0)} & \xleftarrow{d_{(2,0)}} & Q_{(2,0)} & \longleftarrow & \cdots
\end{array}
$$

δ is the projective bimodule resolution depending on the relations $X_1^{2n_1}$, $X_2^{2n_2}$, $X_1^2X_2^2 - X_2^2X_1^2$ and σ is the projective bimodule resolution depending on the relation $X_3^{2n_3}$ as follows.



$\xi_{(i,j,k)}$ **is the projective bimodule resolution depending on the relations** $a_{(1,2)}a_{(2,1)}X_2^{2l_1}a_{(3,1)}$ **and** $a_{(3,2)}a_{(2,2)}X_2^{2l_2}a_{(1,1)}$ **as follows. And** δ' **and** σ' **are similar to** δ **and** σ**.**

For $i, j \geq 1$, we define the projective A-bimodule $Q_{(i,j)}$ as follows:

$$Q_{(i,j)} = A\varepsilon_{(i,j),\langle(2,2),(2,2)\rangle}A \oplus \coprod_{\substack{l_1, l_2 \geq 1 \\ l_1 + l_2 = i}} A\varepsilon_{(i,j),\langle(1,1),(2,2)\rangle,(l_1,l_2)}A$$

$$\oplus \coprod_{\substack{l_1, l_2 \geq 1 \\ l_1 + l_2 = i}} A\varepsilon_{(i,j),\langle(2,2),(1,1)\rangle,(l_1,l_2)}A$$

$$\oplus \begin{cases} A\varepsilon_{(i,j),\langle(1,1),(2,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j),\langle(2,2),(1,1)\rangle,(i,0)}A & \text{if } i, j \text{ are even,} \\ A\varepsilon_{(i,j),\langle(1,2),(3,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j),\langle(3,2),(1,2)\rangle,(i,0)}A & \text{if } i, j \text{ are odd,} \\ A\varepsilon_{(i,j),\langle(1,2),(2,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j),\langle(2,2),(1,2)\rangle,(i,0)}A & \text{if } i \text{ is odd and } j \text{ is even,} \\ A\varepsilon_{(i,j),\langle(1,1),(3,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j),\langle(3,2),(1,1)\rangle,(i,0)}A & \text{if } i \text{ is even and } j \text{ is odd.} \end{cases}$$

where $\varepsilon_{(i,j),\langle(t_1,t_2),(t_3,t_4)\rangle,(l_1,l_2)} = e_{(t_1,t_2)} \otimes e_{(t_3,t_4)}$ and $\varepsilon_{(i,j),\langle(2,2),(2,2)\rangle} = e_{(2,2)} \otimes e_{(2,2)}$.

$$\delta_{(i,j)} : \varepsilon_{(i,j),\{(2,2),(2,2)\}} \mapsto$$

$$\begin{cases} \varepsilon_{(i-1,j),\langle(2,2),(2,2)\langle} X_2^2 - X_2^2 \varepsilon_{(i-1,j),\langle(2,2),(2,2)\rangle} & \text{if } i \text{ is odd,} \\ \displaystyle\sum_{k=0}^{n_2-1} X_2^{2k} \varepsilon_{(i-1,j),\langle(2,2),(2,2)\rangle} X_2^{2(n_2-1-k)} & \text{if } i \text{ is even,} \end{cases}$$

$$\varepsilon_{(i,j),\langle(1,1),(2,2)\rangle,(l_1,l_2)} \mapsto$$

$$\begin{cases} \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1,l_2-1)} X_2^2 - X_2^2 \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1,l_2-1)} & \\ \quad - X_1^2 \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1-1,l_2)} & \text{if } l_1, l_2 \text{ are odd,} \\ \displaystyle\sum_{k=0}^{n_2-1} X_2^{2k} \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1,l_2-1)} X_2^{2(n_2-1-k)} & \\ \quad + X_1^{2(n_1-1)} \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1-1,l_2)} & \text{if } l_1, l_2 \text{ are even,} \\ \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1,l_2-1)} X_2^2 - X_2^2 \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1,l_2-1)} & \\ \quad - X_1^{2(n_1-1)} \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1-1,l_2)} & \text{if } l_1 \text{ is even and } l_2 \text{ is odd,} \\ \displaystyle\sum_{k=0}^{n_2-1} X_2^{2k} \varepsilon_{(i-1,j),\langle(1,1),(2,2)\rangle,(l_1,l_2-1)} X_2^{2(n_2-1-k)} & \\ \quad + X_1^2 \varepsilon_{(i-1,j),\langle(1,1),(2,2)\langle,(l_1-1,l_2)} & \text{if } l_1 \text{ is odd and } l_2 \text{ is even.} \end{cases}$$

