FILTERED CATEGORIES AND REPRESENTATIONS OF BOXES

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ABSTRACT. Boxes are generalisations of algebras; their representations form exact, but in general not abelian categories. Filtered categories that occur naturally in the context of quasi-hereditary algebras, can be described as representations of certain boxes. This context and how boxes are applied here is described.

1. STANDARDISABLE SETS AND QUASI-HEREDITARY ALGEBRAS

Let k be a field and $\mathcal{C}$ an abelian k-category. Morphisms are taken in $\mathcal{C}$ unless specified otherwise. By D we denote k-duality $Hom_k(-, k)$.

The main objects in the story to be told here are the following:

Definition 1.1. Finitely many objects $\Delta(1), \ldots, \Delta(n)$ in $\mathcal{C}$ form a *standardisable set* Δ if and only if the following conditions are satisfied:
(a) For each i: $End(\Delta(i)) = k$.
(b) For all i, j: $Hom(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j$.
(c) For all i, j: $Ext(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i < j$.
Moreover, all morphism and extension spaces occuring here are supposed to be finite dimensional over k.

Standardisable sets occur rather frequently in algebra and in geometry. Here are some examples:

(1) Exceptional collections in algebraic geometry are standardisable sets in the category of coherent sheaves.
(2) Weyl modules of algebraic groups or of Schur algebras of algebraic groups form standardisable sets and so do Verma modules of semisimple complex Lie algebras.
(3) When A is a finite dimensional algebra, its simple modules form a standardisable set provided there is an ordering such that condition (c) is satisfied. This happens exactly when the quiver of A has no oriented cycles (that is, when A is *directed*).
(4) The standard modules of a quasi-hereditary algebra form a standardisable set:

Definition 1.2. (Cline, Parshall and Scott [4, 5]) A finite dimensional algebra A together with a partial ordering $\leq$ on its set of isomorphism classes of simple modules $S(1), \ldots, S(n)$ is called a *quasi-hereditary algebra* if and only if the following conditions are satisfied:

This is a short introduction to motivation and main results of the joint paper [12] with Julian Külshammer and Sergiy Ovsienko.

Let $P(1), \ldots, P(n)$ be projective covers of the simple modules $S(1), \ldots, S(n)$, respectively and $\Delta(i)$ the largest quotient of $P(i)$such that $[\Delta(i) : S(j)] \neq 0$ implies $i \geq j$. Then:
For each i: $[\Delta(i) : S(i)] = 1$.
The kernel of the surjection $P(i) \to \Delta(i)$ has a filtration with subquotients $\Delta(j)$ where the indices occuring satisfy $j > i$.

This standardisable set $\Delta(1), \ldots, \Delta(n)$ of A often is denoted by Δ_A.

Although the definition only uses a partial order, we write it as a total order. This can be done without loss of generality. The standard modules $\Delta(i)$ are relative projective; in fact, they are projective objects in the category $A - mod[\leq i]$ whose objects have composition factors with indices not bigger than i.

An algebra may be quasi-hereditary for many choices of partial orderings; therefore, a quasi-hereditary algebra more precisely is a pair $(A, \leq)$. Hereditary algebras, for instance path algebras of directed quivers, are quasi-hereditary with any choice of orderings, and they are characterised by this property. Directed algebras are quasi-hereditary, and for a particular ordering the standard modules are simple.

Here is an explicit example of a quasi-hereditary algebra: The algebra A is given by quiver and relations; we also depict the Loewy series of its projective and standard modules.

$$\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet \Big/ \alpha\beta = 0 \qquad P(1) = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \quad \Delta(1) = 1 \quad P(2) = \Delta(2) = \begin{matrix} 2 \\ 1 \end{matrix}$$

This algebra occurs, up to Morita equivalence, rather frequently, for instance over the complex numbers as principal block of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of the simple Lie algebra $\mathfrak{sl}(2)$, and over infinite fields of characteristic two as Schur algebra $S(2,2)$.

