TILTING THEORY OF PREPROJECTIVE ALGEBRAS AND c-SORTABLE ELEMENTS

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Abstract. For a finite acyclic quiver Q and the corresponding preprojective algebra Π, the quotient algebra Π_w of Π associated with an element w in the Coxeter group of Q was introduced by Buan-Iyama-Reiten-Scott [6]. The algebra Π_w is Iwanaga-Gorenstein and has the natural $\mathbb{Z}$-grading. Recently, the author showed that the stable category of the category of graded Cohen-Macaulay Π_w-modules has tilting objects when w is a c-sortable element. In this paper, we study the endomorphism algebra of a tilting object.

1. Introduction

The preprojective algebra Π of Q has been introduced by Gelfand-Ponomarev to study representation theory of the path algebra of Q. Preprojective algebras play important roles in many areas of mathematics. One of them is that preprojective algebras provide 2-Calabi-Yau triangulated categories (2-CY, for short) with cluster tilting objects which have been studied in the view point of categorification of cluster algebras.

In the case when Q is a Dynkin quiver, the preprojective algebra Π of Q is a finite dimensional selfinjective algebra. In this case, Geiss-Leclerc-Schröer showed that the stable category $\underline{\mathsf{mod}}\,\Pi$ is a 2-CY category and $\underline{\mathsf{mod}}\,\Pi$ has cluster tilting objects [7]. In the case when Q is non-Dynkin quiver, Π is an infinite dimensional algebra. In this case, Buan-Iyama-Reiten-Scott introduced and studied the factor algebra Π_w associated with an element w in the Coxeter group of Q [6]. They showed that Π_w is a finite dimensional Iwanaga-Gorenstein algebra of dimension at most one and the stable category of $\mathsf{Sub}\,\Pi_w$ is a 2-CY category and has cluster tilting objects, where $\mathsf{Sub}\,\Pi_w$ is the full subcategory of $\mathsf{mod}\,\Pi_w$ of submodules of finitely generated free Π_w-modules.

There are other classes of 2-CY triangulated categories. Amiot introduced the generalized cluster category $\mathcal{C}_A$ for a finite dimensional algebra A of finite global dimension [1]. If $\mathcal{C}_A$ is Hom-finite, then $\mathcal{C}_A$ is a 2-CY category and has cluster tiling objects. There are close connections between 2-CY categories $\underline{\mathsf{Sub}}\,\Pi_w$ and $\mathcal{C}_A$. Amiot-Reiten-Todorov [3] showed that for any finite acyclic quiver Q and any element w of the Coxeter group, there is a triangle equivalence

$$\underline{\mathsf{Sub}}\,\Pi_w \simeq \mathcal{C}_{A_w}$$

for some finite dimensional algebra A_w of global dimension at most two.

The aim of this paper is to construct a derived category version of this equivalence. We regard Π_w as a $\mathbb{Z}$-graded algebra and consider the stable category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ of graded Π_w-submodules of graded free Π_w-modules. We construct a tilting object M in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ and calculate the endomorphism algebra of M.

The detailed version of this paper will be submitted for publication elsewhere.

Notation. Through out this paper, let k be an algebraically closed field and Q a finite acyclic quiver. By a module, we mean a left module. For a ring A, we denote by $\mathsf{mod}A$ the category of finitely generated A-modules and by $\mathsf{proj}A$ the category of finitely generated projective A-modules. For $X \in \mathsf{mod}A$, we denote by $\mathsf{Sub}X$ the full subcategory of $\mathsf{mod}A$ whose objects are submodules of finite direct sums of copies of X. For $X \in \mathsf{mod}A$, we denote by $\mathsf{add}X$ the full subcategory of $\mathsf{mod}A$ whose objects are direct summands of finite direct sums of copies of X. For two arrows α, β of a quiver such that the target point of α is the start point of β, we denote by $\alpha\beta$ the composition of α and β.

2. Preliminaries

In this section, we give definitions used in this paper and recall some result of [6]. We first define preprojective algebras and Coxeter groups of Q.

