COTORSION PAIRS ON TRIANGULATED AND EXACT CATEGORIES

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ABSTRACT. We study hearts of cotorsion pairs in triangulated and exact categories. We show that they are equivalent to functor categories over cohearts of the cotorsion pairs.

1. INTRODUCTION

The notion of cotorsion pair in triangulated and exact categories is a general framework to study important structures in representation theory. Recently the notion of hearts of cotorsion pairs was introduced in [8] and [6], and they are proved to be abelian categories, which were known for the heart of t-structure [2] and the quotient category by cluster tilting subcategory. We refer to [7] and [1] for more results on hearts of cotorsion pairs.

In this talk, we give an equivalence between hearts and the functor categories over cohearts. For the details of functor category, see [4, Definition 2.9].

For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on a triangulated category $\mathcal{T}$, we introduce the notion of *cohearts* of a cotorsion pair, denote by

$$\mathcal{C} = \mathcal{U}[-1] \cap {}^{\perp}\mathcal{U}.$$

This is a generalization of coheart of a co-t-structure, which plays an important role in [5]. We have the following theorem in triangulated category.

Theorem 1. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on a triangulated category $\mathcal{T}$. If $\mathcal{U}[-1] \subseteq \mathcal{C}{*}\mathcal{U}$, then the heart of $(\mathcal{U}, \mathcal{V})$ has enough projectives, and moreover it is equivalent to the functor category* mod$\mathcal{C}$.

This generalizes [3, Theorem 3.4] which is for t-structure. One standard example of this theorem is the following: let A be a Noetherian ring with finite global dimension, then the standard t-structure of $\mathsf{D}^b(\text{mod}A)$ has a heart mod$A$ with co-heart proj$A$, and we have an equivalence mod$A \simeq$ mod(proj$A$) in this case.

For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on an exact category $\mathcal{E}$, we denote

$$\mathcal{C} = \mathcal{U} \cap {}^{\perp_1}\mathcal{U}$$

the *coheart* of $(\mathcal{U}, \mathcal{V})$. We have the following theorem in exact category.

Theorem 2. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on an exact category $\mathcal{E}$ with enough projectives and injectives, if for any any object $U \in \mathcal{U}$, there exists an exact sequence $0 \to U' \to C \to U \to 0$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then the heart of $(\mathcal{U}, \mathcal{V})$ has enough projectives, and moreover it is equivalent to the functor category* mod$(\mathcal{C}/\mathcal{P})$*, where $\mathcal{P}$ is the subcategory of projetive objects on $\mathcal{E}$.*

The detailed version of this paper will be submitted for publication elsewhere.

2. Hearts on triangulated categories

Let $\mathcal{T}$ be a triangulated category.

Definition 3. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{T}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{T}}(\mathcal{U},\mathcal{V})=0$.
(b) For any object $T\in\mathcal{T}$, there exist two short exact sequences

$$T[-1]\to V_T\to U_T\to T,\quad T\to V^T\to U^T\to T[1]$$

satisfying $U_T, U^T\in\mathcal{U}$ and $V_T, V^T\in\mathcal{V}$.

For a cotorsion pairs $(\mathcal{U},\mathcal{V})$, let $\mathcal{W}:=\mathcal{U}\cap\mathcal{V}$. We denote the quotient of $\mathcal{T}$ by $\mathcal{W}$ as $\underline{\mathcal{T}}:=\mathcal{T}/\mathcal{W}$. Let

$$\mathcal{T}^+:=\{T\in\mathcal{T}\mid U_T\in\mathcal{W}\},\quad \mathcal{T}^-:=\{T\in\mathcal{T}\mid V^T\in\mathcal{W}\}.$$

Let

$$\mathcal{H}:=\mathcal{T}^+\cap\mathcal{T}^-$$

we call the additive subcategory $\underline{\mathcal{H}}$ the *heart* of cotorsion pair $(\mathcal{U},\mathcal{V})$. Under these settings, Abe, Nakaoka [1] introduced the homological functor $H:\mathcal{T}\to\underline{\mathcal{H}}$ associated with $(\mathcal{U},\mathcal{V})$. We often use the following property of H: $H(\mathcal{U})=0=H(\mathcal{V})$.

Let's start with an important property for H.

Proposition 4. *The functor $H:\mathcal{C}\to H(\mathcal{C})$ is an equivalence.*

Hence it is enough to show that $\underline{\mathcal{H}}\simeq \mathrm{mod}(H(\mathcal{C}))$.

Then we give the following theorem.

