Crossed products for matrix rings

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Notations

1. R : a ring
2. $\mathrm{Aut}(R)$: the group of ring automorphisms of R
3. R^σ: the subring of R consisting of all $x \in R$ with $\sigma(x) = x$ for $\sigma \in \mathrm{Aut}(R)$
4. $\mathrm{M}_n(R)$: the ring of $n \times n$ full matrices over R for $n \geq 2$

Aim We provide a systematic way to define new multiplications on $\mathrm{M}_n(R)$. To do so, we divide the construction into two steps, i.e., we will construct two ring extensions A/R and Λ/A such that $\Lambda \cong \mathrm{M}_n(R)$ as right R-modules.

Construction

Let

1. $I = \{0, 1, \ldots, n-1\}$ be a set of integers
2. $\mathbb{Z}_+$ the set of non-negative integers.

We fix a pair (q, χ) of an integer $q \in \mathbb{Z}$ and a mapping $\chi : \mathbb{Z}_+ \to \mathbb{Z}$ satisfying the following conditions:
(X0) $\chi(0) = 0$;
(X1) $\chi(i + kn) = \chi(i) + kq$ for all $(i, k) \in I \times \mathbb{Z}_+$;
(X2) $\chi(i) + \chi(j) \geq \chi(i + j)$ for all $i, j \in \mathbb{Z}_+$.
Also, we fix a triple (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R^\sigma$ satisfying the following condition:

$$(*) \quad xc = c\sigma(x), xt = t\sigma^q(x) \text{ for all } x \in R.$$

It should be noted that $ct = tc$.

At first, we will construct a ring extension A/R.
Let A be a free right R-module with a basis $\{u_i\}_{i \in I}$.
We set

$$u_{i+kn} = u_i t^k$$

for $(i, k) \in I \times \mathbb{Z}_+$ and

$$\omega(i,j) = \chi(i) + \chi(j) - \chi(i+j)$$

for $i, j \in \mathbb{Z}_+$. Note that

1. ω is symmetric, i.e., $\omega(i,j) = \omega(j,i)$ for all $i, j \in \mathbb{Z}_+$
2. $\omega(i+kn, j+ln) = \omega(i,j)$ for all $(i,k), (j,l) \in I \times \mathbb{Z}_+$.

Definition 2.1

We define a multiplication on A subject to the following axioms:
(A1) $u_i u_j = u_{i+j} c^{\omega(i,j)}$ for all $i, j \in \mathbb{Z}_+$;
(A2) $x u_i = u_i \sigma^{\chi(i)}(x)$ for all $x \in R$ and $i \in \mathbb{Z}_+$,

We denote by $\{\delta_i\}_{i \in I}$ the dual basis of $\{u_i\}_{i \in I}$ for the free left R-module $\mathrm{Hom}_R(A, R)$,
(i.e., $a = \sum_{i \in I} u_i \delta_i(a)$ for all $a \in A$.)
Then for any $a, b \in A$ we have

$$ab = \sum_{i,j \in I} u_{i+j} c^{\omega(i,j)} \sigma^{\chi(j)}(\delta_i(a)) \delta_j(b).$$

Proposition 2.2 (Hoshino-K-Koga, 2015)

A is an associative ring with $1 = u_0$ and contains R as a subring via the injective ring homomorphism $R \to A, x \mapsto u_0 x$.

Next, we will construct a ring extension Λ/A.
(To do so, we need the group structure of I.)
We fix a cyclic permutation of I

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

and make I a cyclic group with 0 the unit element by the law of composition $I \times I \to I, (i,j) \mapsto \pi^j(i)$.
It should be noted that

$$i + j = \begin{cases} \pi^j(i) & \text{if } i + j < n, \\ \pi^j(i) + n & \text{if } i + j \geq n \end{cases}$$

for all $i, j \in I$.

Setting $A_i = u_i R$ for $i \in I$, $A = \oplus_{i \in I} A_i$ yields an I-graded ring with $A_0 = R$.
We denote by $\varepsilon_i : A \to A_i, a \mapsto u_i \delta_i(a)$ the projection for each $i \in I$.

Then the following conditions are satisfied:
(E1) $\varepsilon_i \varepsilon_j = 0$ unless $i = j$ and $\sum_{i \in I} \varepsilon_i = \mathrm{id}_A$;
(E2) $\varepsilon_i(a)\varepsilon_j(b) = \varepsilon_{\pi^j(i)}(\varepsilon_i(a)b)$ for all $a, b \in A$ and $i, j \in I$.

Let Λ be a free right A-module with a basis $\{v_i\}_{i \in I}$.

Definition 2.3

we define a multiplication on Λ subject to the following axioms:
(L1) $v_i v_j = 0$ unless $i = j$ and $v_i^2 = v_i$ for all $i \in I$;
(L2) $a v_i = \sum_{j \in I} v_j \varepsilon_{\pi^{-i}(j)}(a)$ for all $a \in A$ and $i \in I$.

Let us denote by $\{\gamma_i\}_{i \in I}$ the dual basis of $\{v_i\}_{i \in I}$ for the free left A-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{i \in I} v_i \gamma_i(\lambda)$ for all $\lambda \in \Lambda$.
It is not difficult to see that

$$\lambda\mu = \sum_{i,j \in I} v_i \varepsilon_{\pi^{-j}(i)}(\gamma_i(\lambda))\gamma_j(\mu)$$

for all $\lambda, \mu \in \Lambda$.

Proposition 2.4

Λ is an associative ring with $1 = \sum_{i \in I} v_i$ and contains A as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{i \in I} v_i a$.