Definition 1. Let Q be a finite acyclic quiver.

(1) The *double quiver* $\overline{Q}$ of Q is a quiver obtained from Q by adding an arrow $\alpha^* : v \to u$ for each arrow $\alpha : u \to v$ of Q.
(2) We define the *preprojective algebra* Π of Q by

$$\Pi := k\overline{Q}/\langle \sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha \rangle.$$

Let Q be a connected quiver. It is known that the preprojective algebra Π of Q is does not depend on the orientation of Q and that Π is finite dimensional and selfinjective if and only if Q is a Dynkin quiver.

Definition 2. The *Coxeter group* $W = W_Q$ of a quiver Q is the group generated by the set $\{s_u \mid u \in Q_0\}$ with relations

- $s_u^2 = 1$,
- $s_v s_u = s_u s_v$ if there exist no arrows between u and v,
- $s_u s_v s_u = s_v s_u s_v$ if there exists exactly one arrow between u and v.

An expression $w = s_{u_1} s_{u_2} \dots s_{u_l}$ is *reduced* if for any other expression $w = s_{v_1} s_{v_2} \cdots s_{v_m}$, we have $l \leq m$. For a reduced expression $w = s_{u_1} s_{u_2} \dots s_{u_l}$, let $\mathsf{Supp}(w) = \{u_1, \cdots, u_l\}$.

Note that, $\mathsf{Supp}(w)$ is independent of the choice of a reduced expression of w. Let Q be a connected quiver. It is known that W_Q is a finite group if and only if Q is a Dynkin quiver.

Next we define a two-sided ideal of Π and recall some result of [6]. For a vertex $u \in Q_0$, we define a two-sided ideal I_u of Π by

$$I_u = \Pi(1 - e_u)\Pi,$$

where e_u is the idempotent of Π for u. For a reduced expression $w = s_{u_1} s_{u_2} \dots s_{u_l}$, we define a two-sided ideal I_w of Π by

$$I_w := I_{u_1} I_{u_2} \cdots I_{u_l}.$$

Note that, an ideal I_w is independent of the choice of a reduced expression of w by [6].

A finite dimensional algebra A is said to be *Iwanaga-Gorenstein of dimension at most one* if $\mathsf{inj.dim}({}_A A) \leq 1$. In this case, it is known that the category $\mathsf{Sub}A$ is a Frobenius category.

Proposition 3. [6] *For any element $w \in W_Q$, the algebra Π_w is finite dimensional and Iwanaga-Gorenstein of dimension at most one.*

Next we introduce a grading of a preprojective algebra. The path algebra $k\overline{Q}$ is regarded as a $\mathbb{Z}$-graded algebra by the following grading:

$$\deg \beta = \begin{cases} 1 & \beta = \alpha^*, \alpha \in Q_1 \\ 0 & \beta = \alpha, \alpha \in Q_1. \end{cases}$$

Since the element $\sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha)$ in $k\overline{Q}$ is homogeneous of degree one, the grading of $k\overline{Q}$ naturally gives a grading on the preprojective algebra $\Pi = \bigoplus_{i \geq 0} \Pi_i$. Since Π_0 is spanned by all paths of degree zero, we have $\Pi_0 = kQ$. For any $w \in W$ the ideal I_w of Π is a graded ideal of Π since so is each I_u. In particular, the quotient algebra Π_w is a graded algebra.

For a graded module $X = \bigoplus_{i \in \mathbb{Z}} X_i$ and an integer j, we define a new graded module $X(j)$ by $(X(j))_i = X_{i+j}$. For any integer j, we define a graded submodule $X_{\geq j}$ of M by

$$(X_{\geq j})_i = \begin{cases} X_i & i \geq j \\ 0 & \text{else} \end{cases}$$

and a graded quotient module of X by $X_{\leq j} = X / X_{\geq j+1}$.

