Almost Gorenstein Rees algebras

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Notations

A : a commutative Noetherian local ring

- $\mathfrak{m}$: the unique maximal ideal, $k = A/\mathfrak{m}$

M: a finitely generated A-module (Note: A is an A-module)

- $\ell_A(M)$: the length of M
- $\mu_A(M)$: the minimal numbers of generators of M
- $\dim M$: the Krull dimension of M
- $e_I^0(M)$: the multiplicity of M w.r.t. an ideal I

$$e_I^0(M) = \lim_{n\to\infty} \frac{\ell_A(M/I^{n+1}M)}{n^d} \times d!, \quad \text{where} \quad d = \dim M$$

Basic notions

- $\mathbf{emb}(A) := \mu_A(\mathfrak{m})$: the embedding dimension of A.

A is regular $\Leftrightarrow \mathbf{gldim} A < \infty \Leftrightarrow \mathbf{emb}(A) = \dim A$

Assume that $A = S/I$, where S is a regular local ring with $\mathbf{emb}(A) = \dim S$. Then one can choose a minimal free resolution of A over S:

$$0 \to S^{\beta_p} \to S^{\beta_{p-1}} \to \cdots \to S^{\beta_1} \to S \to A \to 0 \quad (ex)$$

A is Cohen-Macaulay $\Leftrightarrow p = \mathbf{ht} I$.

Let K_A denote the canonical module of A. When A is Cohen-Macaulay,

A is Gorenstein $\Leftrightarrow \beta_p = 1 \Leftrightarrow A \cong K_A$

Almost Gorenstein local rings

Defn

R : an almost Gorenstein local ring

$\overset{\text{def}}{\Longleftrightarrow}$ R is a Cohen-Macaulay local ring with canonical module K_R and there exists an exact sequence of R-modules:

$$0 \to R \xrightarrow{\varphi} K_R \to C \to 0 \quad \text{s.t. } \mu_R(C) = e^0_{\mathfrak{m}}(C)$$

- R is Gorenstein $\Leftrightarrow C = 0$
- $C \neq 0 \Rightarrow C$ is a Cohen-Macaulay R-module with $\mu_R(C) = e^0_{\mathfrak{m}}(C)$ (i.e. an Ulrich R-module) of dimension $d - 1$.

Almost Gorenstein graded rings

Let $R = \oplus_{n\geq 0} R_n$ be a Cohen-Macaulay graded ring over a local ring $A = R_0$ with graded canonical module K_R.
$a = a(R) = -\min\{n \in \mathbb{Z} \mid [K_R]_n \neq 0\}$: a-invariant of R

Defn

R : an almost Gorenstein graded ring

$\overset{\text{def}}{\Longleftrightarrow}$ There exists an exact sequence of graded R-modules:

$$0 \to R \xrightarrow{\varphi} K_R(-a) \to C \to 0 \quad \text{s.t. } \mu_R(C) = e^0_{\mathfrak{m}}(C)$$

Rmk. $M(a)$ is a graded R-module with $[M(a)]_n = M_{n+a}$

Almost Gor. graded rings vs. almost Gor. local rings

Let $R = \oplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring over a local ring $A = R_0$ with graded canonical module K_R.
Set $\mathfrak{M} = \mathfrak{m}R + R_+$, the unique graded maximal ideal.
Then

Fact

R: almost Gorenstein graded ring
$\Longrightarrow R_{\mathfrak{M}}$: almost Gorenstein local ring.

- The converse is not true in genaral.
- R:Cohen-Macaulay (resp. Gorenstein) $\Longleftrightarrow$ so is $R_{\mathfrak{M}}$.

Examples of almost Gorenstein rings (1)

- $\dim R = 0$ R: almost Gorenstein $\Leftrightarrow$ R: Gorenstein.

- $\dim R = 1$

  $K[[H]]$: almost Gorenstein $\Leftrightarrow$ H is almost symmetric.
  e.g. $H =< 3, a, b >$ with $3 < a < b$ and $\gcd(3, a, b) = 1$.
  Then $a < b \leq 2a - 3$.
  H is almost symmetric $\Leftrightarrow$ $b = 2a - 3$.

  R: finite Cohen-Macaulay representation type/$k = \overline{k}$
  $\Rightarrow$ R: almost Gorenstein local ring

Examples of almost Gorenstein rings (2)

- **$\dim R = 2$**

  R: rational singulairty $\Rightarrow$ R: almost Gorenstein local ring

- **$\dim R \geq 3$**

  There are a few examples of almost Gorenstein rings (Higashitani, Murai-Matsuoka etc.)