In algebraic Lie theory, one often considers categories $\mathcal{C}$ with infinitely many simple objects and infinitely many standard objects. Examples are rational or polynomial modules of reductive algebraic groups - here, Weyl modules are standard modules - and the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of semisimple complex Lie algebras - here, Verma modules are standard modules - and various quantisations and further generalisations. Cline, Parshall and Scott [4] have developed the concept of highest weight categories to deal with these situations . Such categories are built up in a precisely described way from (module categories of) quasi-hereditary algebras.

When Δ is a standardisable set, the full subcategory $\mathcal{F}(\Delta)$ of $\mathcal{C}$ whose objects have finite filtrations with subquotients being objects $\Delta(j)$, is called the *Δ-filtered* (or just filtered) *category*.

The standard modules of quasi-hereditary algebras look like a special example, but this is in fact the most general class of examples, by Dlab and Ringel's standardisation theorem (presented in Kyoto in 1990, at the workshop preceding ICRA at Tsukuba):

Theorem 1.3. *(Dlab and Ringel [7]) Let $\Delta = \Delta(1), \ldots, \Delta(n)$ be a standardisable set in an abelian k-category $\mathcal{C}$. Then there exists a quasi-hereditary algebra $(A, \leq)$ (unique up to Morita equivalence) such that $\mathcal{F}(\Delta) \simeq \mathcal{F}(\Delta_A)$.*

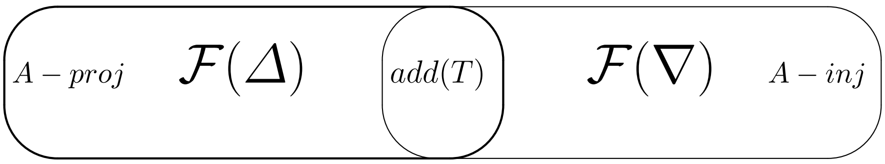
The category $\mathcal{F}(\Delta)$ rarely is abelian - for instance, cokernels of non-trivial maps between standard modules of quasi-hereditary algebras are not Δ-filtered. But it inherits the exact structure of the abelian category $\mathcal{C}$. The equivalence in the standardisation theorem respects the exact structures.

2. Filtered categories and Ringel duality

Let A be a quasi-hereditary algebra. Then the opposite algebra A^{op} is known to be quasi-hereditary, too. Thus, there are modules $\Delta(i, A^{op})$ filtering the projective A^{op}-modules, which are k-dual to the injective A-modules. Therefore, the injective A-modules are filtered by modules $\nabla(i) := D\Delta(i, A^{op})$. They have simple socle $S(i)$ and other properties dual to those of the standard modules Δ. Thus, turning around the partial ordering, the set ∇ turns out to be standardisable as well. Hence, by Dlab and Ringel's standardisation theorem, there must be a quasi-hereditary algebra $R = R(A)$ such that $\mathcal{F}(\nabla) \simeq \mathcal{F}(\Delta, R)$. This algebra now is called the *Ringel dual* of the algebra A; it is unique up to Morita equivalence. At the Tsukuba ICRA in 1990, Ringel presented the following result:

Theorem 2.1. *Let $(A, \leq)$ be a quasi-hereditary algebra. Then there exists a tilting module T such that $\mathcal{F}(\nabla) \cap \mathcal{F}(\Delta) = add(T)$. The module T is called the* characteristic tilting module *of A. Its endomorphism algebra $End_A(T)$ is quasi-hereditary again and it is the Ringel dual R of A.*

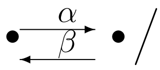
The tilting module T is a full injective object in $\mathcal{F}(\Delta)$ and a full projective object in $\mathcal{F}(\nabla)$. It has finite (but arbitrarily large) projective dimension. Taking the Ringel dual of R produces an algebra that is Morita equivalent to A itself.



This picture illustrates the central role of T in the two filtered categories.The category $add(T)$ coincides with the injectives in $\mathcal{F}(\Delta)$ as well as with the projectives in $\mathcal{F}(\nabla)$ and, up to equivalence, also with the projectives of the Ringel dual R.

