Higher products on Yoneda Ext algberas

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You know

$$0 \to M_1 \to X \to M_2 \to 0$$
$$0 \to M_2 \to Y \to M_3 \to 0$$

Splicing $\Downarrow$

$$0 \to M_1 \to X \to Y \to M_3 \to 0$$

Yoneda product on Ext.

$$0 \to M_1 \to X \to M_2 \to 0$$
$$0 \to M_2 \to Y \to M_3 \to 0$$
$$0 \to M_3 \to Z \to M_4 \to 0$$

$$?????\ \Downarrow$$

$$0 \to M_1 \to U \to V \to M_4 \to 0$$

Massey product on Ext.

DG-algebras and triple Massey products.

DG-algebras 1/3

Let R a dg-algebra.
That is $R := \bigoplus_{i\in\mathbb{Z}} R^i$ a graded algebra equipped with a differential ∂,
which is a morphism of degree 1 satisfying

$$\partial^2 = 0,$$

$$\partial(xy) = \partial(x)y + (-1)^{|x|}x\partial y.$$

where

$$|x| := \deg x.$$

DG-algebras 2/3

$Z(R) := \mathrm{Ker}(\partial)$: the cocycle group.
$B(R) := \mathrm{Im}(\partial)$: the coboundary group.
$H(R) := \dfrac{Z(R)}{B(R)}$: the cohomology group.

For a cocycle $x \in Z(R)$,
$[x]$ denotes the cohomology class, i.e.,
$[x] := x + B(R) \in H(R)$

DG-algebras 3/3

The cohomology group H(R) comes with a graded algebra structure.
The multiplication

$$[x][y] := [xy]$$

gives the cohomolgoy group H(R) a structure of graded algebra.

Example of dg-algebra: the endomorphism DG-algebra

A: an ordinary algebra.
M: a left A-module.
$P^\bullet \xrightarrow{\simeq} M$: a projective resolution of M.
Then
the endomorphism algebra $R = \mathrm{End}_A(P^\bullet)$
has a canonical dg-algebra structure.

$$H(R) \cong \mathrm{Ext}_A(M, M)$$

(An isomorphism of graded algebras.)

The cohomology group H(R) has more structures than associative multiplication. First we introduce triple Massey product, which is partially defined 3-ary operation of degree -1.

Triple Massey product (Outline)

From $\xi, \eta, \zeta \in \mathsf{H}(\mathsf{R})$ such that

$$\xi\eta = 0, \quad \eta\zeta = 0,$$

we construct cocycles w of degree

$$|\mathsf{w}| = |\xi| + |\eta| + |\zeta| - 1.$$

"The cohomology classes [w] are called triple Massey product of ξ, η, ζ".

Triple Massey products 1/4

Let $\xi, \eta, \zeta \in H(R)$ such that

$$\xi\eta = 0, \ \ \eta\zeta = 0.$$

Let $x, y, z \in Z(R)$ be cocycles which represent ξ, η, ζ.

Then $\xi\eta = 0 \implies xy \in B(R)$

$$\exists u \in R^{|\xi|+|\eta|-1} \text{ such that } xy = \partial u.$$

$$\eta\zeta = 0 \implies$$

$$\exists v \in R^{|\eta|+|\zeta|-1} \text{ such that } yz = \partial v.$$

Triple Massey products 2/4

We set

$$\mathbf{w} := (-1)^{|x|+|y|}xv + (-1)^{|y|+1}uz.$$

Claim 2.1

w *is cocycle.*

$\because$ **Using the equations** $\partial x = 0,\ \partial z = 0,$ $xy = \partial u,\ yz = \partial v,$ **we have**

$$\partial(w) = (-1)^{|y|}xyz + (-1)^{|y|+1}xyz = 0$$

Triple Massey products **3/4**

The degree |w| of w is

$$|\mathbf{w}| = |\xi| + |\eta| + |\zeta| - 1$$

Since w is a cocycle,
we can take a cohomology class [w].

Remark 2.2

The cohomology class **[w] does depend** on the choices of **x, y, z, u, v**.

