Cayley graphs over a Finite Chain Ring and GCD-Graphs

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9 September 2015

Joint work with Borworn Suntornpoch

Overview

GCD-Graphs

Let D be a UFD. Let $c \in D$ be a nonzero nonunit element.
Assume that the commutative ring $D/(c)$ is finite.
Let $\mathcal{C}$ be a set of proper divisors of c.
Define the **gcd-graph**, $D_c(\mathcal{C})$, to be a graph whose vertex set is the quotient ring $D/(c)$ and edge set is

$$\{\{x+(c), y+(c)\} : x, y \in D \text{ and } \gcd(x-y, c) \in \mathcal{C}\}.$$

The gcd considered here is unique up to associate.

GCD-Graphs–Some Remarks

1. A gcd-graph generalizes gcd-graphs or integral circulant graph (its adjacency matrix is circulant (commuting with $Z = [\begin{smallmatrix} 0 & I_{n-1} \\ 1 & 0 \end{smallmatrix}]$) and all eigenvalues are integers) defined over $\mathbb{Z}$.

   W. So, Integral circulant graphs, *Discrete Math.*, 2006.

   W. Klotz and T. Sander, Some properties of unitary Cayley graphs, *The Electronic J. Comb.*, 2007.

   If G is a simple undirected graph on vertices $v_1, v_2, \ldots, v_n$, then the **adjacency matrix** of G is the matrix $A(G) = [a_{ij}]_{n\times n}$ given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent.} \end{cases}$$

GCD-Graphs–Some Remarks

2. If $\mathcal{C} = \{1\}$, then

$$\{x + (c), y + (c)\} \text{ is an edge} \Leftrightarrow x - y \text{ is a unit modulo } c,$$

so $D_c(\{1\}) = \text{Cay}(D/(c), (D/(c))^\times)$, called the **unitary Cayley graph**.

W. Klotz and T. Sander, Some properties of unitary Cayley graphs, *The Electronic J. Comb.*, 2007.

A. Ilić,The energy of unitary Cayley graphs, *Linear Algebra Appl.*, 2009.

Kiani D., Aghaei M.M.H., Meemark Y. and Suntornpoch B., Energy of unitary Cayley graphs and gcd-graphs, *Linear Algebra Appl.*, 2011.

GCD-Graphs–Some Remarks

3. Gcd-graphs are circulant $\Leftrightarrow$ $D/(c)$ is cyclic under addition. This is the case for $D = \mathbb{Z}$ and we can apply the Gauss sum to compute the eigenvalues, eigenvectors and energy.

W. So, Integral circulant graphs, *Discrete Math.*, 2006.

Kiani D., Aghaei M.M.H., Meemark Y. and Suntornpoch B., Energy of unitary Cayley graphs and gcd-graphs, *Linear Algebra Appl.*, 2011.

The sum of absolute values of all eigenvalues of a graph G is called the **energy of** G and denoted by $E(G)$. The energy is a graph parameter introduced by Gutman arising from the Hückel molecular orbital approximation for the total π-electron energy. Nowadays, the energy of graph is studied for purely mathematical interest.

GCD-Graphs

Write $c = p_1^{s_1} \dots p_k^{s_k}$ as a product of irreducible elements. Suppose that for each $i \in \{1, 2, \dots, k\}$, there exists a set $\mathcal{C}_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \dots, p_i^{a_{ir_i}}\}$, with $0 \leq a_{i1} < a_{i2} < \cdots < a_{ir_i} \leq s_i - 1$ so that

$$\mathcal{C} = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} \ : \ t_i \in \{1, 2, \dots, r_i\} \text{ for all } i \in \{1, 2, \dots, k\}\}.$$

Then for $x, y \in D/(c)$,

$$x \text{ is adjacent to } y \Leftrightarrow \gcd(x - y, p_i^{s_i}) \in D^\times \mathcal{C}_i \text{ for all } i.$$

GCD-Graphs

This implies that

$$D_c(\mathcal{C}) = \mathrm{Cay}(D/(p_1^{s_1}), \mathcal{C}_1) \otimes \cdots \otimes \mathrm{Cay}(D/(p_k^{s_k}), \mathcal{C}_k),$$

where each factor on the right is the Cayley graph over the finite chain ring $D/(p_i^{s_i})$.

For two graphs G and H, their **tensor product** $G \otimes H$ is the graph with vertex-set $V(G) \times V(H)$, where (u, v) is adjacent to $(u', v') \Leftrightarrow u$ is adjacent to u' in G and v is adjacent to v' in H. The adjacency matrix of $G \otimes H$ is the Kronecker product of $A(G)$ and $A(H)$, i.e., $A(G \otimes H) = A(G) \otimes A(H)$. Hence, $E(G \otimes H) = E(G)E(H)$.

