Cayley graphs over a Finite Chain Ring and GCD-Graphs

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Overview

1. GCD Graphs

2. Finite Chain Rings

3. Energy
GCD-Graphs

Let $D$ be a UFD. Let $c \in D$ be a nonzero nonunit element. Assume that the commutative ring $D/(c)$ is finite. Let $C$ be a set of proper divisors of $c$. Define the **gcd-graph**, $D_c(C)$, to be a graph whose vertex set is the quotient ring $D/(c)$ and edge set is

$$\{\{x + (c), y + (c)\} : x, y \in D \text{ and } \gcd(x - y, c) \in C\}.$$

The gcd considered here is unique up to associate.
GCD-Graphs–Some Remarks

A gcd-graph generalizes gcd-graphs or integral circulant graph (its adjacency matrix is circulant (commuting with $Z = \begin{bmatrix} 0 & 1^{n-1} \\ 1 & 0 \end{bmatrix}$) and all eigenvalues are integers) defined over $\mathbb{Z}$.


If $G$ is a simple undirected graph on vertices $v_1, v_2, \ldots, v_n$, then the **adjacency matrix** of $G$ is the matrix $A(G) = [a_{ij}]_{n \times n}$ given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent.} \end{cases}$$
GCD-Graphs–Some Remarks

If $C = \{1\}$, then

$$\{x + (c), y + (c)\} \text{ is an edge } \Leftrightarrow x - y \text{ is a unit modulo } c,$$

so $D_c(\{1\}) = \text{Cay}(D/(c), (D/(c))^\times)$, called the **unitary Cayley graph**.


GCD-Graphs–Some Remarks

Gcd-graphs are circulant $\iff D/(c)$ is cyclic under addition. This is the case for $D = \mathbb{Z}$ and we can apply the Gauss sum to compute the eigenvalues, eigenvectors and energy.


The sum of absolute values of all eigenvalues of a graph $G$ is called the energy of $G$ and denoted by $E(G)$. The energy is a graph parameter introduced by Gutman arising from the Hückel molecular orbital approximation for the total $\pi$-electron energy. Nowadays, the energy of graph is studied for purely mathematical interest.
GCD-Graphs

Write $c = p_1^{s_1} \cdots p_k^{s_k}$ as a product of irreducible elements. Suppose that for each $i \in \{1, 2, \ldots, k\}$, there exists a set $C_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{iri}}\}$, with $0 \leq a_{i1} < a_{i2} < \cdots < a_{iri} \leq s_i - 1$ so that

$$C = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} : t_i \in \{1, 2, \ldots, r_i\} \text{ for all } i \in \{1, 2, \ldots, k\}\}.$$

Then for $x, y \in D/(c),$

$x$ is adjacent to $y \iff \gcd(x - y, p_i^{s_i}) \in D^\times C_i$ for all $i.$
This implies that

\[ D_c(C) = \text{Cay}(D/(p_1^{s_1}), C_1) \otimes \cdots \otimes \text{Cay}(D/(p_k^{s_k}), C_k), \]

where each factor on the right is the Cayley graph over the finite chain ring \( D/(p_i^{s_i}) \).

For two graphs \( G \) and \( H \), their tensor product \( G \otimes H \) is the graph with vertex-set \( V(G) \times V(H) \), where \((u, v)\) is adjacent to \((u', v')\) \iff \( u \) is adjacent to \( u' \) in \( G \) and \( v \) is adjacent to \( v' \) in \( H \). The adjacency matrix of \( G \otimes H \) is the Kronecker product of \( A(G) \) and \( A(H) \), i.e.,

\[ A(G \otimes H) = A(G) \otimes A(H). \]

Hence, \( E(G \otimes H) = E(G)E(H) \).
Theorem

Let $D$ be a UFD and a nonzero nonunit $c = p_1^{s_1} \ldots p_k^{s_k} \in D$ factored as a product of irreducible elements. Assume that $D/(c)$ is finite and for each $i \in \{1, 2, \ldots, k\}$, there exists a set $C_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{iri}}\}$, with $0 \leq a_{i1} < a_{i2} < \cdots < a_{iri} \leq s_i - 1$ such that

$$C = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} : t_i \in \{1, 2, \ldots, r_i\} \text{ for all } i \in \{1, 2, \ldots, k\}\}.$$  

Then

$$E(D_c(C)) = E(D_{p_1^{s_1}}(C_1)) \cdots E(D_{p_k^{s_k}}(C_k)).$$
A ring is called a **chain ring** if all its ideals form a chain under inclusion.

For example, $\mathbb{Z}_{p^n}$, $p$ a prime and $n \in \mathbb{N}$, is a chain ring. Also, every field is a chain ring.