Let $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ be the category of finitely generated $\mathbb{Z}$-graded Π_w-modules with degree zero morphisms. We denote by $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ the full subcategory of $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ of submodules of graded free Π_w-modules, that is,

$$\mathsf{Sub}^{\mathbb{Z}}\Pi_w = \left\{ X \in \mathsf{mod}^{\mathbb{Z}}\Pi_w \mid X \subset \bigoplus_{j=1}^{m} \Pi_w(i_j),\ i_j \in \mathbb{Z} \right\}.$$

Since Proposition 3, $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ is also a Frobenius category. Therefore we have a triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. In this paper, we get a tilting object in this category and calculate the endomorphism algebra of it.

3. c-sortable elements and tilting modules

In this section, we define c-sortable elements. Throughout this section, we denote by W the Coxeter group of Q.

Definition 4. Let Q be a finite acyclic quiver with $Q_0 = \{1, 2, \ldots, n\}$.

(1) An element c in W is called a *Coxeter element* if c has an expression $c = s_{u_1}s_{u_2}\ldots s_{u_n}$, where $u_1, \ldots, u_n$ is a permutation of $1, \ldots, n$.

(2) A Coxeter element $c = s_{u_1}s_{u_2}\ldots s_{u_n}$ in W is said to be *admissible with respect to the orientation of* Q if c satisfies $e_{u_j}(kQ)e_{u_i} = 0$ for $i < j$.

Since Q is acyclic, W has a Coxeter element c admissible with respect to the orientation of Q. There are several expressions of $c = s_{u_1}s_{u_2}\ldots s_{u_n}$ satisfying $\{u_1, \ldots, u_n\} = \{1, \ldots, n\}$ and $e_{u_j}(kQ)e_{u_i} = 0$ for $i < j$. However, it is shown that c is uniquely determined as an element of W. ¿From now on, we call a Coxeter element admissible with respect to the orientation of Q simply a Coxeter element. We define a c-sortable elements.

Definition 5. Let c be a Coxeter element of W. An element $w \in W$ is said to be *c-sortable* if there is a reduced expression $w = s_{u_1} \cdots s_{u_l} = c^{(0)}c^{(1)} \cdots c^{(m)}$, where each $c^{(i)}$ is subsequence of c and

$$\mathsf{Supp}(c^{(m)}) \subset \mathsf{Supp}(c^{(m-1)}) \subset \cdots \subset \mathsf{Supp}(c^{(0)}) \subset Q_0.$$

Example 6. Let $Q = \begin{smallmatrix} & 1 & \\ \swarrow & & \searrow \\ 2 & \longrightarrow & 3 \end{smallmatrix}$. A Coxeter element is $c = s_1s_2s_3$. Then an element $w = s_1s_2s_3s_1s_2s_1$ is a c-sortable element. Actually, $c^{(0)} = s_1s_2s_3$, $c^{(1)} = s_1s_2$, and $c^{(2)} = s_1$. The element $w' = s_1s_2s_3s_1s_3$ is also a c-sortable element. Actually, $c^{(0)} = s_1s_2s_3$ and $c^{(1)} = s_1s_3$.

4. A tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$

In this section, we construct a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. Let $\mathcal{T}$ be a triangulated category. An object M in $\mathcal{T}$ is called a *tilting object* if the following holds.

- $\mathsf{Hom}_{\mathcal{T}}(M, M[j]) = 0$ for any $j \neq 0$,
- $\mathsf{thick}M = \mathcal{T}$, where $\mathsf{thick}M$ is the smallest triangulated full subcategory of $\mathcal{T}$ containing M and closed under direct summands.

Let $\mathcal{T}$ be the stable category of a Frobenius category, and assume that $\mathcal{T}$ is Krull-Schmidt. If there is a tilting object M in $\mathcal{T}$, then it follows from [8, (4.3)] that there exists a triangle equivalence

$$\mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\underline{\mathsf{End}}_{\mathcal{T}}(M)),$$

where $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathsf{End}_{\mathcal{T}}(M))$ is the homotopy category of bounded complexes of projective $\mathsf{End}_{\mathcal{T}}(M)$-modules.

