HIGHER PRODUCTS ON YONEDA EXT ALGEBRAS..

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Abstract. We show that Massey products on a relative split Ext algebra can be computed in terms of relative split exact sequences. As an application we provide a proof and generalization of the result due to Gugenheim-May and Keller which states that in a suitable situation the Ext algebra $E = \mathrm{Ext}_A(S, S)$ of the direct sum S of all simple modules is generated by the degree 1-part E^1 as an algebra with "higher products". In our proof we see that this proposition is a trivial consequence of the elementary fact that every finite length module has composition series.

1. Introduction

We would like to recall a basic fact of Homological algebra. The n-th extension group $\mathrm{Ext}^n_A(N, M)$ the n-th derived functor of Hom functor has a description

$$\mathrm{Ext}^n_A(N, M) = \{0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0\} /(\text{equivalence})$$

and that under this description, the multiplication on the Ext algebra $\mathrm{Ext}_A(M, M) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_A(M, M)$ corresponds to splicing exact sequences which represents corresponding elements.

Since the Ext algebra $\mathrm{Ext}^n_A(M, M)$ is the cohomology algebra of the endomorphism dg-algebra $\mathbf{R}\mathrm{Hom}_A(P, P)$ for a projective (or injective) resolution P of M, it has more structure than merely a graded associative multiplication. These are A_∞-structure and (higher, matric) Massey products, which are not 2-ary operations but multi-ary operations and are collectively called higher products. Such structure was found in topology and has been studied in many area. Recently, higher products have been becoming to play important role in representation theory (see e.g., [3]).

Now we meet a simple question that under the above description of Ext algebra by exact sequences, what operations for exact sequences correspond to higher products. It is not so obvious at the first sight. For example, presence of triple product tells us that there exists a way to construct an exact sequence $0 \to M \to X \to Y \to M \to 0$ from three short exact sequences $0 \to M \to U \to M \to 0, 0 \to M \to V \to M \to 0, 0 \to M \to W \to M \to 0$.

In this note, an answer is given for Massey products. Namely we show that Massey products on a Ext algebra can be computed in terms of exact sequences. As an application we provide a proof and generalization of the result due to Gugenheim-May and Keller which states that in a suitable situation the Ext algebra $E = \mathrm{Ext}_A(S, S)$ of the direct sum S of all simple modules is generated by the degree 1-part E^1 as an algebra with higher products.

In our proof we see that this proposition is a trivial consequence of the elementary fact that every finite length module has composition series.

The detailed version of this paper will be submitted for publication elsewhere.

2. Massey product

We recall the definition of Massey products (for detail see e.g. [5]). Let $\mathsf{C} = (\bigoplus_{d\in\mathbb{Z}} \mathsf{C}, \partial)$ be a dg-algebra and $\alpha_{i,i+1}$ be a homogeneous element of the cohomology algebra $\mathrm{H}(\mathsf{C})$ of degree $d_{i,i+1}$. We assume that $\alpha_{i,i+1}\alpha_{i+1,i+2} = 0$ for $i = 0, \dots, n-2$. We define the integers d_{ij} for $0 \leq i+1 < j \leq n$ by the formula $d_{ij} = (\sum_{k=i+1}^{j-1} d_{ik}d_{kj}) - 1$ by using induction on $j - i$.

In this situation we say that the Massey product $\langle \alpha_{01}, \alpha_{12}, \dots, \alpha_{n-1,n}\rangle$ is defined if there exist $\rho_{ij} \in \mathsf{C}^{d_{ij}}$ for $0 \leq i < j \leq n$ with $(i,j) \neq (0,n)$ such that (1) $\rho_{i,i+1}$ is a cocycle such that $[\rho_{i,i+1}] = \alpha_{i,i+1}$ and that (2) the following equations are satisfied:

$$\sum_{k=i+1}^{j-1} (-1)^{d_{ik}+1} \rho_{ik}\rho_{kj} = \partial \rho_{ij}.$$

In the case where the Massey product $\langle \alpha_{01}, \alpha_{12}, \dots, \alpha_{n-1,n}\rangle$ is defined, the set of the Massey product $\langle \alpha_{01}, \alpha_{12}, \dots, \alpha_{n-1,n}\rangle$ of $\alpha_{01}, \dots, \alpha_{n-1,n}$ is defined to be a subset of $\mathrm{H}(\mathsf{C})$ which consists of the cohomology class $[\rho_{0n}]$ of the cocycles ρ_{0n} such that there exists a collection $(\rho_{ij})_{0\leq i<j\leq n,(i,j)\neq(0,n)}$ satisfying the above conditions such that

$$\rho_{0n} = \sum_{k=1}^{n-1} (-1)^{d_{0k}+1} \rho_{0k}\rho_{kn}.$$

3. Computation of Massey products of Yoneda Ext algebra in terms of exact sequences

For simplicity we deal with algebras A over a field k.

To compute Massey products on $\mathrm{Ext}_A(M,N)$, we use the model

$$\mathsf{C}(M,N) = \mathrm{Hom}_{A-A}(\mathsf{B}(A), \mathrm{Hom}_k(M,N))$$

where $\mathsf{B}(A)$ is the Bar resolution of A. We recall two things: the n-th cohomology group $\mathrm{H}^i(\mathsf{C}(M,N))$ is naturally isomorphic to the i-th extension group $\mathrm{Ext}^i_A(M,N)$. The coalgebra structure on $\mathsf{B}(A)$ induces the product

$$\mathsf{C}(M,N) \times \mathsf{C}(L,M) \to \mathsf{C}(L,N)$$

which is a morphism of complexes that become the Yoneda product after taking the cohomology group.