Theorem (GTT)

R: *almost Gorenstein local ring with* $\mathrm{emb}(R) = e^0_{\mathfrak{m}}(R) + d - 1$

$\Rightarrow G = \oplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$ *is an almost Gorenstein graded ring*

Fundamental properties of AG rings

Proposition (GTT)

1. R/fR: *almost Gorenstein local ring*
   $f \in \mathfrak{m}$: *nonzero divisor*
   $\Rightarrow R$: *almost Gorenstein local ring*
2. R: *almost Gorenstein local ring with* $d \geq 2$
   f: *superficial for* C
   $\Rightarrow R/fR$: *almost Gorenstein local ring*

When f:NZD, R:CM (resp. Gor.) $\Leftrightarrow$ R/fR:CM (resp. Gor)

Rees algebras

$(A, \mathfrak{m})$: a Noetherian local domain
$I (\neq 0)$ an ideal of A, t: an indeterminate over A

Defn

The graded ring

$$\mathcal{R} := \mathcal{R}(I) = \sum_{n \geq 0} \mathcal{R}_n := \sum_{n \geq 0} I^n t^n \subset A[t]$$

is called the Rees algebra of I.

- $\mathfrak{M} := \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ is the unique graded maximal ideal of $\mathcal{R}$.
- $\dim \mathcal{R} = d + 1$

Main Problem

Question

Let A be a Cohen-Macaulay local domain, and I an ideal of A. Set $\mathcal{R} = \mathcal{R}(I)$ and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$. Then

1. When is $\mathcal{R}$ an almost Gorenstein graded ring?
2. When is $\mathcal{R}_{\mathfrak{M}}$ an almost Gorenstein local ring?

Answers:

- parameter ideal $\cdots$ AG local but not AG graded
- p_g-ideal $\cdots$ AG graded (and thus AG local)
- socle ideal $\cdots$ not AG local

Parameter ideals

Assume: $(A, \mathfrak{m})$: a Cohen-Macaulay local ring of dimension d

- $a_1, a_2, \ldots, a_d$: a system of parameters (s.o.p.)
$\Leftrightarrow A/(a_1, \ldots, a_d)$ has finite length

- $a_1, a_2, \ldots, a_r$: a subsystem of parameters (s.s.o.p.)
$\Leftrightarrow$ a part of a system of parameters

Fact

Put $Q = (a_1, \ldots, a_r)A$, where $a_1, a_2, \ldots, a_r$ be a s.s.o.p. with $r \geq 2$. Then

1. *$\mathcal{R}(Q)$ is Cohen-Macaulay.*
2. *$\mathcal{R}(Q)$ is a Gorenstein $\Leftrightarrow$ A is Gorenstein and $r = 2$.*

Rees algebras of an ideal generated by s.s.o.p. (AG local, $r = 2$)

First we consider the case $r = 2$.

Proposition

Let $Q = (a_1, a_2)$ be an ideal generated by s.s.o.p. Then TFAE:

1. *A is Gorenstein.*
2. *$\mathcal{R}(Q)$ is Gorenstein.*
3. *$\mathcal{R}(Q)_{\mathfrak{M}}$ is an almost Gorenstein local ring.*
4. *$\mathcal{R}(Q)$ is an almost Gorenstein graded ring.*

In fact, $\mathcal{R}(Q) \cong A[T_1, T_2]/(a_2T_1 - a_1T_2)$.

Rees algebras of an ideal generated by s.s.o.p. (AG local, $r \geq 3$)

The following theorem provides us many examples of higher dimensional almost Gorenstein local rings.

Theorem

Assume: A *is Gorenstein.*
Let $Q = (a_1, \ldots, a_r)$ *be an ideal generated by a s.s.o.p. with* $r \geq 3$.
Then TFAE:

1. $\mathcal{R}(Q)_{\mathfrak{M}}$ *is an almost Gorenstein local ring.*
2. A *is a regular local ring.*

Rees alegbras of parameter ideals (graded AG)

Theorem

Assume: A *is a Gorenstein local ring.*
Let $Q = (a_1, \ldots, a_r)$ *be an ideal generated by a s.s.o.p. with* $r \geq 3$.
Then TFAE

1. $\mathcal{R}(Q)$ *is an almost Gorenstein graded ring.*
2. A *is a regular local ring,*
   and $a_1, \ldots, a_r$ *is a part of a regular sytem of parameters.*

Question

How about the case where A is a Cohen-Macaulay local ring?

Example: Rees algebra AG local but not AG graded

Ex.

