On weakly separable extensions and weakly quasi-separable extensions

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Introduction

The notions of weakly separable extensions and weakly quasi-separable extensions was introduced by N. Hamaguchi and A. Nakajima in their paper [On generalizations of separable polynomials over rings (2013)]. The porpose of this talk is to give refinements and sharpenings of Hamaguchi and Nakajima's results.

1. Definitions and notations
2. Weakly separable polynomials over a commutative ring
3. Weakly separable polynomials and weakly quasi-separable polynomials in skew polynomials rings

Let A/B be a ring extension (that is, B is a subring of A) and M a A-A-bimodule.

Definition 1.1

An additive map $\delta : A \longrightarrow M$ is called a B-derivation of A to M if

$$\delta(xy) = \delta(x)y + x\delta(y)\ (x, y \in A) \quad \text{and} \quad \delta(\alpha) = 0\ (\alpha \in B).$$

Moreover, δ is called central if

$$\delta(x)y = y\delta(x) \text{ for any } x, y \in A,$$

and δ is called inner if

$$\delta(x) = mx - xm\ (x \in A) \quad \text{for some fixed element } m \in M.$$

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Definition 1.2

A/B is called a separable extension if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \mapsto ab$ splits.

Lemma 1.3 (S. Elliger, 1975)

The following are equivalent.

1. A/B is separable.
2. For any A-A-bimodule M, every B-derivation of A to M is inner.

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In 1961, Y. Nakai introduced the notion of a quasi-separable extension of commutative rings by using the module differentials, and in the noncommutative case, it was characterized by H. Komatsu as follows:

Lemma 1.4 (K. Komatsu, 2001)

The following are equivalent.

1. A/B is quasi-separable.
2. For any A-A-bimodule M, every central B-derivation of A to M is zero.

Recently, N. Hamaguchi and A. Nakajima generalized the notion of separable extensions and that of quasi-separable extensions as follows:

Definition 1.5 (N. Hamaguchi and A. Nakajima, 2013)

1. A/B is called separable if for any A-A-bimodule M, every B-derivation of A to M is inner.
2. A/B is called quasi-separable if for any A-A-bimodule M, every central B-derivation of A to M is zero.
3. A/B is called weakly separable if every B-derivation of A to A is inner.
4. A/B is called weakly quasi-separable if every central B-derivation of A to A is zero.

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Example 1.6 (N. Hamaguchi and A. Nakajima, 2013)

Let A be a commutative ring and $M_2(A)$ the 2×2-matrix ring over A. Consider the following subset of $M_2(A)$:

$$T_2 = \left\{ \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \middle| r,\, s,\, t \in A \right\} \text{ and } R_2 = \left\{ \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \middle| r \in A \right\}.$$

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Then T_2/R_2 is not separable, but weakly separable and weakly quasi-separable.

- B : a ring with identity element 1.
- ρ : an automorphism of B.
- D : a ρ-derivation of B (i.e. D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\rho(\beta) + \alpha D(\beta)$ $(\alpha, \beta \in B)$).

Let $B[X; \rho, D]$ be the skew polynomial ring in which the multiplication is given by

$$\alpha X = X\rho(\alpha) + D(\alpha) \quad (\alpha \in B).$$

We set $B[X; \rho] = B[X; \rho, 0]$ (Automorphism type) and $B[X; D] = B[X; 1, D]$ (Derivation type).

We set

$$B[X;\rho,D]_{(0)} = \{g \in B[X;\rho,D] \mid g \text{ is monic, } gB[X;\rho,D] = B[X;\rho,D]g\}.$$

Definition 1.7

Let f be in $B[X;\rho,D]_{(0)}$. Then f is called separable (resp. weakly separable, weakly quasi-separable) in $B[X;\rho,D]$ if $B[X;\rho,D]/fB[X;\rho,D]$ is separable (resp. weakly separable, weakly quasi-separable) over B.

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Weakly separable polynomials over a commutative ring

For a ring extension A/B, we set

$$(A \otimes_B A)^A = \{\mu \in A \otimes_B A \,|\, x\mu = \mu x \ \text{ for any } \ x \in A\}.$$

Lemma 2.1 (K. Hirata and K. Sugano, 1966)

A/B is separable if and only if there exists $\sum_j x_j \otimes y_j \in (A \otimes_B A)^A$ such that $\sum_j x_j y_j = 1$.

Lemma 2.2

Let A/B be a commutative ring extension.
If there exists $\sum_j x_j \otimes y_j \in (A \otimes_B A)^A$ such that $\sum_j x_j y_j$ is a non-zero-divisor in A, then A/B is weakly separable.

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Example

Let B be a ring, G a group of order n, and $A = B[G]$, that is, A is a group ring of G over B.