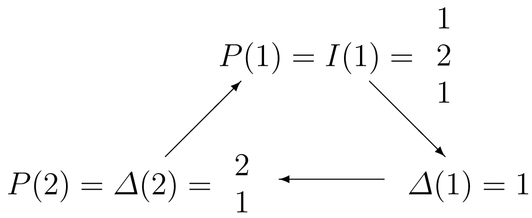
Ringel duality has become very popular and useful in applications of quasi-hereditary algebras and highest weight categories. Soergel [15] has shown that BGG-category $\mathcal{O}$ is Ringel self-dual. Donkin [8] has shown that classical Schur algebras $S(n, r)$ with $n \geq r$ are Ringel self-dual; in this situation, the characteristic tilting module is a direct sum of tensor products of exterior powers of the natural module over GL_n - a description that yields direct applications to invariant theory [9].

In our example, the characteristic tilting module is a direct sum of the projective-injective module $P(1) = I(1)$ and the simple module $1 = \Delta(1) = \nabla(1)$.



$/\alpha\beta = 0 \qquad T = \begin{matrix}1\\2\\1\end{matrix} \oplus 1 \qquad \nabla(1) = 1 \qquad I(2) = \nabla(2) = \begin{matrix}1\\2\end{matrix}$

The Ringel dual $R = End_A(T)$ is isomorphic to A itself, so the algebra A is Ringel self-dual. The category $\mathcal{F}(\Delta)$ has three indecomposable objects, up to isomorphism, and its Auslander-Reiten quiver looks as follows:



The shape of this Auslander-Reiten quiver is very similar to the Auslander-Reiten quiver of the path algebra of type A_2, whose quiver is just one arrow. The difference is that here, there is a map going from right to left, the inclusion $\Delta(1) \to \Delta(2)$. The cokernel of this map is not in $\mathcal{F}(\Delta)$, which just inherits an exact structure from $A-mod$, but not an abelian one. The standard modules $\Delta(1)$ and $\Delta(2)$ look like simple objects in this category, but there are non-trivialmorphisms relating them; so, Schur's lemma is not valid in this situation.

As we have seen, the category $\mathcal{F}(\Delta)$ has more structure than just being exact. It has enough projective objects and enough injective objects. Ringel [14] proved that it has almost split sequences and that it is functorially finite in $A - mod$. It also can be shown that it is closed under kernels of epimorphisms; but it is usually not closed under cokernels of monomorphisms. So, it cannot be a module category, although it has many properties of a module category.

Questions. *(a) What is the structure of the exact category $\mathcal{F}(\Delta)$?*
(b) How is this category related to the algebra A, or rather to its Morita equivalence class?
(c) How to interpret Ringel duality in terms of the standardisable system Δ?

The main result to be reported below answers the first question by interpreting $\mathcal{F}(\Delta)$ as a generalised module category in a precise sense. The second question is, of course, addressed by Dlab and Ringel's standardisation theorem. The main result will provide another, rather different answer, and at the same time an answer to the third question.

3. Boxes

Boxes - originally called bocses (for bimodule over category with coalgebra structure) - have been introduced by the Kiev school around Roiter, Drozd and Ovsienko, in various versions. Alternative approaches to similar concepts are called corings [2] or ditalgebras [1]. The version used in [12] is more general than in the original applications of boxes. The most prominent result based on the theory of boxes is Drozd's tame and wild theorem [10], reproved by Crawley-Boevey [6]. Boxes also have beenused in several attempts to prove Brauer-Thrall type conjectures.

Boxes can be seen as generalisations of algebras. They also have representation categories, which can be studied using methods of representation theory of algebras.

Definition 3.1. A *box* $\mathcal{B}$ is a quadruple (B, W, μ, ϵ), where B is a category, W is a $B-B$-bimodule, $\mu : W \to W \otimes_B W$ is a B-bimodule map that is a coassociative comultiplication, and $\epsilon : W \to B$ is a B-bilinear map that is a counit for μ.

In our context, the category B is just a finite dimensional basic algebra given by quiver and relations. A box $\mathcal{B}$ is called *directed* when the algebra B is directed, which is the same as quasi-hereditary with simple standard modules, and moreover W is a sum of 'directed' projective bimodules (see [12]). Another condition in our context is that the kernel of ϵ has to be a finitely generated projective bimodule.