Let $M_0, M_1, \dots, M_n$ be A-modules. A finite exhaustive filter $F : 0 = F_{-1} \subsetneq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = M$ of an A-module M is said to be of type $(M_0, M_1, \dots, M_n)$ if we have isomorphisms $F_i/F_{i-1} \cong M_i$. Thus a filtered module M of type $(M_0, M_1, \dots, M_n)$ is isomorphic to the direct sum $\bigoplus_{i=0}^n M_i$ as k-modules.

Theorem 1. *Let ξ be the following exact sequence*

$$0 \to M_0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{a-1}} X_a \xrightarrow{f_a} N \to 0.$$

Assume that the module X_1 has a filter of type $(M_0, M_1, \dots, M_n)$ for some A-module $M_1, \dots, M_n$. Then let X_1' be the kernel of the projection $\pi : X_1 \to M_n$ and $X_2' :=$

$X_2/f_1(X'_1)$. *Let η be the exact sequence*

$$0 \to M_n \xrightarrow{g_1} X'_2 \xrightarrow{g_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{a-1}} X_a \xrightarrow{f_a} N \to 0$$

where g_1 and g_2 are induced morphisms. Let $\lambda_{i,i+1} \in \mathrm{Z}^1(\mathsf{C}(M_{i+1}, M_i))$ for $i = 0, 1, \ldots, n-1$ be the cocycle represent the extension induced from the filtration of X_1. Then the Massey product $\langle [\lambda_{01}], [\lambda_{12}], \ldots, [\lambda_{n-1n}], [\eta] \rangle$ is defined and contains $(-1)^{n+1}[\xi]$.

4. Applications: Generating condition of Ext algebra as algebra with higher products

Thanks to Theorem 1, the following theorems are nothing but a consequences the elementary fact that every finite length module has a composition series.

Theorem 2 ([1, Corollary 5.17],[2, 2.2.1.(b)])**.** *Let A be a locally finite non-negatively graded algebra over a field and S be a direct sum of all simple modules. Then the extension algebra $\mathrm{Ext}_A(S, S)$ is generated by $\mathrm{Ext}^0_A(S, S)$ and $\mathrm{Ext}^1_A(S, S)$ using Massey products.*

Proof. Once we recall the fact that any element $\alpha \in \mathrm{Ext}^n_A(S, S)$ is represented by an exact sequence ξ such that each X_i is of finite length.

$$\xi : 0 \to S \to X_1 \to X_2 \to \cdots \to X_n \to S \to 0$$

Then by using Theorem 1, we can show that ξ is obtained as a consequence of iteration of Massey products from $\mathrm{Ext}^1_A(S, S)$. □

Remark 3. (1) The proof of [1] required that the dg-algebra $\mathsf{C}(S, S)$ is augmented and the degree 0-part $\mathsf{C}^0(S, S)$ is semi-simple. The authors showed that the extension algebra $\mathrm{Ext}_A(S, S)$ is generated by $\mathrm{Ext}^0_A(S, S)$ and $\mathrm{Ext}^1_A(S, S)$ using Matric Massey products.

(2) In [2], any proof is not written. The author stated that Then the extension algebra $\mathrm{Ext}_A(S, S)$ is generated by $\mathrm{Ext}^0_A(S, S)$ and $\mathrm{Ext}^1_A(S, S)$ using A_∞ products.

(3) By [4], for the conditions for a graded algebra E below we have the implications (1) ⇒ (2) ⇒ (3):
   (1) E is generated by using Massey products by E^0 and E^1.
   (2) E is generated by using Matric Massey products by E^0 and E^1.
   (3) E is generated by using A_∞-products by E^0 and E^1.
   I don't know how is the converse.

Corollary 4. *Let A be a locally finite non-negatively graded algebra such that the augmentation algebra A^0 is semi-simple. Then the followings are equivalent:*

1. *A is Koszul.*
2. *the Ext algebra $E = \mathrm{Ext}_A(A_0, A_0)$ is generated by E^0 and E^1 as an ordinary algebra.*
3. *the higher A_∞-product on the Ext algebra $\mathrm{Ext}_A(A_0, A_0)$ vanish.*
4. *the Matric Massey products on the Ext algebra $\mathrm{Ext}_A(A_0, A_0)$ vanish.*
5. *the Massey products on the Ext algebra $\mathrm{Ext}_A(A_0, A_0)$ vanish.*

In the similar way we can prove the following theorem. We recall that a ring R is called a *Noetherian algebra* if the center $Z(R)$ is a (commutative) Noetherian ring and R is a finite $Z(R)$-module.

Theorem 5. *Let R be a Noetherian algebra with the center $Z = Z(R)$ and $\mathfrak{p}$ a maximal ideal of Z. We set $\kappa(\mathfrak{p}) := R \otimes_Z Z_\mathfrak{p}/\mathfrak{p}Z_\mathfrak{p}$. Then the Ext algebra $E = \mathrm{Ext}_R(\kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$ is generated by E^0 and E^1 using Massey products.*

This can be proved by using the fact that every exact sequence which has $\kappa(\mathfrak{p})$ the most left term and the right most term is equivalent to a exact sequence with finite length middle terms X_i.

$$0 \to \kappa(\mathfrak{p}) \to X_1 \to X_2 \to \cdots \to X_n \to \kappa(\mathfrak{p}) \to 0$$

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