Triple Massey product **4/4**

Definition 1

The triple Massey product $\langle \xi, \eta, \zeta \rangle$ of ξ, η, ζ is a subset of cohomology algebra

$$\langle \xi, \eta, \zeta \rangle \subset \mathsf{H}^{|\xi|+|\eta|+|\zeta|-1}(\mathbf{R})$$

consisting of $[\mathbf{w}]$ obtained in the above way.

Question

Recall

$$\mathrm{Ext}^n_A(M, M) = \frac{\{0 \to M \to X_1 \to \cdots \to X_n \to M \to 0\}}{\sim}.$$

The multiplication on $\mathrm{Ext}_A(M, M)$ splices the exact sequences.

Can we express triple Massey product on $\mathrm{Ext}_A(M, M)$ in terms of exact sequences?

Triple Massey products of three short exact sequences.

Outline 1/2

Let A be an algebra over a field k.
For a short exact sequence x

$$\mathbf{x : 0 \to M_1 \to X \to M_2 \to 0}$$

$\{\mathbf{x}\} \in \mathbf{Ext}^1_{\mathbf{A}}(\mathbf{M}_2, \mathbf{M}_1)$ **denotes the corresponding cohomology class.**

Outline 2/2

For short exact sequences x, y, z such that

$$\{x\}\{y\} = 0, \ \{y\}\{z\} = 0,$$

we compute triple Massey product of
$\{x\}, \{y\}, \{z\}$,
in terms of exact sequences.
Since $1 + 1 + 1 - 1 = 2$
$\langle\{x\}, \{y\}, \{z\}\rangle \subset \mathrm{Ext}^2_A(M_4, M_1)$ consist of classes of exact sequence of length 4

$$0 \to M_1 \to U \to V \to M_4 \to 0$$

Recalling basic fact 1/2

Let x, y be exact sequences.

$$x : 0 \to M_1 \to X \xrightarrow{f} M_2 \to 0$$

$$y : 0 \to M_2 \xrightarrow{g} Y \to M_3 \to 0$$

Recall that TFAE:

1. $\{x\}\{y\} = 0$ in $\mathrm{Ext}_A(M_3, M_1)$
2. $\exists \tilde{f} : \widetilde{X} \to Y$ such that $f : X \to M_2$ is the pull-back of $\tilde{f}$ along $g : M_2 \to Y$.
3. $\exists \hat{g} : X \to \widehat{Y}$ such that $g : M_2 \to Y$ is the push-out of $\hat{g}$ along $f : X \to M_2$.

Recalling basic fact **2/2**

The second condition:

$$\begin{CD} 0 @>>> M @>>> X @>f>> M_2 @>>> 0 \\ @. @| @VVV @VVgV @. \\ 0 @>>> M @>>> \widetilde{X} @>\tilde{f}>> Y @>>> 0 \end{CD}$$

The third condition :

$$\begin{CD} 0 @>>> X @>\hat{g}>> \widehat{Y} @>>> M_3 @>>> 0 \\ @. @VfVV @VVV @| @. \\ 0 @>>> M_2 @>g>> Y @>>> M_3 @>>> 0 \end{CD}$$

Triple Massey product of short exact sequences **1/3**.

Let x, y, z be exact sequences

$$x : 0 \to M_1 \to X \xrightarrow{f} M_2 \to 0$$

$$y : 0 \to M_2 \xrightarrow{g} Y \xrightarrow{h} M_3 \to 0$$

$$z : 0 \to M_3 \xrightarrow{k} Z \to M_4 \to 0$$

such that

$$\{x\}\{y\} = 0, \ \{y\}\{z\} = 0.$$

Triple Massey product of short exact sequences **2/3**.

$$\{x\}\{y\} = 0 \implies$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & X & \xrightarrow{f} & M_2 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow g & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & \widetilde{X} & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \end{array}$$

$$\{y\}\{z\} = 0 \implies$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\hat{k}} & \widehat{Z} & \longrightarrow & M_4 & \longrightarrow & 0 \\ & & h \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M_3 & \xrightarrow{k} & Z & \longrightarrow & M_4 & \longrightarrow & 0 \end{array}$$

Triple Massey product of short exact sequences **3/3**.