If $R$ is a finite commutative ring, it can be proven that:
$R$ is a chain ring $\iff$ $R$ is local whose maximal ideal is principal.
Let $R$ be a finite chain ring with unique maximal ideal $M$ and residue field of $q$ elements. Let $s$ be the nilpotency of $R$, that is, the least positive integer such that $M^s = \{0\}$. It can be shown that we have the chain of ideals

$$R \supset M \supset M^2 \supset \cdots \supset M^s = \{0\}.$$ 

Write $R = M^0$. By Lemma 2.4 of Norton, we also have $|M^i| = q^{s-i}$ for all $0 \leq i \leq s$, and so $|M^i/M^{i+1}| = q$ for all $0 \leq i < s$.

Thus, $|R| = q^s$. Moreover, $M$ is principal generated by a $\theta \in M \setminus M^2$ and hence any element $x \in R$ can be written as

$$x = v_0 + v_1 \theta + v_2 \theta^2 + \cdots + v_{s-1} \theta^{s-1},$$

where $v_i \in \mathcal{V} = \{ e_0, e_1, \ldots, e_{p^t-1} \}$, a fixed set of representatives of cosets in $R/M$. Let

$$\mathcal{C} = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

where $0 \leq a_1 < a_2 < \cdots < a_r \leq s - 1.$
Consider the Cayley graph \( \text{Cay}(R, C) \) whose vertex set is \( R \) and \( x, y \in R \) are adjacent if and only if \( x - y \in C \).

This graph generalizes the gcd-graph defined over \( \mathbb{Z}/p^s\mathbb{Z} \) with the set \( C = \{ p^{a_1}, p^{a_2}, \ldots, p^{a_r} \} \) of proper divisors of \( p^s \) where two vertices \( a, b \in \mathbb{Z}/p^s\mathbb{Z} \) are adjacent if and only if \( \gcd(b - a, p^s) = p^{a_i} \) for some \( i \in \{1, 2, \ldots, r\} \).

The adjacency condition can be stated in terms of ideals as \( b - a \in p^{a_i}\mathbb{Z} \setminus p^{a_i+1}\mathbb{Z} \) for some \( i \in \{1, 2, \ldots, r\} \).
Adjacency matrix of Cay\((R, C)\)

For \(x, y \in R\) of the forms

\[
x = v_0 + v_1 \theta + v_2 \theta^2 + \cdots + v_{s-1} \theta^{s-1},
\]
\[
y = u_0 + u_1 \theta + v_2 \theta^2 + \cdots + u_{s-1} \theta^{s-1},
\]

for some \(v_i, u_j \in V\), we have

\[x - y \in R \setminus M \iff v_0 \neq u_0.\]
Adjacency matrix of Cay\((R, C)\)

Then the adjacency matrix for \(\text{Cay}(R, C)\) is

\[
A_0 =
\begin{pmatrix}
e_1 + M & e_2 + M & \cdots & e_q + M \\
A_1 & B_1 & \cdots & B_1 \\
B_1 & A_1 & \cdots & B_1 \\
B_1 & B_1 & \cdots & B_1 \\
\vdots & \vdots & \ddots & \vdots \\
B_1 & B_1 & \cdots & A_1
\end{pmatrix},
\]

where

\[
B_1 = \begin{cases} 
J_{q^{s-1} \times q^{s-1}} & \text{if } R \setminus M \subseteq C \\
0_{q^{s-1} \times q^{s-1}} & \text{if } R \setminus M \not\subseteq C,
\end{cases}
\]

and \(A_1\) is a \(q^{s-1} \times q^{s-1}\) submatrix depending on \(M^i, i \geq 1\).
Adjacency matrix of Cay\((R, \mathcal{C})\)

If \(B_1 = 0_{q^{s-1} \times q^{s-1}}\), then

\[A_0 = I_q \otimes A_1\]  

(Process A)

and if \(B_1 = J_{q^{s-1} \times q^{s-1}}\), we have

\[A_0 = (I_q \otimes \overline{A_1}).\]  

(Process B)

Here, \(J_{n \times n}\) is the \(n \times n\) all 1’s matrix and \(\overline{X} = J - I - X\).
Next, we consider $x, y \in M$ such that

\[ x = v_1 \theta + v_2 \theta^2 + \cdots + v_{s-1} \theta^{s-1}, \]
\[ y = u_1 \theta + v_2 \theta^2 + \cdots + u_{s-1} \theta^{s-1}, \]

for some $v_i, u_j \in V$. Then

\[ x - y \in M \setminus M^2 \iff v_1 \neq u_1. \]
Adjacency matrix of Cay$(R, C)$