Note that

1. $\{v_i u_j\}_{i,j\in I}$ is a basis for the free right R-module Λ with $\{\delta_j \gamma_i\}_{i,j\in I}$ the dual basis for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$,
2. For any $i \in I$, we have $xv_i = v_i x$ for all $x \in R$,
3. Λv_i is a Λ-R-bimodule,
4. $v_i\Lambda$ is an R-Λ-bimodule.

Also, by (L2) $u_k v_j = v_{\pi^k(j)} u_k$ for all $j, k \in I$, so that $v_i \Lambda v_j = v_i u_{\pi^{-j}(i)} R$ and

$$\mathrm{Hom}_\Lambda(v_j\Lambda, v_i\Lambda) \xrightarrow{\sim} R, f \mapsto \delta_{\pi^{-j}(i)}(\gamma_i(f(v_j)))$$

as R-R-bimodules for all $i, j \in I$. In particular,

$$\mathrm{End}_\Lambda(v_i\Lambda) \xrightarrow{\sim} R, f \mapsto \delta_0(\gamma_i(f(v_i)))$$

as rings for all $i \in I$.

Now, setting $e_{ij} = v_i u_{\pi^{-j}(i)}$ for $i, j \in I$, we have a basis $\{e_{ij}\}_{i,j\in I}$ for the free right R-module Λ.

Then, we have

1. $v_i \Lambda v_j = e_{ij} R$ for all $i, j \in I$,
2. $\{\delta_{\pi^{-j}(i)} \gamma_i\}_{i,j\in I}$ is the duel basis of $\{e_{ij}\}_{i,j\in I}$ for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$, i.e.,

$$\lambda = \sum_{i,j\in I} e_{ij} \delta_{\pi^{-j}(i)}(\gamma_i(\lambda))$$

for all $\lambda \in \Lambda$.

In particular,

$$\rho : \Lambda \xrightarrow{\sim} \mathrm{M}_n(R), \lambda \mapsto (\delta_{\pi^{-j}(i)}(\gamma_i(\lambda)))_{i,j\in I}$$

as right R-modules.

Theorem 2.5 (Main Theorem)

The multiplication in Λ *is subject to the following axioms:*
(M1) $e_{ij}e_{kl} = 0$ *unless* $j = k$;
(M2) $e_{ij}e_{jk} = e_{ik}t^{\epsilon(\pi^{-j}(i),\pi^{-k}(j))}c^{\omega(\pi^{-j}(i),\pi^{-k}(j))}$ *for all* $i, j, k \in I$;
(M3) $xe_{ij} = e_{ij}\sigma^{\chi(\pi^{-j}(i))}(x)$ *for all* $x \in R$ *and* $i, j \in I$.

where

$$\epsilon(i,j) = \begin{cases} 0 & \text{if } i + j < n, \\ 1 & \text{if } i + j \geq n \end{cases}$$

for $i, j \in I$

Example 2.6

Let

1. $q = n - 1$
2. $\chi(i + kn) = i + kq$ for all $(i, k) \in I \times \mathbb{Z}_+$.

Then a pair (q, χ) satisfies (X0), (X1) and (X2), and

$$\omega(i,j) = \begin{cases} 0 & \text{if } i + j < n, \\ 1 & \text{otherwise} \end{cases}$$

for $i, j \in I$. Also, $x(tc) = (tc)\sigma^n(x)$ for all $x \in R$.

Let $R[X;\sigma]$ be a right skew polynomial ring with trivial derivation, (the multiplication is defined subject to the following rule: $aX = X\sigma(a)$ for all $a \in R$.)

It then follows that $(X^n - tc) = (X^n - tc)R[X;\sigma]$ is a two-sided ideal of $R[X;\sigma]$ and $A \cong R[X;\sigma]/(X^n - tc)$ as extension rings of R.

Auslander-Gorenstein rings

We recall the notion of Auslander-Gorenstein rings.

Proposition 3.1 (Auslander)

Let R be a left and right noetherian ring. Then for any $n \geq 0$ the following are equivalent.

(1) In a minimal injective resolution $I^\bullet$ of R in $\mathrm{Mod}\text{-}R$, $\mathrm{flat\,dim}\ I^i \leq i$ for all $0 \leq i \leq n$.

(2) In a minimal injective resolution $J^\bullet$ of R in $\mathrm{Mod}\text{-}R^{\mathrm{op}}$, $\mathrm{flat\,dim}\ J^i \leq i$ for all $0 \leq i \leq n$.

(3) For any $1 \leq i \leq n+1$, any $M \in \mathrm{mod}\text{-}R$ and any submodule X of $\mathrm{Ext}^i_R(M, R) \in \mathrm{mod}\text{-}R^{\mathrm{op}}$ we have $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X, R) = 0$ for all $0 \leq j < i$.

(4) For any $1 \leq i \leq n+1$, any $X \in \mathrm{mod}\text{-}R^{\mathrm{op}}$ and any submodule M of $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X, R) \in \mathrm{mod}\text{-}R$ we have $\mathrm{Ext}^j_R(M, R) = 0$ for all $0 \leq j < i$.

Definition 3.2 (Björk)

R: a left and right noetherian ring

1. R satisfies the **Auslander condition** $\overset{def}{\Leftrightarrow}$ R satisfies the equivalent conditions in Proposition 3.1 for all $n \geq 0$,
2. R is an **Auslander-Gorenstein ring** $\overset{def}{\Leftrightarrow}$ $\mathrm{inj\ dim}\ R = \mathrm{inj\ dim}\ R^{\mathrm{op}} < \infty$ and it satisfies the Auslander condition.

Corllary 3.3

The following are equivalent.

(1) R is an Auslander-Gorenstein ring;

(2) A is an Auslander-Gorenstein ring;

(3) Λ is an Auslander-Gorenstein ring.

Thank you for your attension.