CONSTRUCTION OF TWO-SIDED TILTING COMPLEXES FOR BRAUER TREE ALGEBRAS

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Abstract. In this note, we explain how to construct two-sided tilting complexes corresponding to one-sided tilting complexes for Brauer tree algebras.

1. Introduction

For finite dimensional symmetric algebras Γ and Λ over an algebraically closed field k, the following is known.

Theorem 1. [2, 3, 4] *Let Γ and Λ be symmetric k-algebras. Then the following are equivalent.*

(1) Γ *and* Λ *are derived equivalent.*
(2) *There exists a complex T of* $K^b(\Gamma\text{-proj})$ *which satisfies the following conditions.*
  (i) $\mathrm{Hom}_{K^b(\Gamma\text{-proj})}(T, T[n]) = 0 \ \ (0 \neq \forall n \in \mathbb{Z})$.
  (ii) $\mathrm{add}(T)$ *generates* $K^b(\Gamma\text{-proj})$ *as a triangulated category.*
  (iii) $\mathrm{End}_{K^b(\Gamma\text{-proj})}(T) \cong \Lambda$.
(3) *There exists a complex C of* $K^b(\Gamma\otimes_k\Lambda^{op}\text{-mod})$ *which satisfies the following conditions.*
  (i) *All terms of C are projective as Γ-modules and as Λ^{op}-modules.*
  (ii) $C^* \otimes_\Gamma C \cong \Lambda$ *in* $K^b(\Lambda \otimes_k \Lambda^{op})$.

Definition 2. A complex T over Γ is called a one-sided tilting complex if it satisfies the conditions (i) and (ii) in Theorem 1 (2). A complex C over $\Gamma \otimes_k \Lambda^{op}$ is called a two-sided tilting complex if it satisfies the conditions (i) and (ii) in Theorem 1 (3).

Let C be a two-sided tilting complex over $\Gamma\otimes_k\Lambda^{op}$. It is known that if we consider C as a one-sided complex over Γ, then it is a one-sided tilting complex with endomorphism ring Λ. However it is difficult in general to construct the two-sided tilting complex corresponding to a one-sided tilting complex.

Let k be an algebraically closed field. Let A be a Brauer tree algebra over k associated to a Brauer tree with e edges and multiplicity μ of the exceptional vertex. Let $B(e,\mu)$ be a Brauer tree algebra over k with respect to a "star" with e edges and exceptional vertex with multiplicity μ in the center (or equivalently is a self-injective Nakayama algebra over k with e simple modules and the nilpotency degree of the radical being $e\mu + 1$). In [2], Rickard showed that A is derived equivalent to the algebra $B(e,\mu)$ by constructing a one-sided tilting complex T over A with endomorphism algebra $B(e,\mu)$. Our aim in this note is to construct a two-sided tilting complex C over $A \otimes_k B(e,\mu)^{op}$ corresponding to the one-sided tilting complex T constructed by Rickard in [2].

The detailed version of this paper will be submitted for publication elsewhere.

2. CONSTRUCTION OF TWO-SIDED TILTING COMPLEXES

Throughout this note algebras are of finite dimensional. First we recall the definition of stable equivalences of Morita type and properties of them.

Definition 3. Let Γ and Λ be symmetric k-algebras. Then Γ and Λ are said to be stably equivalent of Morita type if there exists a $\Gamma \otimes_k \Lambda^{op}$-module M such that

(1) M is projective as a Γ-module and as a Λ-module,
(2) $M \otimes_\Lambda M^* \cong \Gamma \oplus P$ as $\Gamma \otimes_k \Gamma^{op}$-modules, where P is a finitely generated projective $\Gamma \otimes_k \Gamma^{op}$-module and where $M^* = \mathrm{Hom}_k(M, k)$.

Proposition 4. [1, 3] *Two derived equivalent symmetric algebras are stably equivalent of Morita type.*

We know the Brauer tree algebras A and $B(e, \mu)$ defined in Section 1 are derived equivalent. Hence they are stably equivalent of Morita type by Proposition 4 since Brauer tree algebras are symmetric. Therefore there exists an $A \otimes_k B(e, \mu)^{op}$-module M inducing a stable equivalence of Morita type between A and $B(e, \mu)$.

Second we fix the notation to construct the two-sided tilting complex as mentioned above. For an edge corresponding to a simple A-module T, we define a positive integer $d(T)$ as the distance from the exceptional vertex to the furthest vertex of the edge. On this definition we put $m := \max\{d(T) \,|\, T : \text{simple } A\text{-module}\}$. Moreover let S be a simple A-module such that $d(S) = m$.

We need the next lemma later.

Lemma 5. *There exists an $A \otimes_k B(e, \mu)^{op}$-module M inducing a stable equivalence of Morita type between A and $B(e, \mu)$ such that $M^* \otimes_A S$ is simple.*

We construct the two-sided tilting complex by deleting some direct summand from each term of the projective resolution of M described in the Lemma5. Hence we consider the minimal projective resolution of M.