Modules over a box $\mathcal{B}$ are, by definition, B-modules (which we always assume to be finite dimensional); morphisms are, however, different:

Definition 3.2. Let $\mathcal{B} = (B, W, \mu, \epsilon)$ be a box. A representation of $\mathcal{B}$ is a B-module. Let X and Y be representations of $\mathcal{B}$. Then the morphism space is defined to be $Hom_{\mathcal{B}}(X, Y) := Hom_{B \otimes B^{op}}(W, Hom_k(X, Y))$. Composition of $f : X \to Y$ with $g : Y \to Z$ is defined as follows: f is given by a map $f : W \to Hom_k(X, Y)$ and g is given by a map $g : W \to Hom_k(Y, Z)$. The composition $g \circ f$ is given by a map $W \to Hom_k(X, Z)$ which is the composition $W \stackrel{\mu}{\to} W \otimes_B W \stackrel{g \otimes f}{\to} Hom_k(Y, Z) \otimes_B Hom_k(X, Y) \stackrel{composition}{\longrightarrow} Hom_k(X, Z)$.

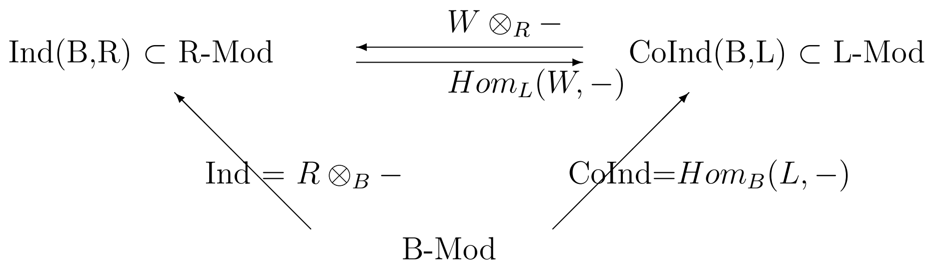
Our definition of homomorphisms is not the usual one, but related to that by adjointness.

The category of representations of a box is not, in general, an abelian category any more. It can, however, be given an exact structure, at least under some assumptions (triangularity of the box).

A very interesting point in changing the definition of morphisms is that also endomorphisms are changing. In particular, since the algebra B is a $\mathcal{B}$-module, it has an endomorphism ring that in general is quite different from Bitself. This is at the basis of a theory developed by Burt and Butler and presented at the Tsukuba ICRA in 1990.

Definition 3.3. Let $\mathcal{B} = (B, W, \mu, \epsilon)$ be a box. The algebras $R_{\mathcal{B}} := End_{\mathcal{B}}(B)^{op} \simeq Hom_B({}_BW, {}_BB)$ and $L_{\mathcal{B}} := End_{\mathcal{B}^{op}}(B) \simeq Hom_B(W_B, B_B)$ are called the *left and right Burt-Butler algebras* of $\mathcal{B}$, respectively.

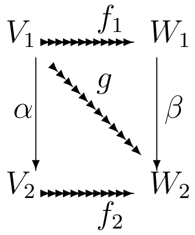
Here is a brief account of Burt-Butler theory: The bimodule W allows to defineinduction and coinduction functors and hence the following commutative diagram of functors:



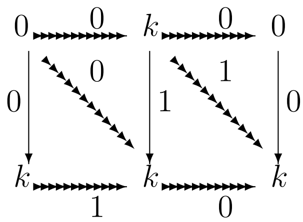
Then there are equivalences of exact categories $rep(\mathcal{B}) \simeq Ind(B,R) \simeq CoInd(B,L)$ provided by restricting the above functors and defining the categories $Ind(B,R)$ and $CoInd(B,L)$ by the images of the respective functors.

As a consequence, the representation category $rep(\mathcal{B})$ has almost split sequences [3]. Moreover, there are double centraliser properties $L_{\mathcal{B}} \simeq End_{R^{op}}(W)$ and $R^{op} \simeq End_L(W)$.