Theorem 5. *If $\mathcal{U}[-1]\subseteq\mathcal{C}*\mathcal{U}$, then $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$.*

Now we have the main result of this section.

Theorem 6. *If $\mathcal{U}[-1]\subseteq\mathcal{C}*\mathcal{U}$, then $\underline{\mathcal{H}}\simeq\mathrm{mod}(H(\mathcal{C}))$.*

Note that the condition $\mathcal{U}[-1]\subseteq\mathcal{C}*\mathcal{U}$ is satisfied in many cases. The following proposition is given as an example.

Proposition 7. *If $\mathcal{U}$ is covariantly finite and $\mathcal{T}$ is Krull-Schmidt, then $\mathcal{U}[-1]\subseteq\mathcal{C}*\mathcal{U}$.*

3. Hearts on exact categories

Let $\mathcal{E}$ be a exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$.

Definition 8. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{E}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{U},\mathcal{V})=0$.
(b) For any object $B\in\mathcal{E}$, there exits two short exact sequences

$$0\to V_B\to U_B\to B\to 0,\quad 0\to B\to V^B\to U^B\to 0$$

satisfying $U_B, U^B\in\mathcal{U}$ and $V_B, V^B\in\mathcal{V}$.

For a cotorsion pairs $(\mathcal{U}, \mathcal{V})$, we denote the quotient of $\mathcal{E}$ by $\mathcal{U} \cap \mathcal{V}$ as $\underline{\mathcal{E}} := \mathcal{E}/\mathcal{U} \cap \mathcal{V}$. Let

$$\mathcal{E}^+ := \{B \in \mathcal{E} \mid U_B \in \mathcal{W}\}, \quad \mathcal{E}^- := \{B \in \mathcal{E} \mid V^B \in \mathcal{W}\}.$$

Let

$$\mathcal{H} := \mathcal{E}^+ \cap \mathcal{E}^-$$

we denote the additive subcategory $\underline{\mathcal{H}}$ the *heart* of cotorsion pair $(\mathcal{U}, \mathcal{V})$. Let $H : \mathcal{E} \to \underline{\mathcal{H}}$ be the half exact functor associated with $(\mathcal{U}, \mathcal{V})$ [7]. We often use the following property of H: $H(\mathcal{U}) = 0 = H(\mathcal{V})$. Since $\mathcal{P} \subseteq \mathcal{U}$ and $\mathcal{I} \subseteq \mathcal{V}$, we have $H(\mathcal{P}) = 0 = H(\mathcal{I})$.

Let $\Omega\mathcal{C} = \{X \in \mathcal{E} \mid$ X admits $0 \to X \to P \to C \to 0$ where $P \in \mathcal{P}$ and $C \in \mathcal{C}\}$. Let $\pi : \Omega\mathcal{C} \to \Omega\mathcal{C}/\mathcal{P}$ be the quotient functor, since $H(\mathcal{P}) = 0$, we have a functor $\overline{H}$: $H(\Omega\mathcal{C}) \to \Omega\mathcal{C}/\mathcal{P}$ such that $\overline{H}\pi = H$.

As in the last section, we have the following proposition.

Proposition 9. $\overline{H} : \Omega\mathcal{C}/\mathcal{P} \to H(\Omega\mathcal{C})$ *is an equivalence.*

Since $\mathcal{C}/\mathcal{P} \simeq \Omega\mathcal{C}/\mathcal{P}$, it is enough to show that the heart of $(\mathcal{U}, \mathcal{V})$ to $\mathrm{mod}(H(\Omega\mathcal{C}))$.

Now we are ready to give the main theorem of this section.

Theorem 10. *If for any any object $U \in \mathcal{U}$, there exists an exact sequence $0 \to U' \to C \to U \to 0$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then $\underline{\mathcal{H}}$ has enough projectives $H(\Omega\mathcal{C})$.*

Theorem 11. *If for any object $U \in \mathcal{U}$, there exists an exact sequence $0 \to U' \to C \to U \to 0$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then $\underline{\mathcal{H}} \simeq \mathrm{mod}(H(\Omega\mathcal{C}))$.*

The following proposition shows that the assumption of Theorem 10 and 11 is satisfied in many cases.

Proposition 12. *If $\mathcal{U}$ is covariantly finite and contains $\mathcal{I}$, $\mathcal{E}$ is Krull-Schmidt , then for any object $U \in \mathcal{U}$, there exists an exact sequence $0 \to U' \to C \to U \to 0$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$.*

References

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