For a reduced expression $w = s_{u_1} \cdots s_{u_l}$ and $1 \leq i \leq l$, let m_i be the number of elements in $\{1 \leq j \leq i-1 \mid u_j = u_i\}$, that is,

$$m_i = \sharp\{1 \leq j \leq i-1 \mid u_j = u_i\}, \qquad \text{for } 1 \leq i \leq l.$$

Moreover, for $1 \leq i \leq l$, put

$$M^i = (\Pi/I_{u_1 \dots u_i})e_{u_i}(m_i),$$

$$M = \bigoplus_{i=1}^{l} M^i.$$

Then we have the following theorem.

Theorem 7. [9] *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then the object M is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$.*

5. The endomorphism algebra of a tilting object

In this section, we calculate the endomorphism algebra of a tilting object which is constructed in Section 4. Throughout this section, for simplicity, assume that a c-sortable element w satisfies $\mathsf{Supp}(w) = Q_0$. Since $\Pi_0 = kQ$, for any graded Π_w-module X, X_0 is a kQ-module. The following theorem is one of the main theorem of [2].

Theorem 8. [2] *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element and M be a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ which is constructed in Section 4. Then there exists an unique tilting kQ-module T_w which satisfies $\mathsf{add}M_0 = \mathsf{Sub}T_w$.*

Using Theorem 8, we calculate the endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$. We have the following morphism of algebras:

$$F : \mathsf{End}^{\mathbb{Z}}_{\Pi_w}(M) \to \mathsf{End}_{kQ}(M_0) \quad f \mapsto f|_{M_0}.$$

Theorem 9. [9] *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then the morphism F induces an isomorphism of algebras:*

$$\underline{F} : \underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M) \xrightarrow{\sim} \mathsf{End}_{kQ}(M_0)/[T_w],$$

where $[T_w]$ is an ideal consisting of morphisms which factors through $\mathsf{add}T_w$.

We can show that the global dimension of the algebra $\mathsf{End}_{kQ}(M_0)/[T_w]$ is at most two. Actually, we can show the following theorem. Let A be a finite dimensional algebra and T a cotilting A-module of finite injective dimension. We denote by ${}^{\perp_{>0}}T$ the full subcategory of $\mathsf{mod}A$ consisting of modules X satisfying $\mathsf{Ext}^i_A(X, T) = 0$ for any $i > 0$.

Theorem 10. [9] *Assume that the global dimension of A is at most n and that ${}^{\perp_{>0}}T$ has an additive generator N. Then the global dimension of $\mathsf{End}_A(N)/[T]$ is at most $3n - 1$.*

Note that $\mathsf{End}_A(M)$ and $\mathsf{End}_A(M)/[T]$ are relative version of Auslander algebras and stable Auslander algebras. It is known that Auslander algebras have global dimension at most two [5], and that stable Auslander algebras have global dimension at most $3n - 1$ [4, Proposition 10.2]. We apply Theorem 10 to our endomorphism algebra.

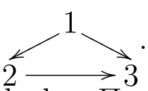
Corollary 11. *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then the global dimension of $\mathsf{End}_{kQ}(M_0)/[T_w]$ is at most two.*

Finally, we have the following theorem.

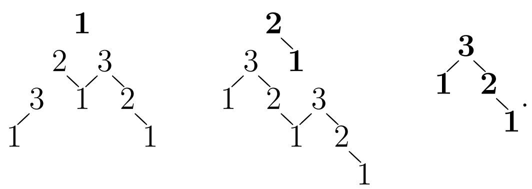
Theorem 12. *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then we have a triangle equivalence $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M))$.*

6. Examples

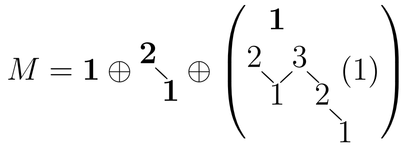
In this section, we calculate some examples.

