The structure of preenvelopes with respect to maximal Cohen-Macaulay modules

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Introduction

R: a d-dimensional Cohen-Macaulay local ring with canonical module ω
MCM: the category of maximal Cohen-Macaulay R-modules
$(-)^\dagger := \mathrm{Hom}_R(-, \omega)$
$\delta_M : M \to M^{\dagger\dagger}$: the biduality homomorphism of M
The functor $(-)^\dagger$ induces a self duality on MCM:

$$(-)^\dagger : \mathrm{MCM} \rightleftarrows \mathrm{MCM}^{op} : (-)^\dagger.$$

δ_M is an isomorphism if $M \in \mathrm{MCM}$.
We assume that all R-modules are finitely generated.

Introduction

Theorem (Auslander-Buchweitz)

For any R-module M, there exists a short exact sequence

$$0 \to Y \to X \xrightarrow{\pi} M \to 0$$

such that $X \in \mathrm{MCM}$ and $\mathrm{id}_R Y < \infty$.

- π is called a maximal Cohen-Macaulay (CM) approximation of M.
- $\mathrm{id}_R Y < \infty \Leftrightarrow \mathrm{Ext}_R^1(\mathrm{MCM}, Y) = 0$.
- special MCM-preenvelope $\overset{\text{dual}}{\longleftrightarrow}$ maximal CM approximation

Introduction

Definition

Let $\mu : M \to X$ be an R-homomorphism with $X \in \mathsf{MCM}$.

(a) μ is called an MCM-preenvelope of M if

$$\mathrm{Hom}_R(\mu, X') : \mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X')$$

is an epimorphism for any $X' \in \mathsf{MCM}$.

(b) μ is called a special MCM-preenvelope of M if μ is an MCM-preenvelope and satisfies $\mathrm{Ext}^1_R(\mathrm{Coker}\,\mu, \mathsf{MCM}) = 0$.

(c) μ is called an MCM-envelope of M if μ is an MCM-preenvelope and every $\phi \in \mathrm{End}_R(X)$ that satisfies $\phi\mu = \mu$ is an automorphism.

The notions of MCM-precover, special MCM-precover, and MCM-cover are defined dually.

Introduction

Remark

1. Every MCM-precover is an epimorphism.
2. a maximal CM approximation = a special MCM-precover
3. an MCM-envelope $\Rightarrow$ a special MCM-preenvelope $\Rightarrow$ an MCM-preenvelope
   The first implication is due to Wakamatsu's lemma.
4. An MCM-envelope is unique up to isomorphism.

Introduction

Theorem

- (Auslander-Buchweitz)
  Every R-module has a special MCM-precover.
- (Yoshino)
  If R is Henselian (e.g. complete), then every R-module has an MCM-cover.

Theorem [Holm]

Every R-module has a special MCM-preenvelope, and if R is Henselian, every R-module has an MCM-envelope.

$\pi : X \to M^\dagger$: a special MCM-precover (resp. an MCM-cover) $\Rightarrow$
$\pi^\dagger \delta_M : M \to X^\dagger$: a special MCM-preenvelope (resp. an MCM-envelope)

Introduction

Let $\pi : X \to M$ be an R-homomorphism such that $X \in \mathrm{MCM}$. By definition, the following are equivalent.

1. π is a special MCM-precover.
2. $\mathrm{Ext}_R^1(\mathrm{MCM}, \mathrm{Ker}\,\pi) = 0$.
3. $\mathrm{id}_R(\mathrm{Ker}\,\pi) < \infty$.

Question

When is a given homomorphism $\mu : M \to X$ with $X \in MCM$ a special MCM-preenvelope?

We give an answer to this question by using its kernel and cokernel.

The structure of MCM-preenvelopes

Theorem A

Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathrm{MCM}$. TFAE

(a) μ is a special MCM-preenvelope of M.

(b) $\operatorname{codim}(\operatorname{Ker}\mu) > 0$ and $\operatorname{Ext}^1_R(\operatorname{Coker}\mu, \mathrm{MCM}) = 0$.

(c) $\operatorname{codim}(\operatorname{Ker}\mu) > 0$, and there exists an exact sequence $0 \to S \to \operatorname{Coker}\mu \to T \to U \to 0$ such that
- $\operatorname{codim} S > 1$, $\operatorname{codim} U > 2$,
- T satisfies Serre's condition (S_2),
- $\operatorname{id}_R T^\dagger < \infty$ and $T^\dagger$ satisfies Serre's condition (S_3).

where, $\operatorname{codim} M := d - \dim M$.

Using Theorem A, we get

- the result of the structure on the special proper MCM-coresolutions, and
- another characterization of special MCM-preenvelopes.