Theorem

Let D be a UFD and a nonzero nonunit $c = p_1^{s_1} \dots p_k^{s_k} \in D$ factored as a product of irreducible elements. Assume that $D/(c)$ is finite and for each $i \in \{1, 2, \dots, k\}$, there exists a set $\mathcal{C}_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \dots, p_i^{a_{ir_i}}\}$, with $0 \leq a_{i1} < a_{i2} < \dots < a_{ir_i} \leq s_i - 1$ such that

$$\mathcal{C} = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} \ : \ t_i \in \{1, 2, \dots, r_i\} \text{ for all } i \in \{1, 2, \dots, k\}\}.$$

Then

$$E(D_c(\mathcal{C})) = E(D_{p_1^{s_1}}(\mathcal{C}_1)) \dots E(D_{p_k^{s_k}}(\mathcal{C}_k)).$$

Chain Rings

A ring is called a **chain ring** if all its ideals form a chain under inclusion.

For example, $\mathbb{Z}_{p^n}$, p a prime and $n \in \mathbb{N}$, is a chain ring. Also, every field is a chain ring.

If R is a finite commutative ring, it can be proven that:
R is a chain ring $\Leftrightarrow$ R is local whose maximal ideal is principal.

Finite Chain Rings

Let R be a finite chain ring with unique maximal ideal M and residue field of q elements. Let s be the nilpotency of R, that is, the least positive integer such that $M^s = \{0\}$. It can be shown that we have the chain of ideals

$$R \supset M \supset M^2 \supset \cdots \supset M^s = \{0\}.$$

Write $R = M^0$. By Lemma 2.4 of Norton, we also have $|M^i| = q^{s-i}$ for all $0 \le i \le s$, and so $|M^i/M^{i+1}| = q$ for all $0 \le i < s$.

G. H. Norton and A. Sălăgean, On the structure of linear and cyclic codes over finite chain rings, *Appl. Algebra Engng. Comm. Comput.*, 2000.

Finite Chain Rings

Thus, $|R| = q^s$. Moreover, M is principal generated by a $\theta \in M \setminus M^2$ and hence any element $x \in R$ can be written as

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$

where $v_i \in \mathcal{V} = \{e_0, e_1, \ldots, e_{p^t-1}\}$, a fixed set of representatives of cosets in R/M. Let

$$\mathcal{C} = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

where $0 \le a_1 < a_2 < \cdots < a_r \le s-1$.

Cayley Graphs

Consider the Cayley graph $\mathrm{Cay}(R, \mathcal{C})$ whose vertex set is $R$and $x, y \in R$ are adjacent if and only if $x - y \in \mathcal{C}$.

This graph generalizes the gcd-graph defined over $\mathbb{Z}/p^s\mathbb{Z}$ with the set $\mathcal{C} = \{p^{a_1}, p^{a_2}, \ldots, p^{a_r}\}$ of proper divisors of p^s where two vertices $a, b \in \mathbb{Z}/p^s\mathbb{Z}$ are adjacent if and only if $\gcd(b - a, p^s) = p^{a_i}$ for some $i \in \{1, 2, \ldots, r\}$.

The adjacency condition can be stated in terms of ideals as $b - a \in p^{a_i}\mathbb{Z} \setminus p^{a_i+1}\mathbb{Z}$ for some $i \in \{1, 2, \ldots, r\}$.

Adjacency matrix of Cay$(R, \mathcal{C})$

For $x, y \in R$ of the forms

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$
$$y = u_0 + u_1\theta + v_2\theta^2 + \cdots + u_{s-1}\theta^{s-1},$$

for some $v_i, u_j \in \mathcal{V}$, we have

$$x - y \in R \setminus M \Leftrightarrow v_0 \neq u_0.$$

Adjacency matrix of $\mathrm{Cay}(R, \mathcal{C})$

Then the adjacency matrix for $\mathrm{Cay}(R, \mathcal{C})$ is

$$A_0 = \begin{array}{cccc} e_1 + M & e_2 + M & \cdots & e_q + M \\ A_1 & B_1 & \cdots & B_1 \\ B_1 & A_1 & \cdots & B_1 \\ B_1 & B_1 & \cdots & B_1 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_1 & \cdots & A_1 \end{array},$$

where

$$B_1 = \begin{cases} J_{q^{s-1} \times q^{s-1}} & \text{if } R \smallsetminus M \subseteq \mathcal{C} \\ \mathbf{0}_{q^{s-1} \times q^{s-1}} & \text{if } R \smallsetminus M \not\subseteq \mathcal{C}, \end{cases}$$

and A_1 is a $q^{s-1} \times q^{s-1}$ submatrix depending on M^i, $i \geq 1$.

Adjacency matrix of Cay$(R, \mathcal{C})$

If $B_1 = \mathbf{0}_{q^{s-1} \times q^{s-1}}$, then

$$A_0 = I_q \otimes A_1 \qquad \text{(Process A)}$$

and if $B_1 = J_{q^{s-1} \times q^{s-1}}$, we have

$$A_0 = \overline{(I_q \otimes \overline{A}_1)}. \qquad \text{(Process B)}$$

Here, $J_{n \times n}$ is the $n \times n$ all 1's matrix and $\overline{X} = J - I - X$.

