1 Preliminary results

Let S be a Noetherian prime ring with quotient ring Q and A be a fractional S-ideal. We use the following notation:

$$(S:A)_l = \{q \in Q \mid qA \subseteq S\}$$
$$(S:A)_r = \{q \in Q \mid Aq \subseteq S\}$$
$$A_v = (S:(S:A)_l)_r \supseteq A$$
$${}_vA = (S:(S:A)_r)_l \supseteq A$$

A is called a *v-ideal* if

$${}_vA = A = A_v$$

A v-ideal A is said to be *v-invertible* (*invertible*) if

$$_v((S:A)_l A) = S = (A(S:A)_r)_v$$
$$\Big((S:A)_l A = S = A(S:A)_r\Big)$$

Note that if A is v-invertible, then

$$O_r(A) = S = O_l(A) \text{ and } (S:A)_l = A^{-1} = (S:A)_r$$

where

$$\begin{aligned} O_l(A) &= \{q \in Q \mid qA \subseteq A\}, \text{ a } \textit{left order} \text{ of } A \\ O_r(A) &= \{q \in Q \mid Aq \subseteq A\}, \text{ a } \textit{right order} \text{ of } A \\ A^{-1} &= \{q \in Q \mid AqA \subseteq A\} \end{aligned}$$

D : hereditary Noetherian prime ring (an HNP ring for short) with quotient ring K
$R = D[t]$: polynomial ring over D in an indeterminate t.

We put

$$
\begin{aligned}
\cdot\ V_r(R) &= \{\mathfrak{a} :\ \text{ideals} \mid \mathfrak{a} = \mathfrak{a}_v\} \supseteq \\
V_{(m,r)}(R) &= \{\mathfrak{a} \in V_r(R) \mid \mathfrak{a} \text{ is maximal in } V_r(R)\} \\
\cdot\ V_l(R) &= \{\mathfrak{a} :\ \text{ideals} \mid \mathfrak{a} = {}_v\mathfrak{a}\} \supseteq \\
V_{(m,l)}(R) &= \{\mathfrak{a} \in V_l(R) \mid \mathfrak{a} \text{ is maximal in } V_l(R)\} \\
\cdot \mathrm{Spec}_0(R) &= \{\mathfrak{b} :\ \text{prime ideals} \mid \mathfrak{b} \cap D = (0) \text{ and} \\
&\qquad\qquad \mathfrak{b} \text{ is a v-ideal}\}.
\end{aligned}
$$

Proposition 1 *(1)* $V_{(m,r)}(R) = V_{(m,l)}(R)$ *and is equal to*

$$V_m(R) = \{\mathfrak{m}[t], \mathfrak{b} \mid \mathfrak{m} \text{ runs over all maximal ideals of } D \text{ and } \mathfrak{b} \in Spec_0(R)\}$$

(2) If $\mathfrak{b} \in Spec_0(R)$*, then* $\mathfrak{b}$ *is invertible.*

(3) $\{\mathfrak{p}[t] = \mathfrak{m}_1[t] \cap \cdots \cap \mathfrak{m}_k[t], \mathfrak{b} \mid \mathfrak{m}_1, \ldots, \mathfrak{m}_k$ *is a cycle of D,* $k \geq 1, \mathfrak{b} \in Spec_0(R)\}$
is the full set of maximal invertible ideals of R.
($\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ *is called a cycle of D if* $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ *are maximal ideals of D and*

$$O_r(\mathfrak{m}_1) = O_l(\mathfrak{m}_2), \ldots, O_r(\mathfrak{m}_{k-1}) = O_l(\mathfrak{m}_k), \quad O_r(\mathfrak{m}_k) = O_l(\mathfrak{m}_1).)$$

(4) The invertible ideals of R generate an Abelian group whose generators are maximal invertible ideals.

2 Examples

D : HNP ring $\subseteq K$: quotient ring satisfying the following:

(a) there is a cycle $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ such that

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = aD = Da$$

for some $a \in D$.

(b) any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$ is invertible.

Example 1 *Let* $D = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$*, where* $\mathbb{Z}$ *is the ring of integers and* p *is a prime number. Then*

$$\mathfrak{m}_1 = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix},\ \mathfrak{m}_2 = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & p\mathbb{Z} \end{pmatrix}$$

is a cycle and $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \begin{pmatrix} p & p \\ 1 & 0 \end{pmatrix} D = D \begin{pmatrix} p & p \\ 1 & 0 \end{pmatrix}$*, and*

$$\left\{ \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} D \,\middle|\, q :\ \textit{prime number}\ \neq p \right\}$$

is the full set of maximal ideals of D different from $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$.

We define a skew derivation (σ, δ) on D by

$$\sigma(r) = ara^{-1} \text{ and } \delta(r) = 0 \text{ for all } r \in D$$

Let $R = D[t]$ be the polynomial ring over D in an indeterminate t.

(σ, δ) on D is extended to a skew derivation on R by

$$\sigma(t) = t \text{ and } \delta(t) = a$$

Note

- $\sigma(\mathfrak{m}_i) = \mathfrak{m}_{i+1}\ (1 \leq i \leq n-1),\ \sigma(\mathfrak{m}_n) = \mathfrak{m}_1$
- $\sigma(\mathfrak{n}) = \mathfrak{n}\ \ (\forall\ \mathfrak{n}\ (\neq \mathfrak{m}_1, \ldots, \mathfrak{m}_n)$: maximal ideal)
- $V_m(R) = \{\mathfrak{m}_i[t],\ \mathfrak{n}[t],\ \mathfrak{b} \mid \mathfrak{n} \neq \mathfrak{m}_i$ and $\mathfrak{b} \in \mathrm{Spec}_0(R)\}$
- $I_m(R) = \{\mathfrak{p}[t],\ \mathfrak{n}[t],\ \mathfrak{b} \mid \mathfrak{p} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n, \mathfrak{n} \neq \mathfrak{m}_i$ and $\mathfrak{b} \in \mathrm{Spec}_0(R)\}$
  is the set of all maximal invertible ideals of R.