The structure of MCM-preenvelopes

Example

1. Let M be an R-module with $\operatorname{codim} M > 0$. Then $\mu : M \to 0$ is a special MCM-preenvelope.
2. Let $\underline{x} = x_1, x_2, \ldots, x_n$ be an R-regular sequence with $n \geq 3$. Consider an exact sequence

$$0 \to M \xrightarrow{\mu} R^{\oplus n} \xrightarrow{(x_1,\ldots,x_n)} R \to R/(\underline{x}) \to 0.$$

   Then μ is a special MCM-preenvelope.
3. Let K and C be R-modules with $\operatorname{codim} K > 0, \operatorname{codim} C > 1$ and $\sigma \in \operatorname{Ext}_R^2(C, K)$. σ defines an exact sequence

$$0 \to K \to M \xrightarrow{\mu} F \to C \to 0$$

   with F a free R-module. Then μ is a special MCM-preenvelope of M.

The structure of special proper MCM-coresolutions

Definition

Let M be an R-module, and

$$0 \to M \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} \cdots \qquad (*)$$

be an R-complex with $X^i \in \mathsf{MCM}$ for each i. Put $\mu^0 := \delta^0$ and let $\mu^i : \mathrm{Coker}\,\delta^{i-1} \to X^i$ be the induced morphism from δ^i for $i > 0$.

- If each μ^i is a special MCM-preenvelope (resp. an MCM-envelope), then we call (*) a special proper MCM-coresolution (resp. a minimal proper MCM-coresolution) of M.
- For a minimal proper MCM-coresolution (*), $\mathrm{Coker}\,\mu^{i-1}$ is called an i-th minimal MCM-cosyzygy of M, and it is denoted by $\mathrm{Cosyz}_{\mathsf{MCM}}{}^i M$.
- We say a minimal proper MCM-coresolution (*) has length at most n if $X^{n+1} = 0$.

The structure of special proper MCM-coresolutions

$$0 \longrightarrow M \xrightarrow{\delta^0=\mu^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} X^2 \longrightarrow \cdots (*)$$

$$X^0 \longrightarrow \operatorname{Coker}\delta^0 \xrightarrow{\mu^1} X^1, \qquad X^1 \longrightarrow \operatorname{Coker}\delta^1 \xrightarrow{\mu^2} X^2$$

Remark

Suppose R is Henselian.

- A special proper MCM-coresolution and a minimal proper MCM-coresolution exist for any R-module.
- For a special proper MCM-coresolution (*),
  - $\operatorname{Coker}\mu^i$ are unique up to free summands, and
  - $\operatorname{Ker}\mu^i$ are unique up to isomorphism.
- Minimal MCM-cosyzygies are unique up to isomorphism.

The structure of special proper MCM-coresolutions

Theorem B

Suppose R is Henselian. Let M be an R-module and

$$0 \to M \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} \cdots$$

a special proper MCM-coresolution of M. Put $\mu^0 := \delta^0$ and let $\mu^i : \operatorname{Coker} \delta^{i-1} \to X^i$ be the induced homomorphisms. Then for each $i \geq 0$, one has

1. $\operatorname{codim}(\operatorname{Ker} \mu^i) > i$,
2. there exists an exact sequence

$$0 \to S^i \to \operatorname{Coker} \mu^i \to T^i \to U^i \to 0$$

   such that

   - $\operatorname{codim} S^i > i + 1$, $\operatorname{codim} U^i > i + 2$,
   - T^i satisfies (S_2),
   - $\operatorname{id}_R (T^i)^\dagger < \infty$ and $(T^i)^\dagger$ satisfies (S_{i+3}).

The structure of special proper MCM-coresolutions

Suppose R is Henselian.

Corollary

For any R-module M,

- ${\mathrm{Cosyz}_{\mathrm{MCM}}}^{d} M = 0$ and
- ${\mathrm{Cosyz}_{\mathrm{MCM}}}^{d-1} M$ has finite length.

In particular, for any R-module M, the minimal proper MCM-coresolution of M has length at most $\min\{0, d-2\}$.

Remark

This corollary refines a theorem of Holm:
For any R-module M, the minimal proper MCM-coresolution of M has length at most $\min\{0, d-1\}$.

Another characterization of special MCM-preenvelopes

Let $\pi : X \to M$ be an R-homomorphism such that $X \in \mathrm{MCM}$. TFAE

(1) π is a special MCM-precover of M.