Lemma 6. [5] *Let Γ and Λ be symmetric k-algebras, and let $\mathcal{M}$ be a $\Gamma \otimes_k \Lambda^{op}$-module which is projective as a Γ-module and as a Λ-module. Then the projective cover of $\mathcal{M}$ is given by*

$$\bigoplus_W P(\mathcal{M} \otimes_\Lambda W) \otimes_k P(W)^*$$

where W runs over a complete set of representatives of isomorphism classes of simple Λ-modules.

If M is projective as an A-module and as a $B(e, \mu)^{op}$-module then $\Omega^n M$ is projective as an A-module and as a $B(e, \mu)^{op}$-module too for any integer n. Hence by Lemma 6 we obtain the minimal projective resolution of M:

$$
\begin{aligned}
\cdots \to & \bigoplus_{0\le i\le e-1} P(\Omega^{n-1}M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* \\
\to & \cdots \\
\to & \bigoplus_{0\le i\le e-1} P(\Omega M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* \\
\xrightarrow{\pi_1} & \bigoplus_{0\le i\le e-1} P(M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* \\
\xrightarrow{\pi_0} & M
\end{aligned}
$$

where V is the simple B-module $M^* \otimes_A S$ (see Lemma 5).

Lemma 7. *For the above projective resolution of M and $1 \le l \le m-2$,*

$$\pi_l\Big(\bigoplus_{d(\mathrm{top}(\Omega^l M\otimes_B \Omega^{2i}V))\le m-l-1} P(\Omega^l M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^*\Big)$$

is contained in

$$\bigoplus_{d(\mathrm{top}(\Omega^{l-1} M\otimes_B \Omega^{2i}V))\le m-l} P(\Omega^{l-1} M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^*.$$

We can construct the two-sided complex $C = (C_n, d)$ by deleting a direct summand in each term of the projective resolution of M as follows:

$$
\begin{cases}
C_0 = M \\
C_n = \displaystyle\bigoplus_{d(\mathrm{top}(\Omega^{n-1} M\otimes_B \Omega^{2i}V))\le m-n} P(\Omega^{n-1} M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* & (1 \le n \le m-1) \\
C_n = 0 & (\text{otherwise})
\end{cases}
$$

and letting d_l be the restriction of π_l to C_l. By Lemma 7 we have that d_l is well-defined for each l. This two-sided complex C is a two-sided tilting complex and if we restrict action of $A \otimes_k B(e,\mu)^{op}$ to A then it coinsides with the one-sided tilting complex T constructed by Rickard in [2].

Theorem 8. *Let A be a Brauer tree algebra with e edges and multiplicity μ and let $B(e,\mu)$ be a Brauer tree algebra for a star with e edges and exceptional vertex with multiplicity μ in the center of the star. Then there exists a two-sided tilting complex C over $A \otimes_k B(e,\mu)^{op}$ such that when restricted to A, this complex coinsides with the one-sided tilting complex T constructed by Rickard in* [2].

3. OUTLINE OF THE PROOF

In this section, let C be the two-sided tilting complex over $A \otimes_k B(e,\mu)^{op}$ constructed in Section 2 and let $T = \bigoplus_{1\le i\le e} T_i$ be the one-sided tilting complex constructed by Rickard in [2], where T_i is the indecomposable summand for each $1 \le i \le e$.

To show that the two-sided complex C is a tilting complex, we show the following lemma.

Lemma 9. *For the two-sided complex C, the following hold.*

- $\mathrm{Hom}_{D^b(B\otimes_k B^{op})}(C^* \otimes_A C, V_i \otimes_k V_j) \cong \delta_{ij}k$
- $\mathrm{Hom}_{D^b(B\otimes_k B^{op})}(C^* \otimes_A C, V_i \otimes_k V_j[-n]) = 0$ *for* $1 \leq n \leq m-1$

where $\{V_1, \cdots, V_e\}$ *are the complete set of representatives of the isomorphism classes of simple* $B(e, \mu)$*-modules.*

This lemma shows that $C^* \otimes_A C \cong B$ in $K^b(B \otimes_k B^{op})$. Therefore C is the two-sided tilting complex.

Next, to show that if we restrict C to A then it coincides with T in the derived category $D^b(A)$, we show the following condition.

Lemma 10. *For the two-sided tilting complex C and the simple B-module V_i, there exists an indecomposable summand T_i of T such that it satisfies the following conditions:*

- $\mathrm{Hom}_{D^b(A)}(T_i, C \otimes_B V_i) \cong k$.
- *For any simple B-module V_j which is not isomorphic to V_i,*
$$\mathrm{Hom}_{D^b(A)}(T_i, C \otimes_B V_j) = 0.$$
- *For any nonzero integer n and any simple B-module U,*
$$\mathrm{Hom}_{D^b(A)}(T_i, C \otimes_B U[n]) = 0.$$

This lemma shows that $C \otimes_B B \cong T$ in the derived category $D^b(A)$. In other words, if we restrict C to A, then it coincides with T in the derived category $D^b(A)$. Therefore C is the requiered two-sided tilting complex.

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