Let A be a regular local ring with $d = \dim A \geq 3$, and $Q = (a_1, \ldots, a_d) \neq \mathfrak{m}$ a parameter ideal. Then

1. $\mathcal{R}(Q)_{\mathfrak{M}}$ is an almost Gorenstein local ring.
2. $\mathcal{R}(Q)$ is not an almost Gorenstein graded ring.

In particular, if $A = K[x_1, x_2, x_3]$, $Q = (x_1, x_2, x_3^k)$ $(k \geq 2)$, then

$$\mathcal{R}(Q) \cong K[x_1, x_2, x_3, y_1, y_2, y_3]/I_2\begin{pmatrix} x_1 & x_2 & x_3^k \\ y_1 & y_2 & y_3 \end{pmatrix}$$

is an almost Gorenstein normal local domain (after localization), but not an almost Gorenstein graded ring.

Idea of the proof

$Q = (a_1, \ldots, a_r)A$: generated by s.s.o.p. in a Gor. local ring A
$\Psi : S = R[X_1, \ldots, X_r] \to \mathcal{R} := \mathcal{R}(Q)$
$\mathrm{Ker}\Psi = I_2(\mathbb{A})$, where $\mathbb{A} = \begin{pmatrix} X_1 & X_2 & \cdots & X_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$
Eagon-Northcott complex associated with the matrix $\mathbb{A}$

$$\mathcal{C}_\bullet : \ 0 \to C_r \to C_{r-1} \to \cdots \to C_0 = S$$

gives a graded minimal free resolution of $\mathcal{R}$ over S.
Taking $S(-r)$-Dual, we have the following presentation of $K_{\mathcal{R}}$:

$$\bigoplus_{i=1}^{r-2} S(-(i+1))^{\oplus r} \to \bigoplus_{i=1}^{r-1} S(-i) \to K_{\mathcal{R}} \to 0 (ex)$$

P_g-ideals (1)

Let A be a 2-dimensional excellent normal local domain.

Assume that $\exists f : X \to \operatorname{Spec} A$: resolution of singularities.

$p_g(A) = \ell_A(H^1(X, \mathcal{O}_X))$: the geometric genus of A.

Fact

Any $\mathfrak{m}$-primary integrally closed ideal I can be written as $I = I_Z := H^0(X, \mathcal{O}_X(-Z))$ for some res. of sing. $X \to \operatorname{Spec} A$ and some anti-nef cycle Z on X such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$.

P_g-ideal (2)

Theorem (Okuma-Watanabe-Y.)

Assume that $\mathcal{O}_X(-Z)$ has no fixed component. Then

$$\ell_A(H^1(X, \mathcal{O}_X(-Z)) \leq p_g(A).$$

If equality holds true, then $\mathcal{O}_X(-Z)$ is generated.

Defn (OWY)

$I = I_Z$ is an p_g-ideal $\Leftrightarrow \ell_A(H^1(X, \mathcal{O}_X(-Z)) = p_g(A)$.

Remark: Any excellent normal local domain of dimension **2** admits a p_g-ideal ([OWY]).

Basic results on p_g-ideals

Theorem (OWY2)

Let I be an $\mathfrak{m}$-primary ideal of A. Then TFAE

1. *I is a p_g-ideal.*
2. *$I^2 = QI$ for some parameter ideal $Q \subset I$, and I^n is integrally closed for every $n \geq 1$.*
3. *$\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.*

Theorem (OWY2)

Assume that I, J are p_g-ideals.
Then there exist $a \in I$,$b \in J$ such that $IJ = aJ + bI$. In particular, the multi-Rees algebra $R(I, J)$ is also a Cohen-Macaulay normal domain.

Rees algebras of p_g-ideals

Theorem

Assume that A is a Gorenstein excellent normal local domain of dimension 2.
Let I be a p_g-ideal of A.
$\Rightarrow \mathcal{R}(I)$ is an almost Gorenstein graded ring.

Question

How about non-Gorenstein case?

Rees algebra of rational singularities

A is a rational singularity $\Leftrightarrow p_g(A) = 0$.

Fact (cf. Lipman)

If A is a rational singularity, then any $\mathfrak{m}$-primary integrally closed ideal is a p_g-ideal.

Corollary

Assume that A is a Gorenstein rational singularity.
Then $\mathcal{R}(I)$ is an almost Gorenstein normal graded ring for any $\mathfrak{m}$-primary integrally closed ideal $I \subset A$.

Example: Ress algebra that is AG graded

Ex.