Here is our example again, now from a new point of view. Denote by B the quiver algebra of an A_2 quiver. So, representations are pairs of vector spaces, related by a linear map. For a certain choice of W (revealed in Appendix A.1 in [12], where full details are given), one gets a box (B, W, μ, ϵ) whose representations are the B-representations. The maps between B-representations are triples of linear maps $f = (f_1, f_2, g)$. In the following example, $V : V_1 \overset{\alpha}{\to} V_2$ and $W : W_1 \overset{\beta}{\to} W_2$ are representations of the quiver. The morphism $V \to W$ is given by the triple (f_1, f_2, g). The linear maps f_1 and f_2 make the diagram (without g) commutative, as for ordinary quiver representations. The additional map g makes the difference; in this example it can be chosen freely.



Now we specify the representations V and W and get the following homomorphisms form V to W and back:



Because of g, there are now morphisms in both directions, the one on the righthand side not existing on the level of quiver representations, while the left hand morphism is the same as for quiver representations. In fact, this new morphism corresponds exactly to the 'additional' morphism occuring in $\mathcal{F}(\Delta)$, that is, the morphism $\Delta(1) \to \Delta(2)$. This correspondence is a special case of the main result to be formulated next. The quasi-hereditary algebra A occuring here is exactly the algebra of our example.

4. Results in [12]

The main result in [12] characterises quasi-hereditary algebras and at the same time their filtered categories.

Theorem 4.1. *(1) For a finite dimensional algebra A, the following are equivalent:*
(a) A is quasi-hereditary for some partial order $\leq$.

(b) A is Morita equivalent to $L_{\mathcal{B}'}$, the left Burt-Butler algebra of a directed box $\mathcal{B}'$.
(c) A is Morita equivalent to $R_{\mathcal{B}}$, the right Burt-Butler algebraof a directed box $\mathcal{B}$. In this case, $\mathcal{F}(\Delta) \simeq rep(\mathcal{B})$.

(2) Let $\mathcal{B}$ be a directed box. Then $L_{\mathcal{B}}$ and $R_{\mathcal{B}}$ are Ringel dual to each other.

The Theorem is wrong if we replace 'Morita equivalent' by 'isomorphic'.

The second part implies that Ringel duality is a special case of Burt-Butler duality that relates $L_{\mathcal{B}}$ and $R_{\mathcal{B}}$; the latter also can be formulated for boxes that are not directed.

The proof of the theorem uses Keller's description of filtered categories in terms of A_∞-structures, the machinery of twisted stalks and the Maurer-Cartan equation, a connection of A_∞-structures with differential graded algebras and then a connection between differential graded algebras and boxes.

Finally, here is an application that motivated the whole development. Write $\mathcal{F}(\Delta) \simeq rep(\mathcal{B})$. As explained above in the context of Burt-Butler theory, this means $\mathcal{F}(\Delta) = Ind(B, R_B)$. Write $A = R_B$. Then the following are true:
(a) The induction functor $A \otimes_B -$ is an exact functor, that is, A_B isprojective.
(b) For each i, $A \otimes_B S_B(i) \simeq \Delta(i)$, where S_B denotes simple B-modules.
(c) The algebra B is directed.

This means exactly that B is an *exact Borel subalgebra* of the quasi-hereditary algebra A, in the sense of [11]; hence, the problem of existence of exact Borel subalgebras for quasi-hereditary algebras (up to Morita equivalence) has been solved.

Taking this point of view, the quasi-hereditary algebra A (or rather an algebra Morita equivalent to it) satisfies an analogue of the PBW-theorem, which for semisimple complex Lie algebras states a bimodule isomorphism: $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}_+) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{n}_-)$. This bimodule isomorphism implies that the Lie theoretic Borel subalgebra $\mathcal{U}(\mathfrak{n}_+ \oplus \mathfrak{h}) \simeq \mathcal{U}(\mathfrak{n}_+) \otimes_{\mathbf{C}} \mathcal{U}(\mathfrak{h})$ satisfies (a); condition (b) is the definition of Verma modules, which are the standard modules in this context, and condition (c) is satisfied by definition. Blocks of the BGG-category $\mathcal{O}$ of $\mathfrak{g}$ are Morita equivalent to quasi-hereditary algebras. For these particular algebras, existence of exact Borel subalgebras had been shown already in [11], but in general it had been an open problem.

For proofs and further details, see [12]. For more on the context and for some recent developments see [13].

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