Example 13. Let Q be a quiver 

. Let $w = s_1 s_2 s_3 s_1 s_2 s_1$. This is a c-sortable element. Then we have a graded algebra $\Pi_w = \Pi_w e_1 \oplus \Pi_w e_2 \oplus \Pi_w e_3$,



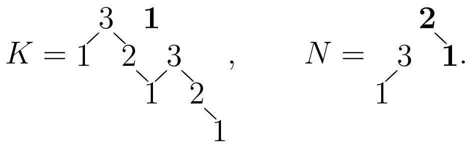
and a tilting module



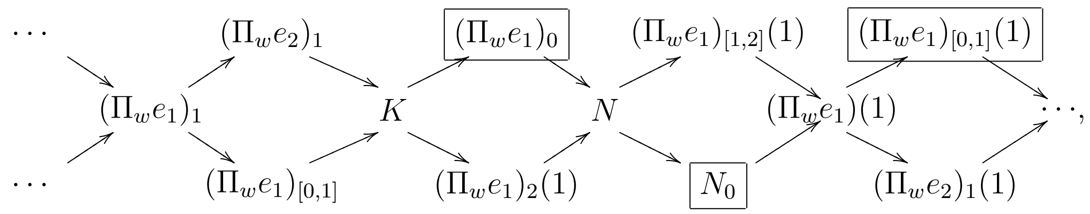
in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$, where graded projective Π_w-modules are removed, and the degree zero parts are denoted by bold numbers. The endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$ of M is given by the following quiver with relations

$$\Delta = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \qquad ab = 0.$$

We can describe the Auslander-Reiten quiver of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. Let K be the kernel of the canonical epimorphism $\Pi_w e_2 \to S_2$, where S_2 is a simple module associated with the vertex 2, and N be the cokernel of an inclusion $(\Pi_w e_1)_1 \to \Pi_w e_2$:

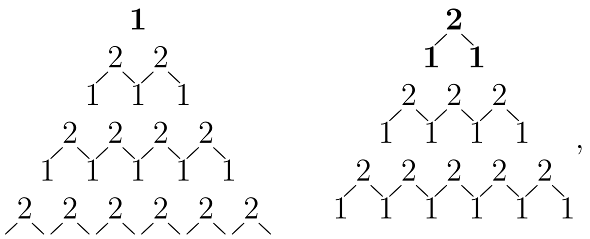


Then the Auslander-Reiten quiver of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ is the following one:



where $M = (\Pi_w e_1)_0 \oplus N_0 \oplus (\Pi_w e_1)_{[0,1]}(1)$.

Example 14. Let Q be a quiver $1 \rightrightarrows 2$. Then we have a graded algebra $\Pi = \Pi e_1 \oplus \Pi e_2$, and these are represented by their radical filtrations as follows:



where the degree zero parts are denoted by bold numbers. Let $c = s_1 s_2$. This is a Coxeter element. Let $w = c^{n+1} = s_1 s_2 s_1 \cdots s_1 s_2$. This is a c-sortable element. We have $(\Pi/I_{c^i})e_1 = (\Pi/J^{2i-1})e_1$, and $(\Pi/I_{c^i})e_2 = (\Pi/J^{2i})e_2$, where J is the Jacobson radical of Π. The object $M = \bigoplus_{i=1}^{n}(\Pi/I_{c^i})(i-1)$ is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$, where graded projective Π_w-modules are removed. The endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$ of M is given by the following quiver with relations

$$\Delta = 1 \underset{b}{\overset{a}{\rightrightarrows}} 2 \underset{b}{\overset{a}{\rightrightarrows}} 3 \underset{b}{\overset{a}{\rightrightarrows}} \quad \cdots\cdots \quad \underset{b}{\overset{a}{\rightrightarrows}} 2n-1 \underset{b}{\overset{a}{\rightrightarrows}} 2n\ , \quad aa = bb.$$

The algebra $k\Delta/\langle aa - bb\rangle$ has global dimension two.

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