By splicing

$$0 \to M_1 \to \widetilde{X} \to Y \to 0$$
$$0 \to Y \to \widehat{Z} \to M_4 \to 0$$

we obtain

$$w : 0 \to M_1 \to \widetilde{X} \to \widehat{Z} \to M_4 \to 0$$

Proposition 3.1

$$-w \in \langle \{x\}, \{y\}, \{z\} \rangle \subset \mathrm{Ext}^2_A(M_4, M_1)$$

A composition series like expression 1/3

The middle term X of

$$x : 0 \to M_1 \to X \to M_2 \to 0$$

has a filtration $F = \{F_1 \subset F_2\}$

$$M_1 = F_1 \subset F_2 = X, \;\; F_2/F_1 = M_2.$$

We express the exact sequence x as

$$0 \longrightarrow M_1 \longrightarrow \begin{matrix} M_2 \\ M_1 \end{matrix} \longrightarrow M_2 \longrightarrow 0 \quad .$$

A composition series like expression 2/3

$$y : 0 \to M_2 \to Y \to M_3 \to 0$$

is

$$\begin{array}{ccccccccc} & & & & M_3 & \searrow & & & \\ 0 & \longrightarrow & M_2 & \longrightarrow & M_2 & & M_3 & \longrightarrow & 0 \end{array}$$

$$0 \to M_1 \to \widetilde{X} \longrightarrow Y \to 0$$

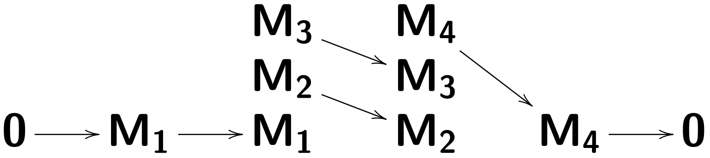
is

$$\begin{array}{ccccccccc} & & & & M_3 & \searrow & & & \\ & & & & M_2 & \searrow & M_3 & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_1 & & M_2 & \longrightarrow & 0 \end{array}$$

A composition series like expression **3/3**

$$\mathbf{w} : 0 \to M_1 \to \widetilde{X} \to \widehat{Z} \to M_4 \to 0$$

is



Prop 3.1 (restate). By triple Massey product

$$0 \longrightarrow M_1 \longrightarrow \begin{matrix} M_3 \\ M_2 \\ M_1 \end{matrix} \quad \begin{matrix} M_4 \\ M_3 \\ M_2 \end{matrix} \quad M_4 \longrightarrow 0$$

is obtained from

$$0 \longrightarrow M_1 \longrightarrow \begin{matrix} M_2 \\ M_1 \end{matrix} \quad M_2 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow \begin{matrix} M_3 \\ M_2 \end{matrix} \quad M_3 \longrightarrow 0$$

$$0 \longrightarrow M_3 \longrightarrow \begin{matrix} M_4 \\ M_3 \end{matrix} \quad M_4 \longrightarrow 0$$

In other words, by reversing triple Massey product

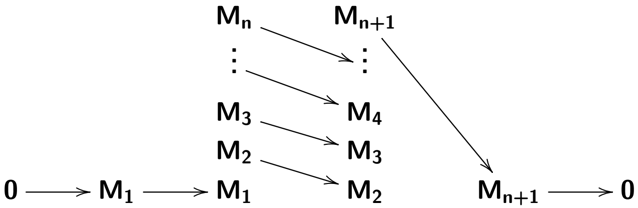
$$0 \longrightarrow M_1 \longrightarrow \begin{array}{c} M_3 \\ M_2 \\ M_1 \end{array} \longrightarrow \begin{array}{c} M_4 \\ M_3 \\ M_2 \end{array} \longrightarrow M_4 \longrightarrow 0$$