Adjacency matrix of Cay$(R, \mathcal{C})$

Next, we consider $x, y \in M$ such that

$$x = v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$
$$y = u_1\theta + v_2\theta^2 + \cdots + u_{s-1}\theta^{s-1},$$

for some $v_i, u_j \in \mathcal{V}$. Then

$$x - y \in M \setminus M^2 \Leftrightarrow v_1 \neq u_1.$$

Adjacency matrix of Cay$(R, \mathcal{C})$

Similarly, we have submatrices

$$B_2 = \begin{cases} J_{q^{s-2}\times q^{s-2}} & \text{if } M \setminus M^2 \subseteq \mathcal{C} \\ \mathbf{0}_{q^{s-2}\times q^{s-2}} & \text{if } M \setminus M^2 \not\subseteq \mathcal{C}, \end{cases}$$

and A_2, which is a $q^{s-2} \times q^{s-2}$ submatrix depending on M^i for $i \geq 2$ such that

$$A_1 = \begin{cases} \frac{I_q \otimes A_2}{(I_q \otimes \overline{A}_2)} & \text{if } B_2 = \mathbf{0}_{q^{s-2}\times q^{s-2}} \\ & \text{if } B_2 = J_{q^{s-2}\times q^{s-2}}. \end{cases}$$

Continuing these processes yields the sets of submatrices $\{A_1, \ldots, A_{s-1}\}$ and $\{B_1, \ldots, B_{s-1}\}$.

If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A of respective multiplicities $m_1, \ldots, m_k$, we use the notation

$$\operatorname{Spec} A = \begin{pmatrix} \lambda_1 & \ldots & \lambda_k \\ m_1 & \ldots & m_k \end{pmatrix}$$

to describe the spectrum of A.

Lemma

Let $i \in \{1, 2, \ldots, s-1\}$. *Assume that* $\operatorname{Spec} A_i = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ m_1 & m_2 & \ldots & m_k \end{pmatrix}$ *with* λ_1 *is the largest eigenvalues. Then*

$$\operatorname{Spec} \overline{(I_q \otimes \overline{A}_i)} = \begin{pmatrix} q^{s-i}(q-1)+\lambda_1 & \lambda_1 - q^{s-i} & \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ 1 & q-1 & q(m_1-1) & qm_2 & \ldots & qm_k \end{pmatrix}.$$

In particular, if $m_1 = 1$*, then*

$$\operatorname{Spec} \overline{(I_q \otimes \overline{A}_i)} = \begin{pmatrix} q^{s-i}(q-1)+\lambda_1 & \lambda_1 - q^{s-i} & \lambda_2 & \ldots & \lambda_k \\ 1 & q-1 & qm_2 & \ldots & qm_k \end{pmatrix}.$$

Repeatedly applying Process A, Process B and this lemma yield the following two lemmas on eigenvalues of $\operatorname{Cay}(R, \mathcal{C})$.

Lemma (Eigenvalues of $\mathrm{Cay}(R, \mathcal{C})$)

Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and of nilpotency s. Let

$$\mathcal{C} = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$. If $a_r = s-1$, then the eigenvalues of $\mathrm{Cay}(R, \mathcal{C})$ are as follows:

1. $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ *with multiplicity* q^{a_1},
2. $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_{k-1}}(q-1)$ *for* $k = 2, \ldots, r$,
3. $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_k-a_{k-1}-1} - q^{a_{k-1}+1}$ *for* $k = 2, \ldots, r$,
4. -1 *with multiplicity* $q^{a_r}(q-1)$.

Lemma (Eigenvalues of $\mathrm{Cay}(R, \mathcal{C})$)

Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and of nilpotency s. Let

$$\mathcal{C} = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$. If $a_r \neq s-1$, then the eigenvalues of $\mathrm{Cay}(R, \mathcal{C})$ are as follows:

1. $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ *with multiplicity* q^{a_1},
2. $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_{k-1}}(q-1)$ *for* $k = 2, \ldots, r$,
3. $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_k} - q^{a_{k-1}+1}$ *for* $k = 2, \ldots, r$,
4. $-q^{s-a_r-1}$ *with multiplicity* $q^{a_r}(q-1)$,
5. 0 *with multiplicity* $q^{a_r+1}(q^{s-a_r-1} - 1)$.

Finally, we compute the energy of the graph $\mathrm{Cay}(R, \mathcal{C})$.

Theorem (Energy of $\mathrm{Cay}(R, \mathcal{C})$)

Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and of nilpotency s. Let

$$\mathcal{C} = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$. Then

$$E(\mathrm{Cay}(R, \mathcal{C})) = 2(q-1)\left(q^{s-1}r - (q-1)\sum_{k=1}^{r-1}\sum_{i=k+1}^{r} q^{s-a_i+a_k-1}\right).$$

Thank You

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