It is easily seen that $\sum_{g\in G} g \otimes g^{-1} \in (A \otimes_B A)^A$, and hence if $n \, (= \sum_{g\in G} gg^{-1})$ is an invertible element then A/B is separable.
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Concerning separable polynomials over a commutative ring, the following is already known.

Lemma 2.3 (T. Nagahara, 1972)

Let B be a commutative ring, and $f(X)$ a monic polynomial in $B[X]$. Then the following are equivalent.

1. $f(X)$ is separable in $B[X]$.
2. $f'(X)$ is invertible in $B[X]$ modulo $(f(X))$.
3. $\delta(f(X))$ is invertible in B.

N. Hamaguchi and A. Nakajima proved that $f(X) = X^m - Xa - b$ is weakly separable in $B[X]$ if and only if $\delta(f(X))$ is a non-zero-divisor in B, or equivalently, $f'(X)$ is a non-zero-divisor in $B[X]$ modulo $(f(X))$.

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Outline of the proof

It is already know that 2 and 3 are equivalent (T. Nagahara, 1972). We shall show that 1 and 2 are equivalent.

Let $f(X) = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X]$, $A = B[X]/(f(X))$, and $x = X + (f(X)) \in A$.

($1 \Longrightarrow 2$) Assume that $f(X)$ is weakly separable, and let $f'(x)g(x) = 0$ for some $g(x) \in A$.

Then there exists a B-derivation D of A such that $D(x) = g(x)$ because $D(f(x)) = f'(x)D(x) = f'(x)g(x) = 0$.

Since $f(X)$ is weakly separable, we have $g(x) = 0$.

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(2 $\Longrightarrow$ 1) Assume that $f'(x)$ is a non-zero-divisor in A.
We have already known that

$$(A \otimes_B A)^A = \{\sum_{j=0}^{m-1} y_j h \otimes x^j \,|\, h \in A\},$$

where $y_j = x^{m-j-1} + x^{m-j-2}a_{m-1} + \cdots + xa_{j+2} + a_{j+1}$ $(0 \leq j \leq m-2)$ and $y_{m-1} = 1$ (S. Ikehata and S. Yamanaka, 2012).
In particular, $\sum_{j=0}^{m-1} y_j \otimes x^j \in (A \otimes_B A)^A$.
Noting that $f'(x) = \sum_{j=0}^{m-1} y_j x^j$, $f(X)$ is weakly separable in $B[X]$ by Lemma 2.2.

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Weakly separable polynomials in skew polynomial rings

N. Hamaguchi and A. Nakajima studied weakly separable polynomials and weakly quasi-separable polynomials in skew polynomial ring in the case that the coefficient ring is a integral domain.
In this section, we shall give some sharpenings of their result for a noncommutative coefficient ring.

We shall study weakly separable polynomials in $B[X;\rho]$.
Let $B^\rho = \{b \in B \mid \rho(b) = b\}$.
We consider a polynomial f in $B[X;\rho]_{(0)} \cap B^\rho[X]$ of the form

$$f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 = \sum_{j=0}^{m} X^j a_j \quad (m \geq 2).$$

Note that f is in $B[X;\rho]_{(0)} \cap B^\rho[X]$ if and only if

$$\alpha a_j = a_j \rho^{m-j}(\alpha) \quad (\alpha \in B,\ 0 \leq j \leq m-1).$$

Let $A = B[X;\rho]/fB[X;\rho]$, $x = X + fB[X;\rho] \in A$, and $\tilde{\rho}$ an automorphism of A which is naturally induced by ρ (that is, $\tilde{\rho}$ is defined by $\tilde{\rho}(\sum_{j=0}^{m-1} x^j c_j) = \sum_{j=0}^{m-1} x^j \rho(c_j)$).

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We shall use the following conventions:

- $V = \{g \in A \,|\, \alpha g = g\alpha \ (\alpha \in B)\}$ (the centralizer of B in A).
- $J_{\rho^k} = \{g \in A \,|\, \alpha g = g\rho^k(\alpha) \ (\alpha \in B)\}$ $(k \geq 1)$.
- $V^{\tilde{\rho}} = \{h \in V \,|\, \tilde{\rho}(h) = h\}$.

We consider a $V^{\tilde{\rho}}$-$V^{\tilde{\rho}}$-homomorphism $\tau : J_\rho \longrightarrow J_{\rho^m}$ defined by

$$\tau(h) = \sum_{k=0}^{m-1} x^k \sum_{j=0}^{k} \tilde{\rho}^j(h) a_{k+1}.$$

Lemma 3.1

If δ is a B-derivation of A, then $\delta(x) \in J_\rho$ and $\tau(\delta(x)) = 0$. Conversely, if $g \in J_\rho$ with $\tau(g) = 0$, then there exists a B-derivation δ of A such that $\delta(x) = g$.