is decomposed into

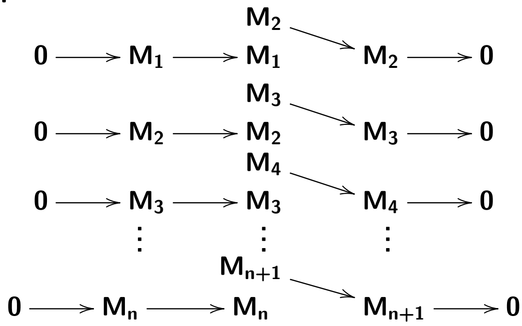
$$0 \longrightarrow M_1 \longrightarrow \begin{array}{c} M_2 \\ M_1 \end{array} \longrightarrow M_2 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow \begin{array}{c} M_3 \\ M_2 \end{array} \longrightarrow M_3 \longrightarrow 0$$

$$0 \longrightarrow M_3 \longrightarrow \begin{array}{c} M_4 \\ M_3 \end{array} \longrightarrow M_4 \longrightarrow 0$$



is obtained from



by higher Massey product.

Application: Generation of $\mathbf{Ext}^2_{\mathbf{A}}(\mathbf{S}, \mathbf{S})$

Proposition 3.2

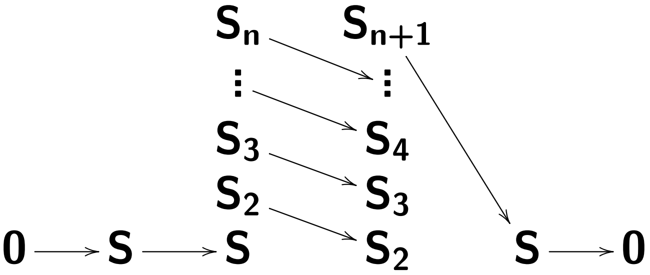
Let $\mathbf{A}$ *be a finite dimensional algebra,*
$\mathbf{S} := \bigoplus(\text{simple modules})$.
Then $\mathbf{Ext}^2_{\mathbf{A}}(\mathbf{S}, \mathbf{S})$ *is generated by* $\mathbf{Ext}^1_{\mathbf{A}}(\mathbf{S}, \mathbf{S})$ *using higher Massey products.*

“Proof”

Every exact sequence

$$\mathbf{0 \to S \to U \to V \to S \to 0}$$

is of the form $\mathbf{S_i}$ are simple modules.



Higher Massey products and A_∞-products on H(R).

Higher Massey product 1/2

For $n \geq 4$, n-th Massey products is partially defined n-ary operation of degree $2 - n$.
For $\xi_1, \xi_2, \ldots, \xi_n \in H(R)$,

$$\xi_1\xi_2 = 0, \xi_2\xi_3 = 0, \ldots, \xi_{n-1}\xi_n = 0$$

is a necessary condition to define n-th Massey product, but not a sufficient condition.

Higher Massey product **2/2**

If n-th Massey product $\langle \xi_1, \dots, \xi_n \rangle$ is defined, it is a subset of cohomology group of degree $|\xi_1| + |\xi_2| + \cdots + |\xi_n| - (n-2)$.

$$\langle \xi_1, \dots, \xi_n \rangle \subset \mathbf{H}^{|\xi_1|+|\xi_2|+\cdots+|\xi_n|-(n-2)}(\mathbf{R}).$$

A_∞-products 1/2

A_∞-product on a graded module H is a collection of morphisms $m_n : H^{\otimes n} \to H$ of degree $2 - n$ for $n \geq 1$ which satisfies the Stasheff identities

$$\sum_{i+j+k=n} (-1)^{ij+k} m_{i+1+k}(\mathrm{id}_H^{\otimes i} \otimes m_j \otimes \mathrm{id}_H^{\otimes k}) = 0$$

for $n \geq 1$

A_∞-products 2/2

1. If $m_n = 0$ for $n \neq 1, 2$,
then m_2 is an associative multiplication
and m_1 is a differential w.r.t m_2,
and H is a DG algebra.
2. If $m_1 = 0$,
then m_2 is an associative multiplication
and H is a graded algebra.
The higher multiplications $m_n \ (n \geq 3)$
can be viewed as an additional structure
on the graded algebra H.