Projective resolution

$$
\varepsilon_{(i,j),\langle(2,2),(1,1)\rangle,(l_1,l_2)} \mapsto
\begin{cases}
\varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1,l_2-1)}X_2^2 - X_2^2\varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1,l_2-1)} \\
\quad - \varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1-1,l_2)}X_1^2 & \text{if } l_1, l_2 \text{ are odd,} \\
\sum_{k=0}^{n_2-1} X_2^{2k}\varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1,l_2-1)}X_2^{2(n_2-1-k)} \\
\quad + \varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1-1,l_2)}X_1^{2(n_1-1)} & \text{if } l_1, l_2 \text{ are even,} \\
\varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1,l_2-1)}X_2^2 - X_2^2\varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1,l_2-1)} \\
\quad - \varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1-1,l_2)}X_1^{2(n_1-1)} & \text{if } l_1 \text{ is even and } l_2 \text{ is odd,} \\
\sum_{k=0}^{n_2-1} X_2^{2k}\varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1,l_2-1)}X_2^{2(n_2-1-k)} \\
\quad + \varepsilon_{(i-1,j),\langle(2,2),(1,1)\rangle,(l_1-1,l_2)}X_1^2 & \text{if } l_1 \text{ is odd and } l_2 \text{ is even.}
\end{cases}
$$

Projective resolution

$$\sigma_{(i,j)} : \varepsilon_{(i,j),\langle(2,2),(2,2)\rangle} \mapsto$$

$$\begin{cases} \varepsilon_{(i,j-1),\langle(2,2),(2,2)\rangle} X_3^2 - X_3^2 \varepsilon_{(i,j),\langle(2,2),(2,2)\rangle} & \text{if } j \text{ is odd,} \\ \sum_{k=0}^{n_3-1} X_3^{2k} \varepsilon_{(i,j-1),\langle(2,2),(2,2)\rangle} X_3^{2(n_3-1-k)} & \text{if } j \text{ is even.} \end{cases}$$

$$\varepsilon_{(i,j),\langle(1,1),(2,2)\rangle,(l_1,l_2)} \mapsto$$

$$\begin{cases} \varepsilon_{(i,j-1),\langle(1,1),(2,2)\rangle,(l_1,l_2)} X_3^2 & \text{if } j \text{ is odd,} \\ \varepsilon_{(i,j-1),\langle(1,1),(2,2)\rangle,(l_1,l_2)} X_3^{2(n_3-1)} & \text{if } j \text{ is even.} \end{cases}$$

$$\varepsilon_{(i,j),\langle(2,2),(1,1)\rangle,(l_1,l_2)} \mapsto$$

$$\begin{cases} X_3^2 \varepsilon_{(i,j-1),\langle(2,2),(1,1)\rangle,(l_1,l_2)} & \text{if } j \text{ is odd,} \\ X_3^{2(n_3-1)} \varepsilon_{(i,j-1),\langle(2,2),(1,1)\rangle,(l_1,l_2)} & \text{if } j \text{ is even.} \end{cases}$$