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Proposition 3.2 (N. Hamaguchi and A. Nakajima, 2013)

Let B be an integral domain, m the order of ρ, and $f = X^m - u$ $(u \neq 0)$ in $B[X;\rho]_{(0)}$. Then f is weakly separable in $B[X;\rho]$ if and only if

$$\{b \in B \mid \sum_{j=0}^{m-1} \rho^j(b) = 0\} = \{\rho(c) - c \,|\, c \in B\}.$$

Theorem 3.3

Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X;\rho]_{(0)} \cap B^\rho[X]$. Then f is weakly separable in $B[X;\rho]$ if and only if

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Outline of the proof

Note that $\{g \in J_\rho \mid \tau(g) = 0\} \supset \{x(\tilde{\rho}(h) - h) \mid h \in V\}$.

Assume that f is weakly separable, and let g be an element in J_ρ such that $\tau(g) = 0$.
Then we can define a B-derivation δ of A such that $\delta(x) = g$ by Lemma 3.1.
Since f is weakly separable, we obtain $g = \delta(x) = hx - xh = x(\tilde{\rho}(h) - h)$ for some $h \in V$.
Thus $g \in \{x(\tilde{\rho}(h) - h) \mid h \in V\}$.

Outline of the proof

Note that $\{g \in J_\rho \,|\, \tau(g) = 0\} \supset \{x(\tilde{\rho}(h) - h) \,|\, h \in V\}$.

Assume that f is weakly separable, and let g be an element in J_ρ such that $\tau(g) = 0$.
Then we can define a B-derivation δ of A such that $\delta(x) = g$ by Lemma 3.1.
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Conversely, assume that $\{g \in J_\rho \,|\, \tau(g) = 0\} = \{x(\tilde{\rho}(h) - h) \,|\, h \in V\}$, and let δ be a B-derivation of A.
Then it follows from Lemma 3.1 that $\delta(x) \in \{g \in J_\rho \,|\, \tau(g) = 0\}$, and hence $\delta(x) = x(\tilde{\rho}(h) - h) = hx - xh$ for some $h \in V$.
This implies $\delta(w) = hw - wh$ for any $w \in A$.
Therefore δ is inner.

□

Theorem 3.4

Let m be the order of ρ, $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ in $B[X;\rho]_{(0)} \cap B^\rho[X]$, $C(A)$ a center of A, and I_x an inner derivation of A by x (that is, $I_x(h) = hx - xh$ for any $h \in A$).

1. *f is weakly separable in $B[X;\rho]$ if and only if the following sequence of $V^{\tilde{\rho}}$-$V^{\tilde{\rho}}$-homomorphisms is exact:*

$$0 \longrightarrow C(A) \xrightarrow{\text{inj}} V \xrightarrow{I_x} J_\rho \xrightarrow{\tau} V^{\tilde{\rho}}.$$

2. *f is separable in $B[X;\rho]$ if and only if the following sequence of $V^{\tilde{\rho}}$-$V^{\tilde{\rho}}$-homomorphisms is exact:*

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Outline of the proof

Note that $\operatorname{Im}\tau \subset V^{\tilde{\rho}}$ because $\tilde{\rho}^j(h)a_j = ha_j$ $(0 \leq j \leq m-1)$.

1. It is obvious by Theorem 3.3.

2. If f is separable then f is always weakly separable, and therefore it suffices to show that $\operatorname{Im}\tau = V^{\tilde{\rho}}$.
It is already known that f is separable in $B[X;\rho]$ if and only if there exists $h \in A$ such that $\rho^{m-1}(\alpha)h = h\alpha$ for any $\alpha \in B$ and $\sum_{j=0}^{m-1} y_j h x^j = 1$, where $y_j = x^{m-j-1} + x^{m-j-2}a_{m-1} + \cdots + xa_{j+2} + a_{j+1}$ $(0 \leq j \leq m-2)$ and $y_{m-1} = 1$ (Y. Miyashita, 1979).
It is obvious that $h \in J_\rho$. Noting that $y_j x^j = \sum_{k=j}^{m-1} x^k a_{k+1}$, we obtain

$$1 = \sum_{j=0}^{m-1} y_j x^j \tilde{\rho}^j(h) = \sum_{k=0}^{m-1} x^k \sum_{j=0}^{k} \tilde{\rho}^j(h) a_{k+1} = \tau(h).$$

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When B is an integral domain, N. Hamaguchi and A. Nakajima proved that every polynomial in $B[X;\rho]_{(0)}$ is weakly quasi-separable.

Proposition 3.5

If $\rho \neq 1$ and $\{\rho(c) - c \mid c \in B\}$ contains a non-zero divisor, then every polynomial in $B[X;\rho]_{(0)}$ is weakly quasi-separable.