Note

- A maximal ideal of $K[t]$ is either

  $tK[t]$ or $\omega K[t]$ for some $\omega = k_l t^l + \cdots + k_0 \in Z(K[t])$

  with $k_l \neq 0, k_0 \neq 0, l \geq 1$, where $Z(K[t])$ is the center of $K[t]$.

- Any $\mathfrak{b} \in \mathrm{Spec}_0(R)$ is either

  $$\mathfrak{b} = tR \;\text{ or }\; \mathfrak{b} = \omega K[t] \cap R,$$

  where $\omega \in Z(K[t])$ and $\omega K[t]$ is a maximal ideal.

A fractional R-ideal $\mathfrak{a}$ is called

- σ-*invariant* if $\sigma(\mathfrak{a}) = \mathfrak{a}$
- δ-*stable* if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$
- (σ, δ)-*stable* if it is σ-invariant and δ-stable.

Lemma 2*(1) Any projective ideal of R is a product of an invertible ideal and an eventually v-idempotent ideal.*

(A v-ideal C is called eventually v-idempotent if $(C^n)_v$ is v-idempotent for some $n \geq 1$, that is, $((C^n)_v{}^2)_v = (C^n)_v$.)

(2) Any eventually v-idempotent ideal is not σ-invariant.

(3) $\mathfrak{n}[t]$ and $\mathfrak{p}[t]$ are (σ, δ)-stable.

(4) Let $\omega = t$ *or* $\omega \in Z(K[t])$ *and let* $\mathfrak{b} = \omega K[t] \cap R$*, which is a maximal invertible ideal of R. Then*

(i) $\mathfrak{b}^n$ *is* σ*-invariant for any* $n \geq 1$*.*

(ii) $\mathfrak{b}^n$ *is* δ*-stable if and only if* $\omega^n K[t]$ *is* δ*-stable if and only if* $\delta(\omega^n) = 0$*.*

(iii)(a) If char $K = 0$*, then* $\mathfrak{b}^n$ *is not* δ*-stable for any n.*

(b) If char $K = p \neq 0$ *and* $\delta(\omega) \neq 0$*, then* $\mathfrak{b}^p$ *is* (σ, δ)*-stable and* $\mathfrak{b}^i$ *is not* (σ, δ)*-stable* $(1 \leq i < p)$*.*

(c) If char $K = p \neq 0$ *and* $\delta(\omega) = 0$*, then* $\mathfrak{b}^n$ *is* (σ, δ)*-stable for all* $n \geq 1$*.*

(5) $\mathfrak{p}[t], \mathfrak{n}[t], \mathfrak{b}$ *(in case* $\delta(\omega) = 0$*) and* $\mathfrak{b}^p$ *(in case* $\delta(\omega) \neq 0$*) are* (σ, δ)*-prime ideals of R.*

In the remainder,

- $S = R[x;\sigma,\delta]$: Ore extension in an indeterminate x

We will prove that S is a maximal order.

Note

For an ideal $\mathfrak{a}$ of R,

$$\mathfrak{a}[x;\sigma,\delta] : \text{ ideal of } S \Longleftrightarrow \mathfrak{a} : (\sigma,\delta)\text{-stable}$$

Lemma 3 *Let A be an ideal of S such that $A = A_v$ and is maximal in $\{B : \text{ ideal } \mid B = B_v\}$.*
If $A \cap R = \mathfrak{a} \neq (0)$, then A is equal to one of

- $P = \mathfrak{p}[t][x;\sigma,\delta]$
- $N = \mathfrak{n}[t][x;\sigma,\delta]$
- $B = \mathfrak{b}[x;\sigma,\delta]$ *(in case $\mathfrak{b}$ is (σ,δ)-stable) or*
  $C = \mathfrak{b}^p[x;\sigma,\delta]$ *(in case $\mathfrak{b}$ is σ-invariant but not δ-stable)*

Lemma 4 *Let A be an ideal of S such that $A = A_v$ and $\mathfrak{a} = A \cap R \neq (0)$. Then $\mathfrak{a}$ is a (σ, δ)-stable invertible ideal and $A = \mathfrak{a}[x; \sigma, \delta]$.*

Lemma 5 *Let A be an ideal of S such that $A = A_v$ and $A \cap R = (0)$. Then A is v-invertible.*

Theorem 6 *$S = R[x; \sigma, \delta]$ is a maximal order and R is not a maximal order.*

Proof. Let A be any ideal of S.
Since $S \subseteq O_l(A) \subseteq O_l(A_v)$, in order to prove $O_l(A) = S$, we may assume that $A = A_v$.
By Lemmas 4 and 5, A is (v)-invertible.
Hence $O_l(A) = S$ and similarly $O_r(A) = S$, that is S is a maximal order.
Of course R is not a maximal order. $\square$

A Noetherian prime ring R is called a *unique factorization ring* (a UFR for short) if each prime ideal P with $P = P_v$ (or $P = {}_vP$) is principal, that is, $P = bR = Rb$ for some $b \in R$.

Note
R is a UFR if and only if R is a maximal order and each v-ideal is principal.

Proposition 7 *Suppose char* $D = 0$ *and any maximal ideal* $\mathfrak{n}$ *different from* $\mathfrak{m}_i$ $(1 \leq i \leq n)$ *is principal. Then* $S = R[x; \sigma, \delta]$ *is a UFR but* R *is not a UFR.*