Let A be a regular local ring with $\dim A = 2$. Then $\mathcal{R}(I)$ is an almost Gorenstein graded ring for any integrally closed ideal $I \subset A$.

Ex.

Let $p \geq 1$ be an integer.

1. Let $A = k[[x,y,z]]/(x^2 + y^3 + z^{6p+1})$. Then $I_k = (x,y,z^k)$ is a p_g-ideal for every $k = 1,2,\ldots,3p$.
2. Let $A = k[[x,y,z]]/(x^2 + y^4 + z^{4p+1})$. Then $I_k = (x,y,z^k)$ is a p_g-ideal for every $k = 2,\ldots,2p$. But $I_1 = \mathfrak{m}$ is not.

When this is the case, $p_g(A) = p$.

Sketch of the proof

Assume that I is a p_g-ideal. Then $J = Q : I$ is also a p_g-ideal ([OWY3]). Hence we can choose $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ such that

$$IJ = gJ + Ih, \qquad \mathfrak{m}J = fJ + \mathfrak{m}h$$

since I, J are p_g-ideals and $\mathfrak{m}$ is integrally closed.
This implies that $\mathfrak{M} \cdot J\mathcal{R} \subset (f, gt)J\mathcal{R} + \mathcal{R}h$.
On the other hand, $K_{\mathcal{R}} = J\mathcal{R}$ and $a(\mathcal{R}) = -1$. Hence

$$\mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0 \text{ (ex)}$$

As $\dim C_{\mathcal{M}} \leq 2 < \dim \mathcal{R}$, φ is injective.
Hence $\mathcal{R}$ is an almost Gorenstein graded ring.

Socle ideals

Let $(A, \mathfrak{m})$ be a regular local ring with $\dim A = d \geq 2$.

Let Q be a parameter ideal of A.and put $I = Q : \mathfrak{m}$. Such an ideal I is called a socle ideal.

Fact

Let $I = Q : \mathfrak{m} \subset A$ be a socle ideal.
If $[d \geq 3]$ or $[d = 2$ and $Q \subset \mathfrak{m}^2]$, then $I^2 = QI$ holds true.
In particular, $\mathcal{R}(I)$ is a Cohen-Macaulay domain.

- We can show that $R(I)$ is not an almost Gorenstein graded ring in many cases.

Rees algebras of socle ideals, the case $d = 2$

Assume that A is a regular local ring of dimension 2 with $\mathfrak{m} = (x, y)$.

Let $Q = (a, b)$ a parameter ideal, and put $I = Q : \mathfrak{m}$. Assume that $Q \subset \mathfrak{m}^2$. Then $I^2 = QI$ and $\mu(I) = 3$. So we can write $I = (a, b, c)$. Since $xc, yc \in Q$, we have two equations

$$f_1 a + f_2 b + xc = 0 \quad \text{and} \quad g_1 a + g_2 b + yc = 0.$$

Theorem

If $(f_1, f_2, g_1, g_2) \subset \mathfrak{m}^2$ (e.g. $Q \subset \mathfrak{m}^3$) then $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring.

Rees algebras of socle ideals, the case $d \geq 3$

Theorem

Assume A is a regular local ring of $d = \dim A \geq 3$. Let Q be a parameter ideal with $Q \neq \mathfrak{m}$, and set $I = Q : \mathfrak{m}$. Then TFAE

1. $\mathcal{R}(I)$ *is an almost Gorenstein graded ring.*
2. *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

Example:Rees algebra that is not AG local

Ex.

Let $A = K[[x,y]]$ and $Q = (x^m, y^n)$ with $2 \leq m \leq n$. Set $I = Q : \mathfrak{m} = (x^m, x^{m-1}y^{n-1}, y^n)$.

- $m \geq 3 \Rightarrow \mathcal{R}(I)$ is not an almost Gorenstein local ring.
- $m = 2 \Rightarrow \mathcal{R}(I)$ is an almost Gorenstein graded ring.

Rmk. If $Q = (x^2, y^4)$, then $I = Q : \mathfrak{m} = (x^2, xy^3, y^4)$ and $\bar{I} = (x^2, xy^2, y^4)$. Hence $\mathcal{R}(I)$ is an almost Gorenstein graded ring but not normal. Indeed, $\overline{\mathcal{R}(I)} = \mathcal{R}((x^2, xy^2, y^4))$.

Problems

1. Examples of almost Gorenstein rings with higher dimension
2. Almost Gorenstein Rees algebras whose base ring is not Gorenstein
3. Almost Gorenstein property for toric algebras, invariant subrings, determinantal rings

Thank you very much for your attention!