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We shall study weakly separable polynomials in $B[X;D]$.
Let $B^D = \{b \in B \mid D(b) = 0\}$.
We consider a polynomial f in $B[X;D]_{(0)}$ of the form

$$f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 = \sum_{j=0}^{m} X^j a_j \quad (m \geq 2).$$

Note that f is in $B[X;D]_{(0)}$ if and only if $a_j \in B^D$ and

$$a_j\alpha = \sum_{k=j}^{m} \binom{k}{j} D^{k-j}(\alpha)a_k \quad (\alpha \in B,\ 0 \leq j \leq m-1).$$

Let $A = B[X;D]/fB[X;D]$, $x = X + fB[X;D] \in A$, and $\tilde{D}$ is an inner derivation of A which is naturally induced by D (that is, $\tilde{D}$ is defined by $\tilde{D}(\sum_{j=0}^{m-1} x^j c_j) = \sum_{j=0}^{m-1} x^j D(c_j)$).

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We shall use the following conventions:

- $V = \{g \in A \,|\, \alpha g = g\alpha \ (\alpha \in B)\}$ (the centralizer of B in A).
- $\tilde{D}(V) = \{\tilde{D}(h) \,|\, h \in V\}$.
- $V^{\tilde{D}} = \{h \in V \,|\, \tilde{D}(h) = 0\}$.

We consider a $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphism $\tau : V \longrightarrow V^{\tilde{D}}$ defined by

$$\tau(h) = \sum_{i=0}^{m-1} x^i \sum_{j=i}^{m-1} \binom{j+1}{i} \tilde{D}^{j-i}(h) a_{j+1}.$$

Lemma 3.6

If δ is a B-derivation of A, then $\delta(x) \in V$ and $\tau(\delta(x)) = 0$. Conversely, if $g \in V$ with $\tau(g) = 0$, then there exists a B-derivation δ of A such that $\delta(x) = g$.

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Proposition 3.7 (N. Hamaguchi and A. Nakajima)

Let B be an integral domain of prime characteristic p, and $f = X^p + Xb_1 + b_0$ in $B[X;D]_{(0)}$. Then f is weakly separable in $B[X;D]$ if and only if

$$\{c \in B \mid D^{p-1}(c) + cb_1 = 0\} = D(B).$$

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Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X;D]_{(0)}$. Then f is weakly separable in $B[X;D]$ if and only if

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Suppose that f is weakly separable, and let g be an element in V such that $\tau(g) = 0$.
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Thus $g \in \tilde{D}(V)$.

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Theorem 3.9

Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X;D]_{(0)}$. Then f is weakly separable in $B[X;D]$ if and only if the following sequence of $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphisms is exact:

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Conjecture

Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X;D]_{(0)}$. f is separable in $B[X;D]$ if and only if the following sequence of $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphisms is exact:

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It is true when B is of prime characteristic p and $f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0$.

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Lemma 3.11

Let $\overline{f'} = f' + fB[X;D] \in A$.
If δ is a central B-derivation of A, then $\overline{f'}\delta(x) = 0$ and $\delta(x)\alpha \in V^{\tilde{D}}$ for any $\alpha \in B$.
Conversely, if $g \in A$ with $\overline{f'}g = 0$ and $g\alpha \in V^{\tilde{D}}$ for any $\alpha \in B$, then there exists a central B-derivation δ of A such that $\delta(x) = g$.

Proposition 3.12

1. *f is weakly quasi-separable in $B[X;D]$ if and only if*
$$\{g \in A \,|\, \overline{f'}g = 0,\ g\alpha \in V^{\tilde{D}}\ (\alpha \in B)\} = 0.$$
2. *If f is weakly separable in $C(B^D)[X]$ then f is weakly quasi-separable in $B[X;D]$, where $C(B^D)$ is the center of B^D.*

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Thank you for your attention !!

Example 3.13

Let $B = \mathbb{Z}$ and m, $n \in \mathbb{Z}$. For the quadratic polynomials in $\mathbb{Z}[X]$ are classified as follows:

1. $X^2 - 2nX - m$ is not separable in $\mathbb{Z}[X]$. It is weakly separable if and only if $m \neq -n^2$.
2. $X^2 - (2n+1)X - m$ is always weakly separable in $\mathbb{Z}[X]$. It is separable if and only if $n^2 + n + m = 0$.

Example 3.14

Let $B = \mathbb{C}$ and ρ an automorphism of $\mathbb{C}$ defined by $\rho(a + bi) = a - bi$ (a, $b \in \mathbb{R}$). Then $f = X^2$ is not weakly separable in $\mathbb{C}[X;\rho]$, but weakly quasi-separable.