In the case of k is odd,

$$
\begin{aligned}
Q_{(i,j,k)} = & \coprod_{\substack{l_1+l_2=i \\ l_1 \geq 2, l_2 \geq 1}} \coprod_{\substack{l_3+l_4=k \\ l_1, l_2 \geq 0}} (A\varepsilon_{(i,j,k),\langle(1,1),(1,1)\rangle,(l_1,l_2),(l_3,l_4)}A \\
& \oplus A\varepsilon_{(i,j,k),\langle(2,2),(2,2)\rangle,(l_1,l_2),(l_3,l_4)}A) \\
& \oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3, l_4 \text{ are odd}}} A\varepsilon_{(i,j,k),\langle(1,1),(1,1)\rangle,(1,i-1),(l_3,l_4)}A \\
& \oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3, l_4 \text{ are even}}} A\varepsilon_{(i,j,k),\langle(2,2),(2,2)\rangle,(1,i-1),(l_3,l_4)}A \\
& \oplus \begin{cases} \coprod_{\substack{l_3+l_4=k \\ l_1, l_2 \geq 0}} (A\varepsilon_{(i,j,k),\langle(1,1),(1,1)\rangle,(i,0),(l_3,l_4)}A \\ \quad \oplus A\varepsilon_{(i,j,k),\langle(2,2),(2,2)\rangle,(i,0),(l_3,l_4)}A) & \text{if } i, j \text{ are even,} \\ A\varepsilon_{(i,j,k),\langle(2,2),(3,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j,k),\langle(3,2),(2,2)\rangle,(i,0)}A \\ \qquad \text{if } i \text{ is even and } j \text{ is odd,} \\ A\varepsilon_{(i,j,k),\langle(1,1),(1,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j,k),\langle(1,2),(1,1)\rangle,(i,0)}A \\ \qquad \text{if } i \text{ is odd and } j \text{ is even,} \end{cases}
\end{aligned}
$$

In the case of k is even,

$$Q_{(i,j,k)} = \coprod_{\substack{l_1+l_2=i \\ l_1\geq 2,\, l_2\geq 1}} \coprod_{\substack{l_3+l_4=k \\ l_1,\, l_2\geq 0}} \big(A\varepsilon_{(i,j,k),\langle(1,1),(2,2)\rangle,(l_1,l_2),(l_3,l_4)}A$$
$$\oplus A\varepsilon_{(i,j,k),\langle(2,2),(1,1)\rangle,(l_1,l_2),(l_3,l_4)}A\big)$$
$$\oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3:\text{even},\, l_4:\text{odd}}} A\varepsilon_{(i,j,k),\langle(1,1),(2,2)\rangle,(1,i-1),(l_3,l_4)}A$$
$$\oplus \coprod_{\substack{l_3+l_4=k+1 \\ l_3:\text{odd},\, l_4:\text{even}}} A\varepsilon_{(i,j,k),\langle(2,2),(1,1)\rangle,(1,i-1),(l_3,l_4)}A$$
$$\oplus \begin{cases} \coprod_{\substack{l_3+l_4=k \\ l_1,\, l_2\geq 0}} \big(A\varepsilon_{(i,j,k),\langle(1,1),(2,2)\rangle,(i,0),(l_3,l_4)}A \\ \oplus A\varepsilon_{(i,j,k),\langle(2,2),(1,1)\rangle,(i,0),(l_3,l_4)}A\big) & \text{if } i,j \text{ are even,} \\ A\varepsilon_{(i,j,k),\langle(1,1),(3,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j,k),\langle(3,2),(1,1)\rangle,(i,0)}A \\ \qquad \text{if } i \text{ is even and } j \text{ is odd,} \\ A\varepsilon_{(i,j,k),\langle(2,2),(1,2)\rangle,(i,0)}A \oplus A\varepsilon_{(i,j,k),\langle(1,2),(2,2)\rangle,(i,0)}A \\ \qquad \text{if } i \text{ is odd and } j \text{ is even,} \end{cases}$$

We have the A^e-homomorphism ξ as follows:

$$\xi_{(i,j,k)} : \begin{cases} \varepsilon_{(i,j,k),\langle(1,1),(2,2)\rangle,(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,j,k-1),\langle(1,1),(1,1)\rangle,(l_1,l_2),(l'_1-1,l'_2)} a_{(2,1)} X_3^2 \\ \quad - X_1^2 a_{(2,1)} \varepsilon_{(i,j,k-1),\langle(2,2),(2,2)\rangle,(l_1,l_2),(l'_1,l'_2-1)}, \\ \varepsilon_{(i,j,k),\langle(2,2),(1,1)\rangle,(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,j,k-1),\langle(2,2),(2,2)\rangle,(l_1,l_2),(l'_1-1,l'_2)} a_{(2,2)} X_1^2 \\ \quad - X_3^2 a_{(2,2)} \varepsilon_{(i,j,k-1),\langle(1,1),(1,1)\rangle,(l_1,l_2),(l'_1,l'_2-1)}, \end{cases} \quad \text{if } k \text{ is even,}$$