Kadeishvili's Theorem

Theorem 2

Let $\mathbf{R}$ *be a DG-algebra and* $\mathbf{H := H(R)}$.
Then $\exists \mathbf{m_n : H^{\otimes n} \to H}$ *for* $\mathbf{n \geq 3}$
which makes $\mathbf{H}$ *an* $\mathbf{A_\infty}$*-algebra such that*
$\mathbf{H}$ *is quasi-isomorphic to* $\mathbf{R}$ *as* $\mathbf{A_\infty}$*-algebras.*
Moreover, such $\mathbf{A_\infty}$*-algebra structure on* $\mathbf{H}$ *is unique up to non-canonical isomorphisms.*

Relation between Massey product and $\mathbf{A_\infty}$-product on $\mathbf{H(R)}$

Theorem 3 (Lu-Palmieri-Wu-Zhang)

Let $\mathbf{R}$ *be a DG-algebra,* $\xi_1, \dots, \xi_n \in \mathbf{H(R)}$*.*
Assume that Massey product $\langle \xi_1, \dots, \xi_n \rangle$ *is defined.*
Then

$$\pm \mathbf{m_n}(\xi_1, \dots, \xi_n) \in \langle \xi_1, \dots, \xi_n \rangle \subset \mathbf{H(R)}.$$

where $\mathbf{A_\infty}$*-products* $\mathbf{m_n}\ \ (\mathbf{n} \geq \mathbf{3})$ *is obtained by the Merkulov construction.*

Applications

Application: Generation of $\mathbf{Ext_A(S, S)}$ by higher products

Theorem 4 (Gugenheim-May, Keller)

Let $\mathbf{A} = \bigoplus_{\mathbf{i} \geq 0} \mathbf{A_i}$ *be a locally finite graded algebra over a field and*
$\mathbf{S} := \bigoplus$ (simple modules)
Then the extension algebra $\mathbf{Ext_A(S, S)}$ *is generated by* $\mathbf{Ext^0_A(S, S)}$ *and* $\mathbf{Ext^1_A(S, S)}$ *using Massey products.*

Remark 5.1

1. Gugenheim-May showed that $\mathbf{Ext_A(S,S)}$ is generated by $\mathbf{Ext^0_A(S,S)}$ and $\mathbf{Ext^1_A(S,S)}$ using Matric Massey products.
   Their main tool was Algebraic Eilenberg-Moore spectral sequence.
2. Keller did not write any proof.
   He stated that $\mathbf{Ext_A(S,S)}$ is generated by $\mathbf{Ext^0_A(S,S)}$ and $\mathbf{Ext^1_A(S,S)}$ using $\mathbf{A}_\infty$ products.

Application of Theorem 4: Koszulity and higher products

Corollary 5 (Keller)

Let $\mathbf{A}$ *be a locally finite non-negatively graded algebra such that the augmentation algebra* $\mathbf{A_0}$ *is semi-simple. Then the followings are equivalent:*

(1) A is Koszul.

(2) the A_∞-product on the Ext algebra $\mathrm{Ext}_A(S, S)$ vanish.

(3) the Matric Massey products on the Ext algebra $\mathrm{Ext}_A(S, S)$ vanish.

(4) the Massey products on the Ext algebra $\mathrm{Ext}_A(S, S)$ vanish.

Application: Generation of $\mathbf{Ext_R}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$ by higher products

Theorem 6

Let $\mathbf{R}$ *be a Noetherian algebra with the center* $\mathbf{Z} = \mathbf{Z}(\mathbf{R})$ *and* $\mathfrak{p}$ *a maximal ideal of* $\mathbf{Z}$*. Set* $\kappa(\mathfrak{p}) := \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{Z}_{\mathfrak{p}}/\mathfrak{p}\mathbf{Z}_{\mathfrak{p}}$*. Then the Ext algebra* $\mathbf{E} = \mathbf{Ext_R}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$ *is generated by* $\mathbf{E}^0$ *and* $\mathbf{E}^1$ *using Massey products.*

Thank you