Similarly, we have submatrices

\[
B_2 = \begin{cases} 
J_{q^{s-2} \times q^{s-2}} & \text{if } M \setminus M^2 \subseteq C \\
0_{q^{s-2} \times q^{s-2}} & \text{if } M \setminus M^2 \not\subseteq C,
\end{cases}
\]

and $A_2$, which is a $q^{s-2} \times q^{s-2}$ submatrix depending on $M^i$ for $i \geq 2$ such that

\[
A_1 = \begin{cases} 
I_q \otimes A_2 & \text{if } B_2 = 0_{q^{s-2} \times q^{s-2}} \\
\frac{I_q \otimes A_2}{(I_q \otimes A_2)} & \text{if } B_2 = J_{q^{s-2} \times q^{s-2}}.
\end{cases}
\]

Continuing these processes yields the sets of submatrices $\{A_1, \ldots, A_{s-1}\}$ and $\{B_1, \ldots, B_{s-1}\}$. 
If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $A$ of respective multiplicities $m_1, \ldots, m_k$, we use the notation

$$\text{Spec } A = \begin{pmatrix} \lambda_1 & \ldots & \lambda_k \\ m_1 & \ldots & m_k \end{pmatrix}$$

to describe the spectrum of $A$. 
Lemma

Let $i \in \{1, 2, \ldots, s - 1\}$. Assume that $\text{Spec } A_i = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}$ with $\lambda_1$ is the largest eigenvalues. Then

$$\text{Spec } (I_q \otimes \overline{A_i}) = \begin{pmatrix} q^{s-i}(q - 1) + \lambda_1 & \lambda_1 - q^{s-i} \\ 1 & q - 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ q(m_1 - 1) & qm_2 & \cdots & qm_k \end{pmatrix}.$$  

In particular, if $m_1 = 1$, then

$$\text{Spec } (I_q \otimes \overline{A_i}) = \begin{pmatrix} q^{s-i}(q - 1) + \lambda_1 & \lambda_1 - q^{s-i} & \lambda_2 & \cdots & \lambda_k \\ 1 & q - 1 & qm_2 & \cdots & qm_k \end{pmatrix}.$$  

Repeatedly applying Process A, Process B and this lemma yield the following two lemmas on eigenvalues of Cay($R, C$).
Lemma (Eigenvalues of $\text{Cay}(R, C)$)

Let $R$ be a finite chain ring with unique maximal ideal $M$, residue field of $q$ elements and of nilpotency $s$. Let

$$C = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \ldots \cup (M^{a_r} \setminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s - 1$. If $a_r = s - 1$, then the eigenvalues of $\text{Cay}(R, C)$ are as follows:

1. $(q - 1) \sum_{i=1}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_1}$,
2. $-q^{s-a_k-1} + (q - 1) \sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_k-1}(q - 1)$ for $k = 2, \ldots, r$,
3. $(q - 1) \sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_k-a_k-1-1} - q^{a_k-1+1}$ for $k = 2, \ldots, r$,
4. $-1$ with multiplicity $q^{a_r}(q - 1)$. 
Lemma (Eigenvalues of Cay$(R, C)$)

Let $R$ be a finite chain ring with unique maximal ideal $M$, residue field of $q$ elements and of nilpotency $s$. Let

$$ C = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}), $$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s - 1$. If $a_r \neq s - 1$, then the eigenvalues of Cay$(R, C)$ are as follows:

1. $(q - 1) \sum_{i=1}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_1}$,

2. $-q^{s-a_{k-1}-1} + (q - 1) \sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_{k-1}}(q - 1)$ for $k = 2, \ldots, r$,

3. $(q - 1) \sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_k} - q^{a_{k-1}+1}$ for $k = 2, \ldots, r$,

4. $-q^{s-a_r-1}$ with multiplicity $q^{a_r}(q - 1)$,

5. $0$ with multiplicity $q^{a_r+1}(q^{s-a_r-1} - 1)$.
Finally, we compute the energy of the graph $\text{Cay}(R, C)$.

**Theorem (Energy of $\text{Cay}(R, C)$)**

Let $R$ be a finite chain ring with unique maximal ideal $M$, residue field of $q$ elements and of nilpotency $s$. Let

$$C = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \ldots \cup (M^{a_r} \setminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s - 1$. Then

$$E(\text{Cay}(R, C)) = 2(q - 1) \left( q^{s-1}r - (q - 1) \sum_{k=1}^{r-1} \sum_{i=k+1}^{r} q^{s-a_i+a_k-1} \right).$$
Thank You

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