$$\xi_{(i,j,k)} : \begin{cases} \varepsilon_{(i,j,k),\langle(1,1),(1,1)\rangle,(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,j,k-1),\langle(1,1),(2,2)\rangle,(l_1,l_2),(l'_1-1,l'_2)} a_{(2,2)} X_1^2 \\ \quad - X_1^2 a_{(2,1)} \varepsilon_{(i,j,k-1),\langle(2,2),(1,1)\rangle,(l_1,l_2),(l'_1,l'_2-1)}, \\ \varepsilon_{(i,j,k),\langle(2,2),(2,2)\rangle,(l_1,l_2),(l'_1,l'_2)} \mapsto \\ \varepsilon_{(i,j,k-1),\langle(2,2),(1,1)\rangle,(l_1,l_2),(l'_1-1,l'_2)} a_{(2,1)} X_3^2 \\ \quad - X_3^2 a_{(2,2)} \varepsilon_{(i,j,k-1),\langle(1,1),(2,2)\rangle,(l_1,l_2),(l'_1,l'_2-1)}, \end{cases} \quad \text{if } k \text{ is odd.}$$

Projective resolution

We denote the total complex of above complexes by

$$\mathbb{P} : 0 \leftarrow A \xleftarrow{\pi} P_0 \xleftarrow{d_1} P_1 \leftarrow \cdots \xleftarrow{d_n} P_n \leftarrow \cdots .$$

We consider the complex $\mathbb{P} \otimes_A A/\operatorname{rad} A$. Then we have the following rusult.

Proposition

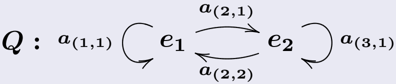
The complex $\mathbb{P} \otimes_A A/\operatorname{rad} A$ is exact.

Therefore $\mathbb{P}$ is a projective bimodule resolution of A by [Theorem 2.8 Green,Snashall(2004)].
Then the Hochschild cohomology ring of A modulo nilpotence is the polynomial ring.

$$\mathrm{HH}^*(A)/\mathcal{N} \simeq K[x_2] \text{ where } x_2 = e_{(1,1)} + e_{(2,2)} \in \mathrm{HH}^2(A).$$

Remark

The Hochschild cohomology ring modulo nilpotence of A corresponds to that of B where $B = KQ/I$ is the quiver algebra defined by the following quiver Q and ideal I.



I : the ideal of KQ generated by

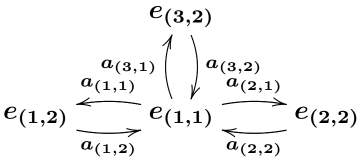
$$a_{(1,1)}^2, (a_{(2,1)} + a_{(2,2)})^4, a_{(3,1)}^2,$$

$$a_{(1,1)}(a_{2,1}a_{2,2}) - (a_{2,1}a_{2,2})a_{(1,1)}, (a_{2,2}a_{2,1})a_{(3,1)} - a_{(3,1)}(a_{2,2}a_{2,1}),$$

$$a_{(1,1)}a_{(2,1)}a_{(3,1)}, a_{(3,1)}a_{(2,2)}a_{(1,1)}.$$

Example 3: Quiver algebra with 3 cycles and quantum-like relations 2

Let Q be the quiver as follows:

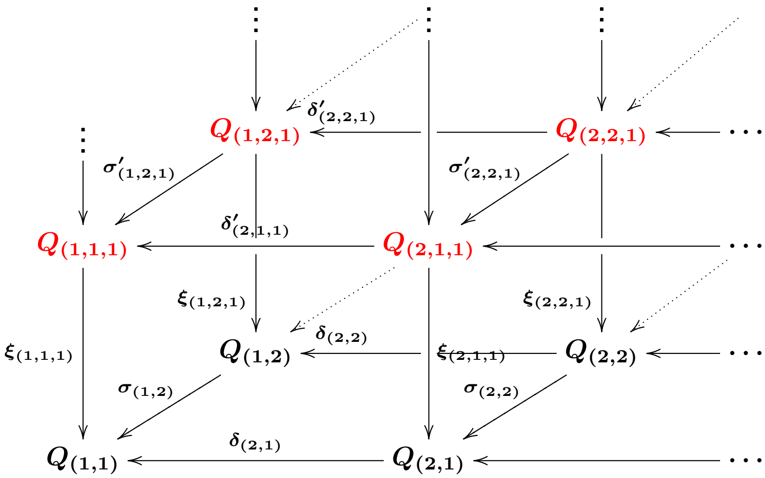


I: the ideal of KQ generated by

$$X_i^{2n_i} \text{ for } 1 \leq i \leq 3, \qquad X_1^2 X_j^2 - X_j^2 X_1^2 \text{ for } 2 \leq j \leq 3,$$

$$a_{(2,2)}a_{(3,1)}, a_{(3,2)}a_{(2,1)}.$$

where $X_i := a_{(i,1)} + a_{(i,2)}$, n_i are integers with $n_i \geq 2$ for $1 \leq i \leq 3$. We consider the quiver algebra $A = KQ/I$. Then, we have the projective bimodule resolution of this algebra as total complex of the following commutative diagram.



$\xi_{(i,j,k)}$ is the projective resolution depending on the relations $a_{(2,2)}a_{(3,1)}$ and $a_{(3,2)}a_{(2,1)}$ as follows. And δ' and σ' are similar to δ and σ.

$$Q_{(i,j,k)} = \coprod_{l=1}^{2} \coprod_{\substack{i,\, j \geq 1 \\ l_1 \geq 0,\, l_2 \geq 1 \\ l_1 + l_2 = i}} A\varepsilon_{(i,j,k),(l_1,l_2),l}A$$

Hochschild cohomology ring modulo nilpotence

$$\xi_{(i,j,k)} : \varepsilon_{(i,j,k),(l_1,l_2),1} \mapsto \begin{cases} \varepsilon_{(i,j,k-1),(l_1,l_2),1} X_2^2 & \text{if } k \text{ is odd,} \\ \varepsilon_{(i,j,k-1),(l_1,l_2),1} X_3^2 & \text{if } k \text{ is even.} \end{cases}$$

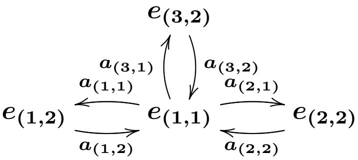
$$\varepsilon_{(i,j,k),(l_1,l_2),2} \mapsto \begin{cases} X_2^2 \varepsilon_{(i,j,k-1),(l_1,l_2),2} & \text{if } k \text{ is odd,} \\ X_3^2 \varepsilon_{(i,j,k-1),(l_1,l_2),2} & \text{if } k \text{ is even.} \end{cases}$$

Then the Hochschild cohomology ring of A modulo nilpotence is the polynomial ring.

$$\mathrm{HH}^*(A)/\mathcal{N} \simeq K[x_1] \text{ where } x_1 = e_{(1,1)} + e_{(1,2)} \in \mathrm{HH}^2(A).$$

Example 4: Quiver algebra with 3 cycles and quantum-like relations 3

Let Q be the quiver as follows:



I: the ideal of KQ generated by

$$X_i^{2n_i} \text{ for } 1 \leq i \leq 3, \quad X_i^2 X_j^2 - X_j^2 X_i^2 \text{ for } 1 \leq i, j \leq 3,$$
$$a_{(1,2)} X_2^{2l_1} a_{(3,1)}, a_{(3,2)} X_2^{2l_2} a_{(1,1)} \text{ for } 0 \leq l_1, l_2 \leq n_2 - 1.$$

where $X_i := a_{(i,1)} + a_{(i,2)}$, n_i are integers with $n_i \geq 2$ for $1 \leq i \leq 3$. We consider the quiver algebra $A = KQ/I$.

The Hochschild cohomology ring of A modulo nilpotence is the polynomial ring.

$$\mathrm{HH}^*(A)/\mathcal{N} \simeq K[x_2] \text{ where } x_2 = e_{(1,1)} + e_{(2,2)} \in \mathrm{HH}^2(A).$$