(2) There exists an R-complex

$$C = (0 \to C_d \xrightarrow{\delta_{d-1}} C_{d-1} \xrightarrow{\delta_{d-2}} \cdots \to C_1 \xrightarrow{\delta_0} C_0 \xrightarrow{\delta_{-1}} C_{-1} \to 0)$$

such that

- C_i is a finite direct sum of ω for $1 \le i \le d$,
- $\delta_{-1} = \pi$,
- C is exact.

Another characterization of special MCM-preenvelopes

Theorem C

Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathrm{MCM}$. TFAE

(1) μ is a special MCM-preenvelope of M.

(2) There exists an R-complex

$$C = (0 \to C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{d-2}} C^{d-1} \to 0)$$

such that

- C^i is free for $1 \le i \le d-1$,
- $\delta^{-1} = \mu$,
- $\operatorname{codim} \mathrm{H}^i(C) > i+1$ for any i.

Outline of the proof of Theorem A

Theorem A

Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathrm{MCM}$. TFAE

(a) μ is a special MCM-preenvelope of M.

(b) $\mathrm{codim}(\mathrm{Ker}\,\mu) > 0$ and $\mathrm{Ext}^1_R(\mathrm{Coker}\,\mu, \mathrm{MCM}) = 0$.

(c) $\mathrm{codim}(\mathrm{Ker}\,\mu) > 0$, and there exists an exact sequence $0 \to S \to \mathrm{Coker}\,\mu \to T \to U \to 0$ such that
- $\mathrm{codim}\, S > 1$, $\mathrm{codim}\, U > 2$,
- T satisfies (S_2),
- $\mathrm{id}_R T^\dagger < \infty$ and $T^\dagger$ satisfies (S_3).

(Outline of the proof of Theorem A)

The main part of the proof is the implication (a) $\Rightarrow$ (c). Therefore we give only the proof of (a) $\Rightarrow$ (c).

Outline of the proof of Theorem A

Lemma 1 (Holm)

If $\mu : M \to X$ is a special MCM-preenvelope, then $\mu^\dagger : X^\dagger \to M^\dagger$ is a special MCM-precover.

Note: The corresponding result for special MCM-precovers does not hold:
π:a special MCM-precover $\not\Rightarrow$ $\pi^\dagger$:a special MCM-preenvelope

Lemma 2

For an R-module M, one has

- $\operatorname{codim}(\operatorname{Ker}\delta_M) > 0$
- $\operatorname{codim}(\operatorname{Coker}\delta_M) > 1$
- $\operatorname{codim}\operatorname{Ext}^1_R(M^\dagger, \omega) > 2$.

Let $\mu : M \to X$ be a special MCM-preenvelope. By Lemma 1, $\mu^\dagger$ is a special MCM-precover. In particular, $\mu^\dagger$ is an epimorphism.

Outline of the proof of Theorem A

Take a short exact sequence $0 \to Y \to X^\dagger \xrightarrow{\mu^\dagger} M^\dagger \to 0$. Applying $(-)^\dagger$ to this sequence, we obtain an exact sequence

$$0 \to M^{\dagger\dagger} \xrightarrow{\mu^{\dagger\dagger}} X^{\dagger\dagger} \to Y^\dagger \to \operatorname{Ext}^1_R(M^\dagger, \omega) \to 0.$$

Note that Y has finite injective dimension and satisfies (S_3).
Because $\delta_X \mu = \mu^{\dagger\dagger}\delta_M$, we get another exact sequence

$$\begin{aligned} 0 &\to \operatorname{Ker}\delta_M \to \operatorname{Ker}\mu \to \operatorname{Ker}(\mu^{\dagger\dagger}) = 0 \\ &\to \operatorname{Coker}\delta_M \to \operatorname{Coker}\mu \to \operatorname{Coker}(\mu^{\dagger\dagger}) \to 0. \end{aligned}$$

This shows $\operatorname{codim}(\operatorname{Ker}\mu) > 0$. Combining these two sequences, we have an exact sequence

$$0 \to \operatorname{Coker}\delta_M \to \operatorname{Coker}\mu \to Y^\dagger \to \operatorname{Ext}^1_R(M^\dagger, \omega) \to 0.$$

We show that this sequence satisfies the condition (c).

Outline of the proof of Theorem A

We have only to show

- $Y^\dagger$ satisfies (S_2),
- $\mathrm{id}_R Y^{\dagger\dagger} < \infty$, and
- $Y^{\dagger\dagger}$ satisfies (S_3).

This follows from the next result.

Lemma 3 (Araya-Iima)

Let M be an R-module. If M satisfies (S_2), then $\delta_M : M \to M^{\dagger\dagger}$ is an isomorphism.

Applying this lemma to Y, we conclude that Y satisfies the above assertion. This shows the statement (c). $\square$

Thank you for your kind attention.